Nils A. Baas · Eric M. Friedlander
Bjørn Jahren · Paul Arne Østvær
Editors

ABEL PRISEN

ABEL SYMPOSIA 4

Algebraic Topology

The Abel Symposium 2007

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Algebraic Topology

The Abel Symposium 2007

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Preface to the Series

The Niels Henrik Abel Memorial Fund was established by the Norwegian government on January 1, 2002. The main objective is to honor the great Norwegian mathematician Niels Henrik Abel by awarding an international prize for outstanding scientific work in the field of mathematics. The prize shall contribute towards raising the status of mathematics in society and stimulate the interest for science among school children and students. In keeping with this objective the board of the Abel fund has decided to finance one or two Abel Symposia each year. The topic may be selected broadly in the area of pure and applied mathematics. The Symposia should be at the highest international level, and serve to build bridges between the national and international research communities. The Norwegian Mathematical Society is responsible for the events. It has also been decided that the contributions from these Symposia should be presented in a series of proceedings, and Springer Verlag has enthusiastically agreed to publish the series. The board of the Niels Henrik Abel Memorial Fund is confident that the series will be a valuable contribution to the mathematical literature.

Ragnar Winther
Chairman of the board of the Niels Henrik Abel Memorial Fund

Preface

The 2007 Abel Symposium took place at the University of Oslo from August 5 to August 10, 2007. The aim of the symposium was to bring together mathematicians whose research efforts have led to recent advances in algebraic geometry, algebraic K-theory, algebraic topology, and mathematical physics. The common theme of this symposium was the development of new perspectives and new constructions with a categorical flavor. As the lectures at the symposium and the papers of this volume demonstrate, these perspectives and constructions have enabled a broadening of vistas, a synergy between once-differentiated subjects, and solutions to mathematical problems both old and new.

This symposium was organized by two complementary groups: an external organizing committee consisting of Eric Friedlander (Northwestern,University of Southern California), Stefan Schwede (Bonn) and Graeme Segal (Oxford) and a local organizing committee consisting of Nils A. Baas (Trondheim), Bjørn Ian Dundas (Bergen), Bjørn Jahren (Oslo) and John Rognes (Oslo).

The webpage of the symposium can be found at http://abelsymposium.no/symp2007/info.html

The interested reader will find titles and abstracts of the talks listed here

		Monday 6th	Tuesday 7th	Wednesday 8th	Thursday 9th
09.30	10.30	F. Morel	S. Stolz	J. Lurie	N. Strickland
11.00	12.00	M. Hopkins	A. Merkurjev	J. Baez	U. Jannsen
13.30	14.20	R. Cohen	M. Behrens	H. Esnault	C. Rezk
14.40	15.30	L. Hesselholt	M. Rost	B. Toën	M. Levine
16.30	17.20	M. Ando		U. Tillmann	D. Freed
17.40	18.30	D. Sullivan		V. Voevodsky	

as well as a few online lecture notes. Moreover, one will find here useful information about our gracious host city, Oslo.

The present volume consists of 12 papers written by participants (and their collaborators). We give a very brief overview of each of these papers.

"The classifying space of a topological 2-group" by J.C. Baez and D. Stevenson: Recent work in higher gauge theory has revealed the importance of categorising the theory of bundles and considering various notions of 2-bundles. The present paper gives a survey on recent work on such generalized bundles and their classification.

"String topology in dimensions two and three" by M. Chas and D. Sullivan describes some applications of string topology in low dimensions for surfaces and 3-manifolds. The authors relate their results to a theorem of W. Jaco and J. Stallings and a group theoretical statement equivalent to the three-dimensional Poincare conjecture.

The paper "Floer homotopy theory, realizing chain complexes by module spectra, and manifolds with corners", by R. Cohen, extends ideas of the author, J.D.S. Jones and G. Segal on realizing the Floer complex as the cellular complex of a natural stable homotopy type. Crucial in their work was a framing condition on certain moduli spaces, and Cohen shows that by replacing this by a certain orientability with respect to a generalized cohomology theory E^*, there is a natural definition of Floer E_*–homology.

In "Relative Chern characters for nilpotent ideals" by G. Cortiñas and C. Weibel, the equality of two relative Chern characters from K-theory to cyclic homology is shown in the case of nilpotent ideals. This important equality has been assumed without proof in various papers in the past 20 years, including recent investigations of negative K-theory.

H. Esnault's paper "Algebraic differential characters of flat connections with nilpotent residues" shows that characteristic classes of flat bundles on quasi-projective varieties lift canonically to classes over a projective completion if the local monodromy at infinity is unipotent. This should facilitate the computations in some situations, for it is sometimes easier to compute such characteristic classes for flat bundles on quasi-projective varieties.

"Norm varieties and the Chain Lemma (after Markus Rost)" by C. Haesemeyer and C. Weibel gives detailed proofs of two results of Markus Rost known as the "Chain Lemma" and the "Norm principle". The authors place these two results in the context of the overall proof of the Bloch–Kato conjecture. The proofs are based on lectures by Rost.

In "On the Whitehead spectrum of the circle" L. Hesselholt extends the known computations of homotopy groups $\pi_q(\mathrm{Wh}^{\mathrm{Top}}(S^1))$ of the Whitehead spectrum of the circle. This problem is of fundamental importance for understanding the homeomorphism groups of manifolds, in particular those admitting Riemannian metrics of negative curvature. The author achieves complete and explicit computations for $q \leq 3$.

J.F. Jardine's paper "Cocycle categories" presents a new approach to defining and manipulating cocycles in right proper model categories. These cocycle methods provide simple new proofs of homotopy classification results for torsors, abelian sheaf cohomology groups, group extensions and gerbes. It is also shown that the algebraic K-theory presheaf of spaces is a simplicial stack associated to a suitably defined parabolic groupoid.

"A survey of elliptic cohomology" by J. Lurie is an expository account of the relationship between elliptic cohomology and derived algebraic geometry. This paper lies at the intersection of homotopy theory and algebraic geometry. Precursors are kept to a minimum, making the paper accessible to readers with a basic background in algebraic geometry, particularly with the theory of elliptic curves. A more comprehensive account with complete definitions and proofs, will appear elsewhere.

"On Voevodsky's algebraic K-theory spectrum" by I. Panin, K. Pimenov and O. Röndigs resolves some K-theoretic questions in the modern setting of motivic homotopy theory. They show that the motivic spectrum BGL, which represents algebraic K-theory in the motivic stable homotopy category, has a unique ring structure over the integers. For general base schemes this structure pulls back to give a distinguished monoidal structure which the authors have employed in their proof of a motivic Conner–Floyd theorem.

The paper "Chern character, loop spaces and derived algebraic geometry" authored by B. Toën and G. Vezzosi presents work in progress dealing with a categorised version of sheaf theory. Central to their work is the new notion of "derived categorical sheaves", which categorises the notion of complexes of sheaves of modules on schemes. Ideas originating in derived algebraic geometry and higher category theory are used to introduce "derived loop spaces" and to construct a Chern character for categorical sheaves with values in cyclic homology. This work can be seen as an attempt to define algebraic analogs of elliptic objects and characteristic classes.

"Voevodsky's lectures on motivic cohomology 2000/2001" consists of four parts, each dealing with foundational material revolving around the proof of the Bloch–Kato conjecture. A motivic equivariant homotopy theory is introduced with the specific aim of extending non-additive functors, such as symmetric products, from schemes to the motivic homotopy category. The text is written by P. Deligne.

We gratefully acknowledge the generous support of the Board for the Niels Henrik Abel Memorial Fund and the Norwegian Mathematical Society. We also thank Ruth Allewelt and Springer-Verlag for constant encouragement and support during the editing of these proceedings.

Trondheim, Los Angeles and Oslo
March 2009

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The Classifying Space of a Topological 2-Group

John C. Baez and Danny Stevenson*

Abstract Categorifying the concept of topological group, one obtains the notion of a "topological 2-group". This in turn allows a theory of "principal 2-bundles" generalizing the usual theory of principal bundles. It is well known that under mild conditions on a topological group G and a space M, principal G-bundles over M are classified by either the Čech cohomology $\check{H}^1(M, G)$ or the set of homotopy classes $[M, BG]$, where BG is the classifying space of G. Here we review work by Bartels, Jurčo, Baas–Bökstedt–Kro, and others generalizing this result to topological 2-groups and even topological 2-categories. We explain various viewpoints on topological 2-groups and the Čech cohomology $\check{H}^1(M, \mathcal{G})$ with coefficients in a topological 2-group $\mathcal{G}$, also known as "nonabelian cohomology". Then we give an elementary proof that under mild conditions on M and $\mathcal{G}$ there is a bijection $\check{H}^1(M, \mathcal{G}) \cong [M, B|\mathcal{G}|]$ where $B|\mathcal{G}|$ is the classifying space of the geometric realization of the nerve of $\mathcal{G}$. Applying this result to the "string 2-group" $\mathrm{String}(G)$ of a simply-connected compact simple Lie group G, it follows that principal $\mathrm{String}(G)$-2-bundles have rational characteristic classes coming from elements of $H^*(BG, \mathbb{Q})/\langle c \rangle$, where c is any generator of $H^4(BG, \mathbb{Q})$.

1 Introduction

Recent work in higher gauge theory has revealed the importance of categorifying the theory of bundles and considering "2-bundles", where the fiber is a topological category instead of a topological space [4]. These structures show up not only in mathematics, where they form a useful generalization of nonabelian gerbes [8], but also in physics, where they can be used to describe parallel transport of strings [29, 30].

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The concepts of "Čech cohomology" and "classifying space" play a well-known and fundamental role in the theory of bundles. For any topological group G, principal G-bundles over a space M are classified by the first Čech cohomology of M with coefficients in G. Furthermore, under some mild conditions, these Čech cohomology classes are in 1–1 correspondence with homotopy classes of maps from M to the classifying space BG. This lets us define characteristic classes for bundles, coming from cohomology classes for BG.

All these concepts and results can be generalized from bundles to 2-bundles. Bartels [5] has defined principal $\mathcal{G}$-2-bundles where $\mathcal{G}$ is a "topological 2-group": roughly speaking, a categorified version of a topological group. Furthermore, his work shows how principal $\mathcal{G}$-2-bundles over M are classified by $\check{H}^1(M, \mathcal{G})$, the first Čech cohomology of M with coefficients in $\mathcal{G}$. This form of cohomology, also known as "nonabelian cohomology", is familiar from work on nonabelian gerbes [7, 17].

In fact, under mild conditions on $\mathcal{G}$ and M, there is a 1–1 correspondence between $\check{H}^1(M, \mathcal{G})$ and the set of homotopy classes of maps from M to a certain space $B|\mathcal{G}|$: the classifying space of the geometric realization of the nerve of $\mathcal{G}$. So, $B|\mathcal{G}|$ serves as a classifying space for the topological 2-group $\mathcal{G}$! This paper seeks to provide an introduction to topological 2-groups and nonabelian cohomology leading up to a self-contained proof of this fact.

In his pioneering work on this subject, Jurčo [20] asserted that a certain space homotopy equivalent to ours is a classifying space for the first Čech cohomology with coefficients in $\mathcal{G}$. However, there are some gaps in his argument for this assertion (see Sect. 5.2 for details).

Later, Baas, Bökstedt and Kro [1] gave the definitive treatment of classifying spaces for 2-bundles. For any "good" topological 2-category $\mathcal{C}$, they construct a classifying space $B\mathcal{C}$. They then show that for any space X with the homotopy type of a CW complex, concordance classes of "charted $\mathcal{C}$-2-bundles" correspond to homotopy classes of maps from M to $B\mathcal{C}$. In particular, a topological 2-group is just a topological 2-category with one object and with all morphisms and 2-morphisms invertible – and in this special case, their result *almost* reduces to the fact mentioned above.

There are some subtleties, however. Most importantly, while their "charted $\mathcal{C}$-2-bundles" reduce precisely to our principal $\mathcal{G}$-2-bundles, their classify these 2-bundles up to concordance, while we classify them up to a superficially different equivalence relation. Two $\mathcal{G}$-2-bundles over a space X are "concordant" if they are restrictions of some $\mathcal{G}$-2-bundle over $X \times [0, 1]$ to the two ends $X \times \{0\}$ and $X \times \{1\}$. This makes it easy to see that homotopic maps from X to the classifying space define concordant $\mathcal{G}$-2-bundles. We instead consider two $\mathcal{G}$-2-bundles to be equivalent if their defining Čech 1-cocycles are cohomologous. In this approach, some work is required to show that homotopic maps from X to the classifying space define equivalent $\mathcal{G}$-2-bundles. A priori, it is not obvious that two $\mathcal{G}$-2-bundles are equivalent in this Čech sense if and only if they are concordant. However, since the classifying space of Baas, Bökstedt and Kro is homotopy equivalent to the one we use, it follows from that these equivalence relations are the same – at least given $\mathcal{G}$ and M satisfying the technical conditions of both their result and ours.

We also discuss an interesting example: the "string 2-group" String(G) of a simply-connected compact simple Lie group G [2, 18]. As its name suggests, this 2-group is of special interest in physics. Mathematically, a key fact is that $|$String $(G)|$– the geometric realization of the nerve of String(G) – is the 3-connected cover of G. Using this, one can compute the rational cohomology of $B|$String$(G)|$. This is nice, because these cohomology classes give "characteristic classes" for principal $\mathcal{G}$-2-bundles, and when M is a manifold one can hope to compute these in terms of a connection and its curvature, much as one does for ordinary principal bundles with a Lie group as structure group.

Section 2 is an overview, starting with a review of the classic results that people are now categorifying. Section 3 reviews four viewpoints on topological 2- groups. Section 4 explains nonabelian cohomology with coefficients in a topological 2-group. Finally, in Sect. 5 we prove the results stated in Sect. 2, and comment a bit further on the work of Jurčo and Baas–Bökstedt–Kro.

2 Overview

Once one knows about "topological 2-groups", it is irresistibly tempting to generalize all ones favorite results about topological groups to these new entities. So, let us begin with a quick review of some classic results about topological groups and their classifying spaces.

Suppose that G is a topological group. The Čech cohomology $\check{H}^1(M, G)$ of a topological space M with coefficients in G is a set carefully designed to be in 1–1 correspondence with the set of isomorphism classes of principal G-bundles on M. Let us recall how this works.

First suppose $\mathcal{U} = \{U_i\}$ is an open cover of M and P is a principal G-bundle over M that is trivial when restricted to each open set U_i. Then by comparing local trivialisations of P over U_i and U_j we can define maps $g_{ij}\colon U_i \cap U_j \to G$: the transition functions of the bundle. On triple intersections $U_i \cap U_j \cap U_k$, these maps satisfy a cocycle condition:

$$g_{ij}(x)g_{jk}(x) = g_{ik}(x)$$

A collection of maps $g_{ij}\colon U_i \cap U_j \to G$ satisfying this condition is called a "Čech 1-cocycle" subordinate to the cover $\mathcal{U}$. Any such 1-cocycle defines a principal G-bundle over M that is trivial over each set U_i.

Next, suppose we have two principal G-bundles over M that are trivial over each set U_i, described by Čech 1-cocycles g_{ij} and g'_{ij}, respectively. These bundles are isomorphic if and only if for some maps $f_i\colon U_i \to G$ we have

$$g_{ij}(x) f_j(x) = f_i(x) g'_{ij}(x)$$

on every double intersection $U_i \cap U_j$. In this case we say the Čech 1-cocycles are "cohomologous". We define $\check{H}^1(\mathcal{U}, G)$ to be the quotient of the set of Čech 1-cocycles subordinate to $\mathcal{U}$ by this equivalence relation.

Recall that a "good" cover of M is an open cover $\mathcal{U}$ for which all the non-empty finite intersections of open sets U_i in $\mathcal{U}$ are contractible. We say a space M *admits good covers* if any cover of M has a good cover that refines it. For example, any (paracompact Hausdorff) smooth manifold admits good covers, as does any simplicial complex.

If M admits good covers, $\check{H}^1(\mathcal{U}, G)$ is independent of the choice of good cover $\mathcal{U}$. So, we can denote it simply by $\check{H}^1(M, G)$. Furthermore, this set $\check{H}^1(M, G)$ is in 1–1 correspondence with the set of isomorphism classes of principal G-bundles over M. The reason is that we can always trivialize any principal G-bundle over the open sets in a good cover.

For more general spaces, we need to define the Čech cohomology more carefully. If M is a paracompact Hausdorff space, we can define it to be the inverse limit

$$\check{H}^1(M, G) = \varprojlim_{\mathcal{U}} \check{H}^1(\mathcal{U}, G)$$

over all open covers, partially ordered by refinement.

It is a classic result in topology that $\check{H}^1(M, G)$ can be understood using homotopy theory with the help of Milnor's construction [13, 26] of the classifying space BG:

Theorem 0. *Let G be a topological group. Then there is a topological space BG with the property that for any paracompact Hausdorff space M, there is a bijection*

$$\check{H}^1(M, G) \cong [M, BG]$$

Here $[X, Y]$ denotes the set of homotopy classes of maps from X into Y. The topological space BG is called the *classifying space* of G. There is a canonical principal G-bundle on BG, called the universal G-bundle, and the theorem above is usually understood as the assertion that every principal G-bundle P on M is obtained by pullback from the universal G-bundle under a certain map $M \to BG$ (the classifying map of P).

Now let us discuss how to generalize all these results to topological 2-groups. First of all, what is a "2-group"? It is like a group, but "categorified". While a group is a *set* equipped with *functions* describing multiplication and inverses, and an identity *element*, a 2-group is a *category* equipped with *functors* describing multiplication and inverses, and an identity *object*. Indeed, 2-groups are also known as "categorical groups".

A down-to-earth way to work with 2-groups involves treating them as "crossed modules". A crossed module consists of a pair of groups H and G, together with a homomorphism $t\colon H \to G$ and an action α of G on H satisfying two conditions, (4) and (5) below. Crossed modules were introduced by Whitehead [35] without the aid of category theory. Mac Lane and Whitehead [22] later proved that just as the fundamental group captures all the homotopy-invariant information about a connected pointed homotopy 1-type, a crossed module captures all the homotopy-invariant information about a connected pointed homotopy 2-type. By the 1960s it was clear to Verdier and others that crossed modules are essentially the same as categorical groups. The first published proof of this may be due to Brown and Spencer [10].

Just as one can define principal G-bundles over a space M for any topological group G, one can define "principal $\mathcal{G}$-2-bundles" over M for any topological 2-group $\mathcal{G}$. Just as a principal G-bundle has a copy of G as fiber, a principal $\mathcal{G}$-2-bundle has a copy of $\mathcal{G}$ as fiber. Readers interested in more details are urged to read Bartels' thesis, available online [5]. We shall have nothing else to say about principal $\mathcal{G}$-2-bundles except that they are classified by a categorified version of Čech cohomology, denoted $\check{H}^1(M,\mathcal{G})$.

As before, we can describe this categorified Čech cohomology as a set of cocycles modulo an equivalence relation. Let $\mathcal{U}$ be a cover of M. If we think of the 2-group $\mathcal{G}$ in terms of its associated crossed module (G,H,t,α), then a cocycle subordinate to $\mathcal{U}$ consists (in part) of maps $g_{ij}\colon U_i \cap U_j \to G$ as before. However, we now "weaken" the cocycle condition and only require that

$$t(h_{ijk})g_{ij}g_{jk} = g_{ik} \tag{1}$$

for some maps $h_{ijk}\colon U_i \cap U_j \cap U_k \to H$. These maps are in turn required to satisfy a cocycle condition of their own on quadruple intersections, namely

$$\alpha(g_{ij})(h_{jkl})h_{ijl} = h_{ijk}h_{ikl} \tag{2}$$

where α is the action of G on H. This mildly intimidating equation will be easier to understand when we draw it as a commuting tetrahedron – see (6) in the next section. The pair (g_{ij}, h_{ijk}) is called a $\mathcal{G}$-valued *Čech 1-cocycle* subordinate to $\mathcal{U}$.

Similarly, we say two cocycles (g_{ij}, h_{ijk}) and (g'_{ij}, h'_{ijk}) are *cohomologous* if

$$t(k_{ij})g_{ij}f_j = f_i g'_{ij} \tag{3}$$

for some maps $f_i\colon U_i \to G$ and $k_{ij}\colon U_i \cap U_j \to H$, which must make a certain prism commute – see (7). We define $\check{H}^1(\mathcal{U},\mathcal{G})$ to be the set of cohomology classes of $\mathcal{G}$-valued Čech 1-cocycles. To capture the entire cohomology set $\check{H}^1(M,\mathcal{G})$, we must next take an inverse limit of the sets $\check{H}^1(\mathcal{U},\mathcal{G})$ as $\mathcal{U}$ ranges over all covers of M. For more details we refer to Sect. 4.

Theorem 0 generalizes nicely from topological groups to topological 2-groups:

Theorem 1. *Suppose that $\mathcal{G}$ is a well-pointed topological 2-group and M is a paracompact Hausdorff space admitting good covers. Then there is a bijection*

$$\check{H}^1(M,\mathcal{G}) \cong [M, B|\mathcal{G}|]$$

where the topological group $|\mathcal{G}|$ is the geometric realization of the nerve of $\mathcal{G}$.

One term here requires explanation. A topological group G is said to be "well pointed" if $(G, 1)$ is an NDR pair, or in other words if the inclusion $\{1\} \hookrightarrow G$ is a closed cofibration. We say that a topological 2-group $\mathcal{G}$ is *well pointed* if the topological groups G and H in its corresponding crossed module are well pointed. For example, any "Lie 2-group" is well pointed: a topological 2-group is called a *Lie 2-group* if G and H are Lie groups and the maps t, α are smooth. More generally, any "Fréchet Lie 2-group" [2] is well pointed. We explain the importance of this notion in Sect. 5.1.

Bartels [5] has already considered two examples of principal $\mathcal{G}$-2-bundles, corresponding to abelian gerbes and nonabelian gerbes. Let us discuss the classification of these before turning to a third, more novel example.

For an abelian gerbe [11], we first choose an abelian topological group H – in practice, usually just $\mathrm{U}(1)$. Then, we form the crossed module with $G = 1$ and this choice of H, with t and α trivial. The corresponding topological 2-group deserves to be called $H[1]$, since it is a "shifted version" of H. Bartels shows that the classification of abelian H-gerbes matches the classification of $H[1]$-2-bundles. It is well known that

$$|H[1]| \simeq BH$$

so the classifying space for abelian H-gerbes is

$$B|H[1]| \simeq B(BH)$$

In the case $H = \mathrm{U}(1)$, this classifying space is just $K(\mathbb{Z}, 3)$. So, in this case, we recover the well-known fact that abelian $\mathrm{U}(1)$-gerbes over M are classified by

$$[M, K(\mathbb{Z}, 3)] \cong H^3(M, \mathbb{Z})$$

just as principal $\mathrm{U}(1)$ bundles are classified by $H^2(M, \mathbb{Z})$.

For a nonabelian gerbe [7,16,17], we fix any topological group H. Then we form the crossed module with $G = Aut(H)$ and this choice of H, where $t\colon H \to G$ sends each element of H to the corresponding inner automorphism, and the action of G on H is the tautologous one. This gives a topological 2-group called $\mathrm{AUT}(H)$. Bartels shows that the classification of nonabelian H-gerbes matches the classification of $\mathrm{AUT}(H)$-2-bundles. It follows that, under suitable conditions on H, nonabelian H-gerbes are classified by homotopy classes of maps into $B|\mathrm{AUT}(H)|$.

A third application of Theorem 1 arises when G is a simply-connected compact simple Lie group. For any such group there is an isomorphism $H^3(G, \mathbb{Z}) \cong \mathbb{Z}$ and the generator $\nu \in H^3(G, \mathbb{Z})$ transgresses to a characteristic class $c \in H^4(BG, \mathbb{Z}) \cong \mathbb{Z}$. Associated to ν is a map $G \to K(\mathbb{Z}, 3)$ and it can be shown that the homotopy fiber of this can be given the structure of a topological group $\hat{G}$. This group $\hat{G}$ is the 3-connected cover of G. When $G = \mathrm{Spin}(n)$, this group $\hat{G}$ is known as $\mathrm{String}(n)$. In general, we might call $\hat{G}$ the *string group* of G. Note that until one picks a specific construction for the homotopy fiber, $\hat{G}$ is only defined up to homotopy – or more precisely, up to equivalence of A_∞-spaces.

In [2], under the above hypotheses on G, a topological 2-group subsequently dubbed the *string 2-group* of G was introduced. Let us denote this by $\mathrm{String}(G)$. A key result about $\mathrm{String}(G)$ is that the topological group $|\mathrm{String}(G)|$ is equivalent to $\hat{G}$. By construction $\mathrm{String}(G)$ is a Fréchet Lie 2-group, hence well pointed. So, from Theorem 1 we immediately conclude:

Corollary 1. *Suppose that G is a simply-connected compact simple Lie group. Suppose M is a paracompact Hausdorff space admitting good covers. Then there are bijections between the following sets:*

- *The set of equivalence classes of principal* $\mathrm{String}(G)$*-2-bundles over M*
- *The set of isomorphism classes of principal $\hat{G}$-bundles over M*
- $\check{H}^1(M, \mathrm{String}(G))$
- $\check{H}^1(M, \hat{G})$
- $[M, B\hat{G}]$

One can describe the rational cohomology of $B\hat{G}$ in terms of the rational cohomology of BG, which is well understood. The following result was pointed out to us by Matt Ando (personal communication), and later discussed by Greg Ginot [15]:

Theorem 2. *Suppose that G is a simply-connected compact simple Lie group, and let $\hat{G}$ be the string group of G. Let $c \in H^4(BG, \mathbb{Q}) = \mathbb{Q}$ denote the transgression of the generator $\nu \in H^3(G, \mathbb{Q}) = \mathbb{Q}$. Then there is a ring isomorphism*

$$H^*(B\hat{G}, \mathbb{Q}) \cong H^*(BG, \mathbb{Q})/\langle c\rangle$$

where $\langle c\rangle$ is the ideal generated by c.

As a result, we obtain characteristic classes for $\mathrm{String}(G)$-2-bundles:

Corollary 2. *Suppose that G is a simply-connected compact simple Lie group and M is a paracompact Hausdorff space admitting good covers. Then an equivalence class of principal* $\mathrm{String}(G)$*-2-bundles over M determines a ring homomorphism*

$$H^*(BG, \mathbb{Q})/\langle c\rangle \to H^*(M, \mathbb{Q})$$

To see this, we use Corollary 1 to reinterpret an equivalence class of principal $\mathcal{G}$-2-bundles over M as a homotopy class of maps $f\colon M \to B|\mathcal{G}|$. Picking any representative f, we obtain a ring homomorphism

$$f^*\colon H^*(B|\mathcal{G}|, \mathbb{Q}) \to H^*(M, \mathbb{Q}).$$

This is independent of the choice of representative. Then, we use Theorem 2.

It is a nice problem to compute the rational characteristic classes of a principal $\mathrm{String}(G)$-2-bundle over a manifold using de Rham cohomology. It should be possible to do this using the curvature of an arbitrary connection on the 2-bundle, just as for ordinary principal bundles with a Lie group as structure group. Sati et al. [29] have recently made excellent progress on solving this problem and its generalizations to n-bundles for higher n.

3 Topological 2-Groups

In this section we recall four useful perspectives on topological 2-groups. For a more detailed account, we refer the reader to [3].

A *topological* 2-*group* is a groupoid in the category of topological groups. In other words, it is a groupoid $\mathcal{G}$ where the set $\mathrm{Ob}(\mathcal{G})$ of objects and the set $\mathrm{Mor}(\mathcal{G})$ of morphisms are each equipped with the structure of a topological group such that the source and target maps $s, t\colon \mathrm{Mor}(\mathcal{G}) \to \mathrm{Ob}(\mathcal{G})$, the map $i\colon \mathrm{Ob}(\mathcal{G}) \to \mathrm{Mor}(\mathcal{G})$ assigning each object its identity morphism, the composition map $\circ\colon \mathrm{Mor}(\mathcal{G}) \times_{\mathrm{Ob}(\mathcal{G})} \mathrm{Mor}(\mathcal{G}) \to \mathrm{Mor}(\mathcal{G})$, and the map sending each morphism to its inverse are all continuous group homomorphisms.

Equivalently, we can think of a topological 2-group as a group in the category of topological groupoids. A *topological groupoid* is a groupoid $\mathcal{G}$ where $\mathrm{Ob}(\mathcal{G})$ and $\mathrm{Mor}(\mathcal{G})$ are topological spaces and all the groupoid operations just listed are continuous maps. We say that a functor $f\colon \mathcal{G} \to \mathcal{G}'$ between topological groupoids is *continuous* if the maps $f\colon \mathrm{Ob}(\mathcal{G}) \to \mathrm{Ob}(\mathcal{G}')$ and $f\colon \mathrm{Mor}(\mathcal{G}) \to \mathrm{Mor}(\mathcal{G}')$ are continuous. A group in the category of topological groupoids is such a thing equipped with continuous functors $m\colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, $\mathrm{inv}\colon \mathcal{G} \to \mathcal{G}$ and a unit object $1 \in \mathcal{G}$ satisfying the usual group axioms, written out as commutative diagrams.

This second viewpoint is useful because any topological groupoid $\mathcal{G}$ has a "nerve" $N\mathcal{G}$, a simplicial space where the space of n-simplices consists of composable strings of morphisms

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} x_{n-1} \xrightarrow{f_n} x_n$$

Taking the geometric realization of this nerve, we obtain a topological space which we denote as $|\mathcal{G}|$ for short. If $\mathcal{G}$ is a topological 2-group, its nerve inherits a group structure, so that $N\mathcal{G}$ is a topological simplicial group. This in turn makes $|\mathcal{G}|$ into a topological group. This passage from the topological 2-group $\mathcal{G}$ to the topological group $|\mathcal{G}|$ will be very important in what follows.

A third way to understand topological 2-groups is to view them as topological crossed modules. Recall that a *topological crossed module* (G, H, t, α) consists of topological groups G and H together with a continuous homomorphism

$$t\colon H \to G$$

and a continuous action

$$\begin{aligned} \alpha\colon G \times H &\to H \\ (g,h) &\mapsto \alpha(g)h \end{aligned}$$

of G as automorphisms of H, satisfying the following two identities:

$$t(\alpha(g)(h)) = gt(h)g^{-1} \tag{4}$$

$$\alpha(t(h))(h') = hh'h^{-1}. \tag{5}$$

The first equation above implies that the map $t\colon H \to G$ is equivariant for the action of G on H defined by α and the action of G on itself by conjugation. The second equation is called the *Peiffer identity*. When no confusion is likely to result, we will sometimes denote the 2-group corresponding to a crossed module (G, H, t, α) simply by $H \to G$.

Every topological crossed module determines a topological 2-group and vice versa. Since there are some choices of convention involved in this construction, we briefly review it to fix our conventions. Given a topological crossed module (G, H, t, α), we define a topological 2-group $\mathcal{G}$ as follows. First, define the group $\mathrm{Ob}(\mathcal{G})$ of objects of $\mathcal{G}$ and the group $\mathrm{Mor}(\mathcal{G})$ of morphisms of $\mathcal{G}$ by

$$\mathrm{Ob}(\mathcal{G}) = G, \qquad \mathrm{Mor}(\mathcal{G}) = H \rtimes G$$

where the semidirect product $H \rtimes G$ is formed using the left action of G on H via α:

$$(h,g)\cdot(h',g') = (h\alpha(g)(h'), gg')$$

for $g, g' \in G$ and $h, h' \in H$. The source and target of a morphism $(h,g) \in \mathrm{Mor}(\mathcal{G})$ are defined by

$$s(h,g) = g \qquad \text{and} \qquad t(h,g) = t(h)g$$

(Denoting both the target map $t\colon \mathrm{Mor}(\mathcal{G}) \to \mathrm{Ob}(\mathcal{G})$ and the homomorphism $t\colon H \to G$ by the same letter should not cause any problems, since the first is the restriction of the second to $H \subseteq \mathrm{Mor}(\mathcal{G})$.) The identity morphism of an object $g \in \mathrm{Ob}(G)$ is defined by

$$i(g) = (1, g).$$

Finally, the composite of the morphisms

$$\alpha = (h,g)\colon g \to t(h)g \qquad \text{and} \qquad \beta = (h', t(h)g)\colon t(h)g \to t(h'h)g'$$

is defined to be

$$\beta \circ \alpha = (h'h, g)\colon g \to t(h'h)g$$

It is easy to check that with these definitions, $\mathcal{G}$ is a 2-group. Conversely, given a topological 2-group $\mathcal{G}$, we define a crossed module (G, H, t, α) by setting G to be

$\mathrm{Ob}(\mathcal{G})$, H to be $\ker(s) \subset \mathrm{Mor}(\mathcal{G})$, t to be the restriction of the target homomorphism $t\colon \mathrm{Mor}(\mathcal{G}) \to \mathrm{Ob}(\mathcal{G})$ to the subgroup $H \subset \mathrm{Mor}(\mathcal{G})$, and setting

$$\alpha(g)(h) = i(g)hi(g)^{-1}$$

If G is any topological group then there is a topological crossed module $1 \to G$ where t and α are trivial. The underlying groupoid of the corresponding topological 2-group has G as its space of objects, and only identity morphisms. We sometimes call this 2-group the *discrete* topological 2-group associated to G – where "discrete" is used in the sense of category theory, not topology!

At the other extreme, if H is a topological group then it follows from the Peiffer identity that $H \to 1$ can be made into topological crossed module if and only if H is abelian, and then in a unique way. This is because a groupoid with one object and H as morphisms can be made into a 2-group precisely when H is abelian. We already mentioned this 2-group in the previous section, where we called it $H[1]$.

We will also need to talk about homomorphisms of 2-groups. We shall understand these in the strictest possible sense. So, we say a *homomorphism* of topological 2-groups is a functor such that $f\colon \mathrm{Ob}(\mathcal{G}) \to \mathrm{Ob}(\mathcal{G}')$ and $f\colon \mathrm{Mor}(\mathcal{G}) \to \mathrm{Mor}(\mathcal{G}')$ are both continuous homomorphisms of topological groups. We can also describe f in terms of the crossed modules (G, H, t, α) and (G', H', t', α') associated to $\mathcal{G}$ and $\mathcal{G}'$ respectively. In these terms the data of the functor f is described by the commutative diagram

$$\begin{tikzcd} H \arrow[r, "f"] \arrow[d, "t"'] & H' \arrow[d, "t'"] \\ G \arrow[r, "f"'] & G' \end{tikzcd}$$

where the upper f denotes the restriction of $f\colon \mathrm{Mor}(\mathcal{G}) \to \mathrm{Mor}(\mathcal{G}')$ to a map from H to H'. (We are using f to mean several different things, but this makes the notation less cluttered, and should not cause any confusion.) The maps $f\colon G \to G'$ and $f\colon H \to H'$ must both be continuous homomorphisms, and moreover must satisfy an equivariance property with respect to the actions of G on H and G' on H': we have

$$f(\alpha(g)(h)) = \alpha(f(g))(f(h))$$

for all $g \in G$ and $h \in H$.

Finally, we will need to talk about short exact sequences of topological groups and 2-groups. Here the topology is important. If G is a topological group and H is a normal topological subgroup of G, then we can define an action of H on G by right translation. In some circumstances, the projection $G \to G/H$ is a Hurewicz fibration. For instance, this is the case if G is a Lie group and H is a closed normal subgroup of G. We define a *short exact sequence* of topological groups to be a sequence

$$1 \to H \to G \to K \to 1$$

of topological groups and continuous homomorphisms such that the underlying sequence of groups is exact and the map underlying the homomorphism $G \to K$ is a Hurewicz fibration.

Similarly, we define a *short exact sequence* of topological 2-groups to be a sequence

$$1 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 1$$

of topological 2-groups and continuous homomorphisms between them such that both the resulting sequences

$$1 \to \mathrm{Ob}(\mathcal{G}') \to \mathrm{Ob}(\mathcal{G}) \to \mathrm{Ob}(\mathcal{G}'') \to 1$$

$$1 \to \mathrm{Mor}(\mathcal{G}') \to \mathrm{Mor}(\mathcal{G}) \to \mathrm{Mor}(\mathcal{G}'') \to 1$$

are short exact sequences of topological groups. Again, we can interpret this in terms of the associated crossed modules: if (G, H, t, α), (G', H', t', α') and $(G'', H'', t'', \alpha'')$ denote the associated crossed modules, then it can be shown that the sequence of topological 2-groups $1 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 1$ is exact if and only if both rows in the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & H' \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & H'' \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & G' \arrow[r] & G \arrow[r] & G'' \arrow[r] & 1
\end{tikzcd}$$

are short exact sequences of topological groups. In this situation we also say we have a short exact sequence of topological crossed modules.

At times we shall also need a fourth viewpoint on topological 2-groups: they are strict topological 2-groupoids with a single object, say $\bullet$. In this approach, what we had been calling "objects" are renamed "morphisms", and what we had been calling "morphisms" are renamed "2-morphisms". This verbal shift can be confusing, so we will not engage in it! However, the 2-groupoid viewpoint is very handy for diagrammatic reasoning in nonabelian cohomology. We draw $g \in \mathrm{Ob}(\mathcal{G})$ as an arrow:

$$\begin{tikzcd}
\bullet \arrow[r, "g"] & \bullet
\end{tikzcd}$$

and draw $(h, g) \in \mathrm{Mor}(\mathcal{G})$ as a bigon:

$$\begin{tikzcd}
\bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow h"] & \bullet
\end{tikzcd}$$

where g' is the target of (h, g), namely $t(h)g$. With our conventions, horizontal composition of 2-morphisms is then given by:

$$\begin{tikzcd}
\bullet \arrow[r, bend left, "g_1"{name=A}] \arrow[r, bend right, "g_1'"'{name=B}] \arrow[from=A, to=B, Rightarrow, "h_1"] & \bullet \arrow[r, bend left, "g_2"{name=C}] \arrow[r, bend right, "g_2'"'{name=D}] \arrow[from=C, to=D, Rightarrow, "h_2"] & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[rr, bend left, "g_1 g_2"{name=E}] \arrow[rr, bend right, "g_1' g_2'"'{name=F}] \arrow[from=E, to=F, Rightarrow, "h_1 \alpha(g_1)(h_2)"] & & \bullet
\end{tikzcd}$$

while vertical composition is given by:

$$\begin{tikzcd}
\bullet \arrow[r, bend left=60, "g"{name=A}] \arrow[r, ""{name=M}] \arrow[r, bend right=60, "g'"'{name=B}] \arrow[from=A, to=M, Rightarrow, "h"] \arrow[from=M, to=B, Rightarrow, "h'"] & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, bend left, "g"{name=C}] \arrow[r, bend right, "g'"'{name=D}] \arrow[from=C, to=D, Rightarrow, "h'h"] & \bullet
\end{tikzcd}$$

4 Nonabelian Cohomology

In Sect. 2 we gave a quick sketch of nonabelian cohomology. The subject deserves a more thorough and more conceptual explanation.

As a warmup, consider the Čech cohomology of a space M with coefficients in a topological group G. In this case, Segal [31] realized that we can reinterpret a Čech 1-cocycle as a *functor*. Suppose $\mathcal{U}$ is an open cover of M. Then there is a topological groupoid $\hat{\mathcal{U}}$ whose objects are pairs (x, i) with $x \in U_i$, and with a single morphism from (x, i) to (x, j) when $x \in U_i \cap U_j$, and none otherwise. We can also think of G as a topological groupoid with a single object $\bullet$. Segal's key observation was that a continuous functor

$$g: \hat{\mathcal{U}} \to G$$

is the same as a normalized Čech 1-cocycle subordinate to $\mathcal{U}$.

To see this, note that a functor $g: \hat{\mathcal{U}} \to G$ maps each object of $\hat{\mathcal{U}}$ to $\bullet$, and each morphism $(x, i) \to (x, j)$ to some $g_{ij}(x) \in G$. For the functor to preserve composition, it is necessary and sufficient to have the cocycle equation

$$g_{ij}(x) g_{jk}(x) = g_{ik}(x)$$

We can draw this suggestively as a commuting triangle in the groupoid G:

$$\begin{tikzcd} & \bullet \arrow[dr, "g_{jk}(x)"] & \\ \bullet \arrow[ur, "g_{ij}(x)"] \arrow[rr, "g_{ik}(x)"'] & & \bullet \end{tikzcd}$$

For the functor to preserve identities, it is necessary and sufficient to have the normalization condition $g_{ii}(x) = 1$.

In fact, even more is true: two cocycles g_{ij} and g'_{ij} subordinate to $\mathcal{U}$ are cohomologous if and only if the corresponding functors g and g' from $\hat{\mathcal{U}}$ to G have a continuous natural isomorphism between them. To see this, note that g_{ij} and g'_{ij} are cohomologous precisely when there are maps $f_i : U_i \to G$ satisfying

$$g_{ij}(x) f_j(x) = f_i(x) g'_{ij}(x)$$

We can draw this equation as a commuting square in the groupoid G:

$$\begin{tikzcd} \bullet \arrow[r, "g_{ij}(x)"] \arrow[d, "f_i(x)"'] & \bullet \arrow[d, "f_j(x)"] \\ \bullet \arrow[r, "g'_{ij}(x)"'] & \bullet \end{tikzcd}$$

This is precisely the naturality square for a natural isomorphism between the functors g and g'.

One can obtain Čech cohomology with coefficients in a 2-group by categorifying Segal's ideas. Suppose $\mathcal{G}$ is a topological 2-group and let (G, H, t, α) be the corresponding topological crossed module. Now $\mathcal{G}$ is the same as a topological 2-groupoid with one object $\bullet$. So, it is no longer appropriate to consider mere *functors* from $\hat{\mathcal{U}}$ into $\mathcal{G}$. Instead, we should consider *weak 2-functors*, also known as "pseudofunctors" [21]. For this, we should think of $\hat{\mathcal{U}}$ as a topological 2-groupoid with only identity 2-morphisms.

Let us sketch how this works. A weak 2-functor $g : \hat{\mathcal{U}} \to \mathcal{G}$ sends each object of $\hat{\mathcal{U}}$ to $\bullet$, and each 1-morphism $(x, i) \to (x, j)$ to some $g_{ij}(x) \in G$. However, composition of 1-morphisms is only weakly preserved. This means the above triangle will now commute only up to isomorphism:

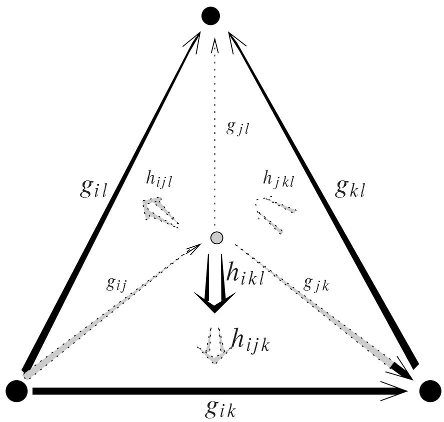
$$\begin{tikzcd} & \bullet \arrow[dr, "g_{jk}"] \arrow[d, phantom, "\Downarrow h_{ijk}"] & \\ \bullet \arrow[ur, "g_{ij}"] \arrow[rr, "g_{ik}"'] & {} & \bullet \end{tikzcd}$$

where for readability we have omitted the dependence on $x \in U_i \cap U_j \cap U_k$. Translated into equations, this triangle says that we have continuous maps $h_{ijk} \colon U_i \cap U_j \cap U_k \to H$ satisfying

$$g_{ik}(x) = t(h_{ijk}(x))g_{ij}(x)g_{jk}(x)$$

This is precisely (1) from Sect. 2.

For a weak 2-functor, it is not merely true that composition is preserved up to isomorphism: this isomorphism is also subject to a coherence law. Namely, the following tetrahedron must commute:



(6)

where again we have omitted the dependence on x. The commutativity of this tetrahedron is equivalent to the following equation:

$$\alpha(g_{ij})(h_{jkl})h_{ijl} = h_{ijk}h_{ikl}$$

holding for all $x \in U_i \cap U_j \cap U_k \cap U_l$. This is (2).

A weak 2-functor may also preserve identity 1-morphisms only up to isomorphism. However, it turns out [5] that without loss of generality we may assume that g preserves identity 1-morphisms *strictly*. Thus we have $g_{ii}(x) = 1$ for all $x \in U_i$. We may also assume $h_{ijk}(x) = 1$ whenever two or more of the indices i, j and k are equal. Finally, just as for the case of an ordinary topological group, we require that g is a *continuous* weak 2-functor We shall not spell this out in detail; suffice it to say that the maps $g_{ij} \colon U_i \cap U_j \to G$ and $h_{ijk} \colon U_i \cap U_j \cap U_k \to H$ should be continuous. We say such continuous weak 2-functors $g \colon \hat{\mathcal{U}} \to \mathcal{G}$ are *Čech 1-cocycles* valued in $\mathcal{G}$, subordinate to the cover $\mathcal{U}$.

We now need to understand when two such cocycles should be considered equivalent. In the case of cohomology with coefficients in an ordinary topological group, we saw that two cocycles were cohomologous precisely when there was a continuous natural isomorphism between the corresponding functors. In our categorified setting we should instead use a "weak natural isomorphism", also called a pseudonatural isomorphism [21]. So, we declare two cocycles to be *cohomologous* if there is a continuous weak natural isomorphism $f : g \Rightarrow g'$ between the corresponding weak 2-functors g and g'.

In a weak natural isomorphism, the usual naturality square commutes only up to isomorphism. So, $f : g \Rightarrow g'$ not only sends every object (x, i) of $\hat{\mathcal{U}}$ to some $f_i(x) \in G$, but also sends every morphism $(x, i) \to (x, j)$ to some $k_{ij}(x) \in H$ filling in this square:

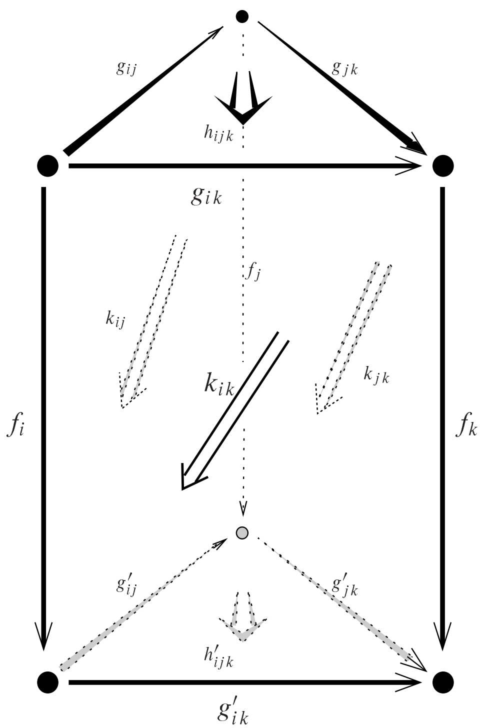
$$\begin{tikzcd} \bullet \arrow[r, "g_{ij}"] \arrow[d, "f_i"'] \arrow[dr, phantom, "\swarrow k_{ij}"] & \bullet \arrow[d, "f_j"] \\ \bullet \arrow[r, "g'_{ij}"'] & \bullet \end{tikzcd}$$

Translated into equations, this square says that

$$t(k_{ij}) g_{ij} f_j = f_i g'_{ij}$$

This is (3).

There is also a coherence law that the k_{ij} must satisfy: they must make the following prism commute:



(7)

At this point, translating the diagrams into equations becomes tiresome and unenlightening.

It can be shown that this notion of "cohomologousness" of Čech 1-cocycles $g:\hat{\mathcal{U}} \to \mathcal{G}$ is an equivalence relation. We denote by $\check{H}^1(\mathcal{U}, \mathcal{G})$ the set of equivalence classes of cocycles obtained in this way. In other words, we let $\check{H}^1(\mathcal{U}, \mathcal{G})$ be the set of continuous weak natural isomorphism classes of continuous weak 2-functors $g:\hat{\mathcal{U}} \to \mathcal{G}$.

Finally, to define $\hat{H}^1(M, \mathcal{G})$, we need to take all covers into account as follows. The set of all open covers of M is a directed set, partially ordered by refinement. By restricting cocycles defined relative to $\mathcal{U}$ to any finer cover $\mathcal{V}$, we obtain a map $\check{H}^1(\mathcal{U}, \mathcal{G}) \to \check{H}^1(\mathcal{V}, \mathcal{G})$. This allows us to define the Čech cohomology $\check{H}^1(M, \mathcal{G})$ as an inverse limit:

Definition 3. Given a topological space M and a topological 2-group $\mathcal{G}$, we define the *first Čech cohomology of* M *with coefficients in* $\mathcal{G}$ to be

$$\check{H}^1(M, \mathcal{G}) = \varprojlim_{\mathcal{U}} \check{H}^1(\mathcal{U}, \mathcal{G})$$

When we want to emphasize the crossed module, we will sometimes use the notation $\check{H}^1(M, H \to G)$ instead of $\check{H}^1(M, \mathcal{G})$. Note that $\check{H}^1(M, \mathcal{G})$ is a pointed set, pointed by the trivial cocycle defined relative to any open cover $\{U_i\}$ by $g_{ij} = 1$, $h_{ijk} = 1$ for all indices i, j and k.

In Theorem 1 we assume M admits good covers, so that every cover $\mathcal{U}$ of M has a refinement by a good cover $\mathcal{V}$. In other words, the directed set of good covers of M is cofinal in the set of all covers of M. As a result, in computing the inverse limit above, it is sufficient to only consider *good* covers $\mathcal{U}$.

Finally, we remark that there is a more refined version of the set $\check{H}^1(M, \mathcal{G})$ defined using the notion of "hypercover" [7, 9, 19]. For a paracompact space M this refined cohomology set $H^1(M, G)$ is isomorphic to the set $\check{H}^1(M, G)$ defined in terms of Čech covers. While the technology of hypercovers is certainly useful, and can simplify some proofs, our approach is sufficient for the applications we have in mind (see also the remark following the proof of Lemma 2 in Sect. 5.4).

5 Proofs

5.1 *Proof of Theorem 1*

First, we need to distinguish between Milnor's [26] original construction of a classifying space for a topological group and a later construction introduced by Milgram [25], Segal [31] and Steenrod [33] and further studied by May [24]. Milnor's construction is very powerful, as witnessed by the generality of Theorem 0. The later construction is conceptually more beautiful: for any topological group G, it constructs BG as the geometric realization of the nerve of the topological groupoid with one object associated to G. But, it seems to give a slightly weaker result: to obtain a bijection

$$\check{H}^1(M,G) \cong [M,BG]$$

all of the above cited works require some extra hypotheses on G: Segal [32] requires that G be locally contractible; May, Milgram and Steenrod require that G be well pointed. This extra hypothesis on G is required in the construction of the universal principal G-bundle EG over BG; to ensure that the bundle is locally trivial we must make one of the above assumptions on G. May's work goes further in this regard: he proves that if G is well pointed then EG is a *numerable* principal G-bundle over BG, and hence $EG \to BG$ is a Hurewicz fibration.

Another feature of this later construction is that EG comes equipped with the structure of a topological group. In the work of May and Segal, this arises from the fact that EG is the geometric realization of the nerve of a topological 2-group. We need the group structure on EG, so we will use this later construction rather than Milnor's. For further comparison of the constructions see tom Dieck [13].

We prove Theorem 1 using three lemmas that are of some interest in their own right. The second, as far as we know, is due to Larry Breen:

Lemma 1. *Let $\mathcal{G}$ be any well-pointed topological 2-group, and let (G,H,t,α) be the corresponding topological crossed module. Then:*

1. *$|\mathcal{G}|$ is a well-pointed topological group.*
2. *There is a topological 2-group $\hat{\mathcal{G}}$ such that $|\hat{\mathcal{G}}|$ fits into a short exact sequence of topological groups*

$$1 \to H \to |\hat{\mathcal{G}}| \xrightarrow{p} |\mathcal{G}| \to 1.$$

3. *G acts continuously via automorphisms on the topological group EH, and there is an isomorphism $|\hat{\mathcal{G}}| \cong G \ltimes EH$. This exhibits $|\mathcal{G}|$ as $G \ltimes_H EH$, the quotient of $G \ltimes EH$ by the normal subgroup H.*

Lemma 2. *If*

$$1 \to H \xrightarrow{t} G \xrightarrow{p} K \to 1$$

is a short exact sequence of topological groups, there is a bijection

$$\check{H}^1(M, H \to G) \cong \check{H}^1(M,K)$$

Here $H \to G$ is our shorthand for the 2-group corresponding to the crossed module (G,H,t,α) where t is the inclusion of the normal subgroup H in G and α is the action of G by conjugation on H.

Lemma 3. *If*

$$1 \to \mathcal{G}_0 \xrightarrow{f} \mathcal{G}_1 \xrightarrow{p} \mathcal{G}_2 \to 1$$

is a short exact sequence of topological 2-groups, then

$$\check{H}^1(M,\mathcal{G}_0) \xrightarrow{f_*} \check{H}^1(M,\mathcal{G}_1) \xrightarrow{p_*} \check{H}^1(M,\mathcal{G}_2)$$

is an exact sequence of pointed sets.

Given these lemmas the proof of Theorem 1 goes as follows. Assume that $\mathcal{G}$ is a well-pointed topological 2-group. From Lemma 1 we see that $|\mathcal{G}|$ is a well-pointed topological group. It follows that we have a bijection

$$\check{H}^1(M,|\mathcal{G}|) \cong [M, B|\mathcal{G}|]$$

So, to prove the theorem, it suffices to construct a bijection

$$\check{H}^1(M,\mathcal{G}) \cong \check{H}^1(M,|\mathcal{G}|)$$

By Lemma 1, $|\mathcal{G}|$ fits into a short exact sequence of topological groups:

$$1 \to H \to G \ltimes EH \to |\mathcal{G}| \to 1$$

We can use Lemma 2 to conclude that there is a bijection

$$\check{H}^1(M, H \to G \ltimes EH) \cong \check{H}^1(M,|\mathcal{G}|)$$

To complete the proof it thus suffices to construct a bijection

$$\check{H}^1(M, H \to G \ltimes EH) \cong \check{H}^1(M,\mathcal{G})$$

For this, observe that we have a short exact sequence of topological crossed modules:

$$\begin{tikzcd}
1 \arrow[r] & 1 \arrow[r] \arrow[d] & H \arrow[r, "1"] \arrow[d] & H \arrow[r] \arrow[d, "t"] & 1 \\
1 \arrow[r] & EH \arrow[r] & G \ltimes EH \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

So, by Lemma 3, we have an exact sequence of sets:

$$\check{H}^1(M, EH) \to \check{H}^1(M, H \to G \ltimes EH) \to \check{H}^1(M, H \to G)$$

Since EH is contractible and M is paracompact Hausdorff, $\check{H}^1(M, EH)$ is easily seen to be trivial, so the map $\check{H}^1(M, H \to G \ltimes EH) \to \check{H}^1(M, H \to G)$ is injective. To see that this map is surjective, note that there is a homomorphism of crossed modules going back:

$$\begin{tikzcd}
1 \arrow[r] & 1 \arrow[r] \arrow[d] & H \arrow[r, "1"] \arrow[d] & H \arrow[r] \arrow[d, "t"] \arrow[l, bend right, "1"'] & 1 \\
1 \arrow[r] & EH \arrow[r] & G \ltimes EH \arrow[r] & G \arrow[r] \arrow[l, bend right, "i"'] & 1
\end{tikzcd}$$

where i is the natural inclusion of G in the semidirect product $G \ltimes EH$. This homomorphism going back "splits" our exact sequence of crossed modules. It follows that $\check{H}^1(M, H \to G \ltimes EH) \to \check{H}^1(M, H \to G)$ is onto, so we have a bijection

$$\check{H}^1(M, H \to G \ltimes EH) \cong \check{H}^1(M, H \to G) = \check{H}^1(M, \mathcal{G})$$

completing the proof.

5.2 Remarks on Theorem 1

Theorem 1, asserting the existence of a classifying space for first Čech cohomology with coefficients in a topological 2-group, was originally stated in a preprint by Jurčo [20]. However, the argument given there was missing some details. In essence, the Jurčo's argument boils down to the following: he constructs a map $\check{H}^1(M, |\mathcal{G}|) \to \check{H}^1(M, \mathcal{G})$ and sketches the construction of a map $\check{H}^1(M, \mathcal{G}) \to \check{H}^1(M, |\mathcal{G}|)$. The construction of the latter map however requires some further justification: for instance, it is not obvious that one can choose a classifying map satisfying the cocycle property listed on the top of page 13 of [20]. Apart from this, it is not demonstrated that these two maps are inverses of each other.

As mentioned earlier, Jurčo and also Baas, Bökstedt and Kro [1] use a different approach to construct a classifying space for a topological 2-group $\mathcal{G}$. In their approach, $\mathcal{G}$ is regarded as a topological 2-groupoid with one object. There is a well-known nerve construction that turns any 2-groupoid (or even any 2-category) into a simplicial set [14]. Internalizing this construction, these authors turn the topological 2-groupoid $\mathcal{G}$ into a simplicial space, and then take the geometric realization of that to obtain a space. Let us denote this space by $B\mathcal{G}$. This is the classifying space used by Jurčo and Baas–Bökstedt–Kro. It should be noted that the assumption that $\mathcal{G}$ is a well-pointed 2-group ensures that the nerve of the 2-groupoid $\mathcal{G}$ is a "good" simplicial space in the sense of Segal; this "goodness" condition is important in the work of Baas, Bökstedt and Kro [1].

Baas, Bökstedt and Kro also consider a third way to construct a classifying space for $\mathcal{G}$. If we take the nerve $N\mathcal{G}$ of $\mathcal{G}$ we get a simplicial group, as described in Sect. 3 above. By thinking of each group of p-simplices $(N\mathcal{G})_p$ as a groupoid with one object, we can think of $N\mathcal{G}$ as a simplicial groupoid. From $N\mathcal{G}$ we can obtain a bisimplicial space $NN\mathcal{G}$ by applying the nerve construction to each groupoid $(N\mathcal{G})_p$. $NN\mathcal{G}$ is sometimes called the "double nerve", since we apply the nerve construction twice. From this bisimplicial space $NN\mathcal{G}$ we can form an ordinary simplicial space $dNN\mathcal{G}$ by taking the diagonal. Taking the geometric realization of this simplicial space, we obtain a space $|dNN\mathcal{G}|$.

It turns out that this space $|dNN\mathcal{G}|$ is homeomorphic to $B|\mathcal{G}|$ [6,28]. It can also be shown that the spaces $|dNN\mathcal{G}|$ and $B\mathcal{G}$ are homotopy equivalent – but although this fact seems well known to experts, we have been unable to find a reference in the case of a *topological* 2-group $\mathcal{G}$. For ordinary 2-groups (without topology) the relation between all three nerves was worked out by Moerdijk and Svensson [4] and Bullejos and Cegarra [12]. In any case, since we do not use these facts in our arguments, we forgo providing the proofs here.

5.3 Proof of Lemma 1

Suppose $\mathcal{G}$ is a well-pointed topological 2-group with topological crossed module (G, H, t, α), and let $|\mathcal{G}|$ be the geometric realization of its nerve. We shall prove that there is a topological 2-group $\hat{\mathcal{G}}$ fitting into a short exact sequence of topological 2-groups

$$1 \to H \to \hat{\mathcal{G}} \to \mathcal{G} \to 1 \tag{8}$$

where H is the discrete topological 2-group associated to the topological group H. On taking nerves and then geometric realizations, this gives an exact sequence of groups:

$$1 \to H \to |\hat{\mathcal{G}}| \to |\mathcal{G}| \to 1$$

Redescribing the 2-group $\hat{\mathcal{G}}$ with the help of some work by Segal, we shall show that $|\hat{\mathcal{G}}| \cong G \ltimes EH$ and thus $|\mathcal{G}| \cong (G \ltimes EH)/H$. Then we prove that the above sequence is an exact sequence of *topological* groups: this requires checking that $|\hat{\mathcal{G}}| \to |\mathcal{G}|$ is a Hurewicz fibration. We conclude by showing that $|\hat{\mathcal{G}}|$ is well pointed.

To build the exact sequence of 2-groups in (8), we construct the corresponding exact sequence of topological crossed modules. This takes the following form:

$$\begin{tikzcd}
1 \arrow[r] & 1 \arrow[r] \arrow[d] & H \arrow[r, "1"] \arrow[d, "t'"] & H \arrow[r] \arrow[d, "t"] & 1 \\
1 \arrow[r] & H \arrow[r, "f"] & G \ltimes H \arrow[r, "f'"] & G \arrow[r] & 1
\end{tikzcd}$$

Here the crossed module $(G \ltimes H, H, t', \alpha')$ is defined as follows:

$$t'(h) = (1, h)$$
$$\alpha'(g, h)(h') = \alpha(t(h)g)(h')$$

while f and f' are given by

$$\begin{aligned} f\colon H &\to G \ltimes H \\ h &\mapsto (t(h), h^{-1}) \end{aligned}$$

$$\begin{aligned} f'\colon G \ltimes H &\to G \\ (g,h) &\mapsto t(h)g \end{aligned}$$

It is easy to check that these formulas define an exact sequence of topological crossed modules. The corresponding exact sequence of topological 2-groups is

$$1 \to H \to \hat{\mathcal{G}} \to \mathcal{G} \to 1$$

where $\hat{\mathcal{G}}$ denotes the topological 2-group associated to the topological crossed module $(G \ltimes H, H, t', \alpha')$.

In more detail, the 2-group $\hat{\mathcal{G}}$ has

$$\begin{aligned} &\mathrm{Ob}(\hat{\mathcal{G}}) = G \ltimes H \\ &\mathrm{Mor}(\hat{\mathcal{G}}) = (G \ltimes H) \ltimes H \\ &s((g,h),h') = (g,h), \qquad t((g,h),h') = (g,h'h) \\ &i(g,h) = ((g,h),1), \qquad ((g,h'h),h'') \circ ((g,h),h') = ((g,h),h''h') \end{aligned}$$

Note that there is an isomorphism $(G \ltimes H) \ltimes H \cong G \ltimes H^2$ sending $((g,h),h')$ to $(g,(h,h'h))$. Here by $G \ltimes H^2$ we mean the semidirect product formed with the diagonal action of G on H^2, namely $g(h,h') = (\alpha(g)(h), \alpha(g)(h'))$. Thus the group $\mathrm{Mor}(\hat{\mathcal{G}})$ is isomorphic to $G \ltimes H^2$.

We can give a clearer description of the 2-group $\hat{\mathcal{G}}$ using the work of Segal [31]. Segal noted that for any topological group H, there is a 2-group $\overline{H}$ with one object for each element of H, and one morphism from any object to any other. In other words, $\overline{H}$ is the 2-group with:

$$\begin{aligned} &\mathrm{Ob}(\overline{H}) = H \\ &\mathrm{Mor}(\overline{H}) = H^2 \\ &s(h,h') = h, \quad t(h,h') = h' \\ &i(h) = (h,h), \quad (h',h'') \circ (h,h') = (h,h'') \end{aligned}$$

Moreover, Segal proved that the geometric realization $|\overline{H}|$ of the nerve of $\overline{H}$ is a model for EH. Since G acts on H by automorphisms, we can define a "semidirect product" 2-group $G \ltimes \overline{H}$ with

$$\begin{aligned} \mathrm{Ob}(G \ltimes \overline{H}) &= G \ltimes H \\ \mathrm{Mor}(G \ltimes \overline{H}) &= G \ltimes H^2 \end{aligned}$$

$$s(g,(h,h')) = (g,h), \qquad t(g,(h,h')) = (g,h')$$
$$i(g,h) = (g,(h,h)), \qquad (g,(h',h'')) \circ (g,(h,h')) = (g,(h,h''))$$

The isomorphism $(G \ltimes H) \ltimes H \cong G \ltimes H^2$ above can then be interpreted as an isomorphism $\mathrm{Mor}(\hat{\mathcal{G}}) \cong \mathrm{Mor}(G \ltimes \overline{H})$. It is easy to check that this isomorphism is compatible with the structure maps for $\hat{\mathcal{G}}$ and $G \ltimes \overline{H}$, so we have an isomorphism of topological 2-groups:

$$\hat{\mathcal{G}} \cong G \ltimes \overline{H}$$

It follows that the nerve $N\hat{\mathcal{G}}$ of $\hat{\mathcal{G}}$ is isomorphic as a simplicial topological group to the nerve of $G \ltimes \overline{H}$. As a simplicial space it is clear that $N(G \ltimes \overline{H}) = G \times N\overline{H}$. We need to identify the simplicial group structure on $G \times N\overline{H}$. From the definition of the products on $\mathrm{Ob}(G \ltimes \overline{H})$ and $\mathrm{Mor}(G \ltimes \overline{H})$, it is clear that the product on $N(G \ltimes \overline{H})$ is given by the simplicial map

$$(G \times N\overline{H}) \times (G \times N\overline{H}) \to G \times N\overline{H}$$

defined on p-simplices by

$$((g,(h_1,\ldots,h_p)),(g',(h'_1,\ldots,h'_p))) \mapsto (gg',(h_1\alpha(g)(h'_1),\ldots,h_p\alpha(g)(h'_p)))$$

Thus one might well call $N(G \ltimes \overline{H})$ the "semidirect product" $G \ltimes N\overline{H}$. Since geometric realization preserves products, it follows that there is an isomorphism of topological groups

$$|\hat{\mathcal{G}}| \cong G \ltimes EH.$$

Here the semidirect product is formed using the action of G on EH induced from the action of G on H. Finally note that H is embedded as a normal subgroup of $G \ltimes EH$ through

$$\begin{aligned} H &\to G \ltimes EH \\ h &\mapsto (t(h), h^{-1}) \end{aligned}$$

It follows that the exact sequence of groups $1 \to H \to |\hat{\mathcal{G}}| \to |\mathcal{G}| \to 1$ can be identified with

$$1 \to H \to G \ltimes EH \to |\mathcal{G}| \to 1 \tag{9}$$

It follows that $|\mathcal{G}|$ is isomorphic to the quotient $G \ltimes_H EH$ of $G \ltimes EH$ by the normal subgroup H. This amounts to factoring out by the action of H on $G \ltimes EH$ given by $h(g,x) = (t(h)g, xh^{-1})$.

Next we need to show that (9) specifies an exact sequence of *topological* groups: in particular, that the map $G \ltimes EH \to |\mathcal{G}| = G \ltimes_H EH$ is a Hurewicz fibration. To do this, we prove that the following diagram is a pullback:

$$\begin{tikzcd} G \ltimes EH \arrow[r] \arrow[d] & EH \arrow[d] \\ G \ltimes_H EH \arrow[r] & BH \end{tikzcd}$$

Since H is well pointed, $EH \to BH$ is a numerable principal bundle (and hence a Hurewicz fibration) by the results of May [24] referred to earlier. The statement above now follows, as Hurewicz fibrations are preserved under pullbacks.

To show the above diagram is a pullback, we construct a homeomorphism

$$\alpha\colon (G \ltimes_H EH) \times_{BH} EH \to G \ltimes EH$$

whose inverse is the canonical map $\beta\colon G \ltimes EH \to (G \ltimes_H EH) \times_{BH} EH$. To do this, suppose that $([g,x],y) \in (G \ltimes_H EH) \times_{BH} EH$. Then x and y belong to the same fiber of EH over BH, so $y^{-1}x \in H$. We set

$$\alpha([g,x],y) = (t(y^{-1}x)g, y)$$

A straightforward calculation shows that α is well defined and that α and β are inverse to one another.

To conclude, we need to show that $|\mathcal{G}|$ is a well-pointed topological group. For this it is sufficient to show that $N\mathcal{G}$ is a "proper" simplicial space in the sense of May [23] (note that we can replace his "strong" NDR pairs with NDR pairs). For, if we follow May and denote by $F_p|\mathcal{G}|$ the image of $\coprod_{i=0}^{p} \Delta^i \times N\mathcal{G}_i$ in $|\mathcal{G}|$, it then follows from his Lemma 11.3 that $(|\mathcal{G}|, F_p|\mathcal{G}|)$ is an NDR pair for all p. In particular $(|\mathcal{G}|, F_0|\mathcal{G}|)$ is an NDR pair. Since $F_0|\mathcal{G}| = G$ and $(G, 1)$ is an NDR pair, it follows that $(|\mathcal{G}|, 1)$ is an NDR pair: that is, $|\mathcal{G}|$ is well pointed. $\square$

5.4 *Proof of Lemma 2*

Suppose that M is a topological space admitting good covers. Also suppose that $1 \to H \xrightarrow{t} G \xrightarrow{p} K \to 1$ is an exact sequence of topological groups.

This data gives rise to a topological crossed module $H \xrightarrow{t} G$ where G acts on H by conjugation. For short we denote this by $H \to G$. The same data also gives a topological crossed module $1 \to K$. There is a homomorphism of crossed modules from $H \to G$ to $1 \to K$, arising from this commuting square:

$$\begin{tikzcd} H \arrow[r] \arrow[d, "t"] & 1 \arrow[d] \\ G \arrow[r, "p"] & K \end{tikzcd}$$

Call this homomorphism α. It yields a map

$$\alpha_*\colon \check{H}^1(M, H \to G) \to \check{H}^1(M, 1 \to K).$$

Note that $\check{H}^1(M, 1 \to K)$ is just the ordinary Čech cohomology $\check{H}^1(M, K)$. To prove Lemma 2, we need to construct an inverse

$$\beta : \check{H}^1(M, K) \to \check{H}^1(M, H \to G).$$

Let $\mathcal{U} = \{U_i\}$ be a good cover of M; then, as noted in Sect. 4 there is a bijection

$$\check{H}^1(M, K) = \check{H}^1(\mathcal{U}, K)$$

Hence to define the map β it is sufficient to define a map $\beta : \check{H}^1(\mathcal{U}, K) \to \check{H}^1(\mathcal{U}, H \to G)$. Let k_{ij} be a K-valued Čech 1-cocycle subordinate to $\mathcal{U}$. Then from it we construct a Čech 1-cocycle (g_{ij}, h_{ijk}) taking values in $H \to G$ as follows. Since the spaces $U_i \cap U_j$ are contractible and $p : G \to K$ is a Hurewicz fibration, we can lift the maps $k_{ij} : U_i \cap U_j \to K$ to maps $g_{ij} : U_i \cap U_j \to G$. The g_{ij} need not satisfy the cocycle condition for ordinary Čech cohomology, but instead we have

$$t(h_{ijk}) g_{ij} g_{jk} = g_{ik}$$

for some unique $h_{ijk} : U_i \cap U_j \cap U_k \to H$. In terms of diagrams, this means we have triangles

$$\begin{tikzcd}
& \bullet \arrow[dr, "g_{jk}"] \arrow[d, phantom, "\Downarrow h_{ijk}"] & \\
\bullet \arrow[ur, "g_{ij}"] \arrow[rr, "g_{ik}"'] & \, & \bullet
\end{tikzcd}$$

The uniqueness of h_{ijk} follows from the fact that the homomorphism $t : H \to G$ is injective. To show that the pair (g_{ij}, h_{ijk}) defines a Čech cocycle we need to check that the tetrahedron (6) commutes. However, this follows from the commutativity of the corresponding tetrahedron built from triangles of this form:

$$\begin{tikzcd}
& \bullet \arrow[dr, "k_{jk}"] \arrow[d, phantom, "\Downarrow 1"] & \\
\bullet \arrow[ur, "k_{ij}"] \arrow[rr, "k_{ik}"'] & \, & \bullet
\end{tikzcd}$$

and the injectivity of t.

Let us show that this construction gives a well-defined map

$$\beta : \check{H}^1(M, K) = \check{H}^1(\mathcal{U}, K) \to \check{H}^1(M, H \to G)$$

sending $[k_{ij}]$ to $[g_{ij}, h_{ijk}]$. Suppose that k'_{ij} is another K-valued Čech 1-cocycle subordinate to $\mathcal{U}$, such that k'_{ij} and k_{ij} are cohomologous. Starting from the cocycle k'_{ij}

we can construct (in the same manner as above) a cocycle (g'_{ij}, h'_{ijk}) taking values in $H \to G$. Our task is to show that (g_{ij}, h_{ijk}) and (g'_{ij}, h'_{ijk}) are cohomologous. Since k_{ij} and k'_{ij} are cohomologous there exists a family of maps $\kappa_i : U_i \to K$ fitting into the naturality square

$$\begin{tikzcd}
\bullet \arrow[r, "k_{ij}(x)"] \arrow[d, "\kappa_i(x)"'] & \bullet \arrow[d, "\kappa_j(x)"] \\
\bullet \arrow[r, "k'_{ij}(x)"'] & \bullet
\end{tikzcd}$$

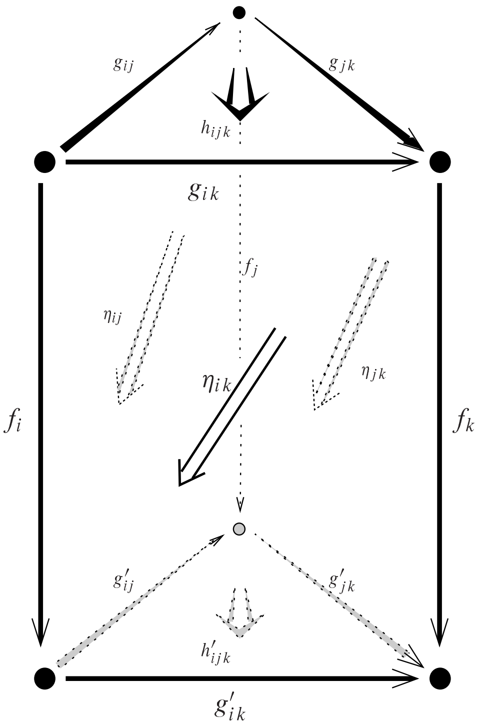
Choose lifts $f_i : U_i \to G$ of the various κ_i. Since $p(g_{ij}) = p(g'_{ij}) = k_{ij}$ and $p(f_i) = \kappa_i$, $p(f_j) = \kappa_j$ there is a unique map $\eta_{ij} : U_i \cap U_j \to H$

$$t(\eta_{ij}) g_{ij} f_j = f_i g'_{ij}.$$

So, in terms of diagrams, we have the following squares:

$$\begin{tikzcd}
\bullet \arrow[r, "g_{ij}"] \arrow[d, "f_i"'] & \bullet \arrow[d, "f_j"] \arrow[dl, Rightarrow, "\eta_{ij}"'] \\
\bullet \arrow[r, "g'_{ij}"'] & \bullet
\end{tikzcd}$$

The triangles and squares defined so far fit together to form prisms:



It follows from the injectivity of the homomorphism t that these prisms commute, and therefore that (g_{ij}, h_{ijk}) and (g'_{ij}, h'_{ijk}) are cohomologous. Therefore we have a well-defined map $\check{H}^1(\mathcal{U}, K) \to \check{H}^1(\mathcal{U}, H \to G)$ and hence a well-defined map $\beta\colon \check{H}^1(M, K) \to \check{H}^1(M, H \to G)$.

Finally we need to check that α and β are inverse to one another. It is obvious that $\alpha \circ \beta$ is the identity on $\check{H}^1(M, K)$. To see that $\beta \circ \alpha$ is the identity on $\check{H}^1(M, H \to G)$ we argue as follows. Choose a cocycle (g_{ij}, h_{ijk}) subordinate to a good cover $\mathcal{U} = \{U_i\}$. Then under α the cocycle $[g_{ij}, h_{ijk}]$ is sent to the K-valued cocycle $[p(g_{ij})]$. But then we may take g_{ij} as our lift of $p(g_{ij})$ in the definition of $\beta(p(g_{ij}))$. It is then clear that $(\beta \circ \alpha)[g_{ij}, h_{ijk}] = [g_{ij}, h_{ijk}]$. $\square$

At this point a remark is in order. The proof of the above lemma is one place where the definition of $\check{H}^1(M, H \to G)$ in terms of hypercovers would lead to simplifications, and would allow us to replace the hypothesis that the map underlying the homomorphism $G \to K$ was a fibration with a less restrictive condition. The homomorphism of crossed modules

$$\begin{tikzcd} H \arrow[r] \arrow[d, "t"] & 1 \arrow[d] \\ G \arrow[r, "p"] & K \end{tikzcd}$$

gives a homomorphism between the associated 2-groups and hence a simplicial map between the nerves of the associated 2-groupoids. It turns out that this simplicial map belongs to a certain class of simplicial maps with respect to which a subcategory of simplicial spaces is localized. In the formalism of hypercovers, for M paracompact, the nonabelian cohomology $\check{H}^1(M, \mathcal{G})$ with coefficients in a 2-group $\mathcal{G}$ is defined as a certain set of morphisms in this localized subcategory. It is then easy to see that the induced map $\check{H}^1(M, H \to G) \to \check{H}^1(M, K)$ is a bijection.

5.5 *Proof of Lemma 3*

Suppose that

$$1 \to \mathcal{G}_0 \xrightarrow{f} \mathcal{G}_1 \xrightarrow{p} \mathcal{G}_2 \to 1$$

is a short exact sequence of topological 2-groups, so that we have a short exact sequence of topological crossed modules:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & H_0 \arrow[r, "f"] \arrow[d, "t_0"'] & H_1 \arrow[r, "p"] \arrow[d, "t_1"'] & H_2 \arrow[r] \arrow[d, "t_2"'] & 1 \arrow[d] \\ 1 \arrow[r] & G_0 \arrow[r, "f"] & G_1 \arrow[r, "p"] & G_2 \arrow[r] & 1 \end{tikzcd}$$

Also suppose that $\mathcal{U} = \{U_i\}$ is a good cover of M, and that (g_{ij}, h_{ijk}) is a cocycle representing a class in $\check{H}^1(\mathcal{U}, \mathcal{G}_1)$. We claim that the image of

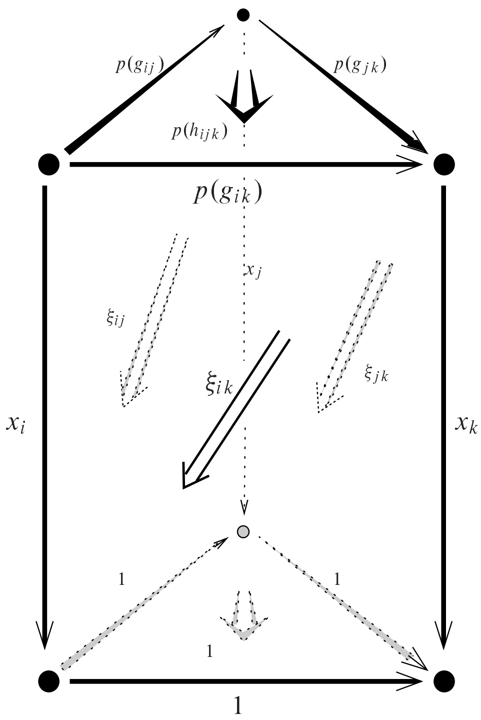
$$f_*\colon \check{H}^1(M, \mathcal{G}_0) \to \check{H}^1(M, \mathcal{G}_1)$$

equals the kernel of

$$p_*\colon \check{H}^1(M, \mathcal{G}_1) \to \check{H}^1(M, \mathcal{G}_2).$$

If the class $[g_{ij}, h_{ijk}]$ is in the image of f_*, it is clearly in the kernel of p_*. Conversely, suppose it is in kernel of p_*. We need to show that it is in the image of f_*.

The pair $(p(g_{ij}), p(h_{ijk}))$ is cohomologous to the trivial cocycle, at least after refining the cover $\mathcal{U}$, so there exist $x_i\colon U_i \to G_2$ and $\xi_{ij}\colon U_i \cap U_j \to H_2$ such that this diagram commutes for all $x \in U_i \cap U_j \cap U_k$:



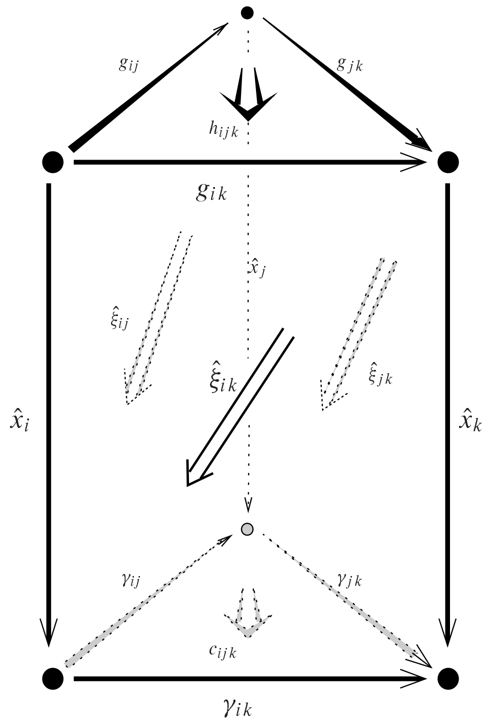
Since $p\colon G_1 \to G_2$ is a fibration and U_i is contractible, we can lift x_i to a map $\hat{x}_i\colon U_i \to G_1$. Similarly, we can lift ξ_{ij} to a map $\hat{\xi}_{ij}\colon U_i \cap U_j \to H_1$. There are then unique maps $\gamma_{ij}\colon U_i \cap U_j \to G_1$ giving squares like this:

$$\begin{tikzcd} \bullet \arrow[r, "g_{ij}"] \arrow[d, "\hat{x}_i"'] & \bullet \arrow[d, "\hat{x}_j"] \arrow[dl, phantom, "\hat{\xi}_{ij}\ \swarrow"] \\ \bullet \arrow[r, "\gamma_{ij}"'] & \bullet \end{tikzcd}$$

namely

$$\gamma_{ij} = \hat{x}_i g_{ij} \hat{x}_j^{-1} t(\hat{\xi}_{ij})$$

Similarly, there are unique maps $c_{ijk}: U_i \cap U_j \cap U_k \to H_1$ making this prism commute:



To define c_{ijk}, we simply compose the 2-morphisms on the sides and top of the prism.

Applying p to the prism above we obtain the previous prism. So, γ_{ij} and c_{ijk} must take values in the kernel of $p: G_1 \to G_2$ and $p: H_1 \to H_2$, respectively. It follows that γ_{ij} and c_{ijk} take values in the image of f.

The above prism says that (γ_{ij}, c_{ijk}) is cohomologous to (g_{ij}, h_{ijk}), and therefore a cocycle in its own right. Since γ_{ij} and c_{ijk} take values in the image of f, they represent a class in the image of

$$f_*: \check{H}^1(M, \mathcal{G}_0) \to \check{H}^1(M, \mathcal{G}_1).$$

So, $[g_{ij}, h_{ijk}] = [\gamma_{ij}, c_{ijk}]$ is in the image of f_*, as was to be shown. $\square$

5.6 Proof of Theorem 2

The following proof was first described to us by Matt Ando (personal communication), and later discussed by Greg Ginot [15].

Suppose that G is a simply-connected, compact, simple Lie group. Then the string group $\hat{G}$ of G fits into a short exact sequence of topological groups

$$1 \to K(\mathbb{Z}, 2) \to \hat{G} \to G \to 1$$

for some realization of the Eilenberg–Mac Lane space $K(\mathbb{Z}, 2)$ as a topological group. Applying the classifying space functor B to this short exact sequence gives rise to a fibration

$$K(\mathbb{Z}, 3) \to B\hat{G} \xrightarrow{p} BG.$$

We want to compute the rational cohomology of $B\hat{G}$.

We can use the Serre spectral sequence to compute $H^*(B\hat{G}, \mathbb{Q})$. Since BG is simply connected the E_2 term of this spectral sequence is

$$E_2^{p,q} = H^p(BG, \mathbb{Q}) \otimes H^q(K(\mathbb{Z}, 3), \mathbb{Q}).$$

Because $K(\mathbb{Z}, 3)$ is rationally indistinguishable from S^3, the first nonzero differential is d_4. Furthermore, the differentials of this spectral sequence are all derivations. It follows that $d_4(y \otimes x_3) = (-1)^p y \otimes d_4(x_3)$ if $y \in H^p(BG, \mathbb{Q})$. It is not hard to identify $d_4(x_3)$ with c, the class in $H^4(BG, \mathbb{Q})$ which is the transgression of the generator ν of $H^3(G, \mathbb{Q}) = \mathbb{Q}$. It follows that the spectral sequence collapses at the E_5 stage with

$$E_5^{p,q} = E_\infty^{p,q} = \begin{cases} 0 & \text{if } q > 0 \\ H^p(BG, \mathbb{Q})/\langle c \rangle & \text{if } q = 0. \end{cases}$$

One checks that all the subcomplexes $F^i H^*(B\hat{G}, \mathbb{Q})$ in the filtration of $H^*(B\hat{G}, \mathbb{Q})$ are zero for $i \geq 1$. Hence $H^p(B\hat{G}, \mathbb{Q}) = E_\infty^{p,0} = H^p(BG, \mathbb{Q})/\langle c \rangle$ and so Theorem 2 is proved. □

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String Topology in Dimensions Two and Three

Moira Chas and Dennis Sullivan

1 Introduction

Let V denote the vector space with basis the conjugacy classes in the fundamental group of an oriented surface S. In 1986 Goldman [1] constructed a Lie bracket $[\,,\,]$ on V. If a and b are conjugacy classes, the bracket $[a,b]$ is defined as the signed sum over intersection points of the conjugacy classes represented by the loop products taken at the intersection points. In 1998 the authors constructed a bracket on higher dimensional manifolds which is part of String Topology [2]. This happened by accident while working on a problem posed by Turaev [3], which was not solved at the time. The problem consisted in characterizing algebraically which conjugacy classes on the surface S are represented by simple closed curves. Turaev was motivated by a theorem of Jaco and Stallings [4,5] that gave a group theoretical statement equivalent to the three dimensional Poincaré conjecture. This statement involved simple conjugacy classes.

Recently a number of results have been achieved which illuminate the area around Turaev's problem. Now that the conjecture of Poincarè has been solved, the statement about groups of Jaco and Stallings is true and one may hope to find a Group Theory proof. Perhaps the results to be described here could play a role in such a proof. See Sect. 3 for some first steps in this direction.

2 Algebraic Characterization of Simple Closed Curves on Surfaces

Let a and b be conjugacy classes in the fundamental group of the surface S. Let $i(a,b)$ denote the minimal number of intersection points with multiplicity of representatives of a and b. Let $M(v)$ for v in V denote the sum of the absolute values

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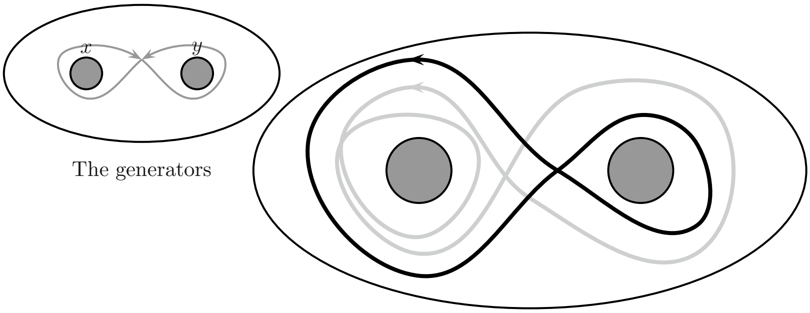

Fig. 1 An example of a pair of nondisjoint curves with zero bracket

of the coefficients of v in the given basis of V. From the definition of the Goldman bracket it is clear that $M([a, b])$ is at most $i(a, b)$. There are examples showing equality may not hold, e.g., $a = xy$ and $b = xyy$ on the disk with two punctures where x and y are the generators running around two of the boundary components, with counterclockwise and clockwise orientations respectively (see Fig. 1). Note both a and b have self-intersections in this example.

Theorem 1. *[6]. If a is represented by a curve without self-intersections then for all conjugacy classes b, $i(a, b) = M([a, b])$. In other words, there is no cancellation in the bracket where at least one factor is simple.*

Corollary. *Let a and b both be simple. Then the function $i(a, b)$ used to compactify Teichmuller space in Thurston's work can be encoded in the algebraic structure of the bracket on V with the given basis.*

To describe the next result note the following: If a is a simple conjugacy class, for all integers n and m the bracket of the n-th power of a with the m-th power of a is zero, $[a^n, a^m] = 0$. This is true because a^n and a^m have disjoint representatives.

Theorem 2. *[7]. Suppose S is a surface with at least one puncture or boundary component and let a be a conjugacy class which is not a power of another class. Then a has a simple representative if and only if $[a^n, a^m] = 0$ for all pairs of positive integers n and m.*

Theorem 3. *[7]. Suppose S is a surface with at least one puncture or boundary component, a is any conjugacy class and m and n are distinct positive integers such that one of them is at least three. Then the bracket $[a^n, a^m]$ counts the minimal number of self intersections of any representative of a.*

Turaev's question concerned a Lie cobracket δ defined on the vector space V', defined as V modulo the one dimensional subspace generated by the trivial conjugacy class (see [3]). The cobracket δ is a sum over self intersection points of a representative of the wedge product of the two classes obtained by oriented reconnecting at the intersection point. (Notice that there are two ways to reconnect, but only one of them yields two loops. Also the order of the loops for defining the wedge is determined by the orientation of the surface).

Theorem 4. *[8]. Let S be a surface with at least one puncture or boundary component and let a be a conjugacy class which is not the power of another class. Then a has a simple representative if and only if $\delta(a^n) = 0$ for all n. Moreover, the equation $\delta(a^n) = 0$ for any fixed n at least three suffices to insure the existence of a simple representative of a.*

Theorem 5. *[8]. Let S be a surface with at least one puncture or boundary component and let n be an integer larger than four. Then for any conjugacy class a the number of terms in $\delta(a^n)$ counts the minimal number of self intersection points of representatives of a.*

Theorem 1 uses the HNN extension or the free product with amalgamation decomposition of the fundamental group associated to a simple element a to compute $[a, b]$ in terms of a canonical form for the conjugacy class b.

Theorems 2–5 uses Combinatorial Group Theory and the presentation of the Goldman–Turaev Lie bialgebra given in [9]. Extending this algorithm and the Theorems 2–5 to closed surfaces is work in progress.

The bracket and cobracket were defined as part of String Topology on the reduced circle equivariant homology of the free loop space of any manifold (generalizing V') (see the 2003 Abel Proceedings [10]).

3 Three Manifolds

3.1 The Statement About Groups for Closed Surfaces and Surfaces with Boundary

We recall the statement about groups that is equivalent to the Poincarè conjecture [4, 5]. Let π_g denote the fundamental group of a closed orientable surface of genus g and let F_g denote the free group on g generators, $g = 2, 3, \ldots$. The statement about groups equivalent to the three dimensional Poincarè conjecture is

$*$ *Every surjection $\pi_g \xrightarrow{h} F_g \times F_g$ contains in its kernel a conjugacy class represented by a simple closed curve.*

Now, Theorems 2 and 4 above, characterized simple conjugacy classes on surfaces with boundary. So here is a variant of $*$ that fits with these theorems. Let Σ'_g be an orientable surface of genus g with one boundary component. Let A denote the conjugacy class of the boundary component in F_{2g} the fundamental group of Σ'_g. ($A = [x_1, y_1] \cdot [x_2, y_2] \cdots [x_g, y_g]$ in the standard generating set of Σ_g).

Theorems 2 and 4 characterize simple conjugacy classes on Σ'_g in terms of the bracket and the cobracket on Σ'_g. The following statement is equivalent to $*$ above.

$**$ *Any surjection* $F_{2g} \xrightarrow{h} F_g \times F_g$ *that sends* $A^{\pm 1} = (\prod [x_i, y_i])^{\pm 1}$ *to the identity also sends a distinct nontrivial simple conjugacy class to the identity.*

Proof that $*$ and $**$ are equivalent. In effect, we are studying free homotopy classes of maps $\Sigma'_g \xrightarrow{s'} \Gamma_g \times \Gamma_g$ and $\Sigma_g \xrightarrow{s} \Gamma_g \times \Gamma_g$ where Σ_g is the closed surface of genus g and Γ_g is a graph whose first homology has rank g. Then a map s' extends to s if and only if s' sends A to the identity. Any two extensions are homotopic because $\Gamma_g \times \Gamma_g$ has contractible universal cover.

Writing Σ_g as Σ'_g union a two-disk we see any nontrivial embedded closed curve in Σ_g can be isotoped into Σ'_g giving a nontrivial embedded curve distinct from the boundary. Conversely, if a nontrivial embedded curve on the surface with boundary is not freely homotopic to the boundary then it is still nontrivial in the closed surface. This proves the equivalence of $*$ and $**$.

3.2 Two Properties of the Kernel

Consider maps $\Sigma'_g \xrightarrow{s} \Gamma_g \times \Gamma_g$ which are surjective on the fundamental group. Denote by $\widehat{s}$ the induced map on conjugacy classes. Denote by s' the map induced by s on π_1, where the basepoint of Σ'_g is taken on the boundary and the basepoint of $\Gamma_g \times \Gamma_g$ is the image by s of the basepoint of Σ'_g.

Let K denote the linear span of the classes of F_{2g} that are sent by $\widehat{s}$ to the trivial class in $F_g \times F_g$. Let J denote the group ring of the kernel of the homomorphism $F_{2g} \longrightarrow F_g \times F_g$ induced by s.

Proposition. *K is a subLie algebra of the Goldman Lie algebra V. For the natural String Topology action of V on the group ring of F_{2g} by derivations [11], K keeps J invariant.*

Proof. Both the bracket and the action are defined as a sum over transversal intersection points of loop compositions with a sign coming from the orientation and where we drop the base point in the bracket case and where we move the base point back to the boundary in the case of the action.The connected sum at an intersection point of a disk on x a generator of K and a disk on y, a generator of J is a disk on that term of the action of x on y. This proves the second part. The proof for the first part is the same. $\square$

3.3 Irreducible Three Manifolds

The results of [12] plus geometrization show that String Topology in dimension three vanishes in a certain sense precisely on closed hyperbolic manifolds. These ideas can be extended to a relationship between the torus decomposition and the Lie algebra structure for irreducible three manifolds. This is also work in progress.

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Floer Homotopy Theory, Realizing Chain Complexes by Module Spectra, and Manifolds with Corners

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Abstract In this paper we describe and continue the study begun in Cohen et al. (Progress in Mathematics, vol. 133, Birkhauser, Boston, 1995, pp. 287–325) of the homotopy theory that underlies Floer theory. In that paper the authors addressed the question of realizing a Floer complex as the cellular chain complex of a CW-spectrum or pro-spectrum, where the attaching maps are determined by the compactified moduli spaces of connecting orbits. The basic obstructions to the existence of this realization are the smoothness of these moduli spaces, and the existence of compatible collections of framings of their stable tangent bundles. In this note we describe a generalization of this, to show that when these moduli spaces are smooth, and are oriented with respect to a generalized cohomology theory E^*, then a Floer E_*-homology theory can be defined. In doing this we describe a functorial viewpoint on how chain complexes can be realized by E-module spectra, generalizing the stable homotopy realization criteria given in Cohen et al. (Progress in Mathematics, vol. 133, Birkhauser, Boston, 1995, pp. 287–325). Since these moduli spaces, if smooth, will be manifolds with corners, we give a discussion about the appropriate notion of orientations of manifolds with corners.

1 Introduction

In [5], the author, J.D.S Jones, and G. Segal began the study of the homotopy theoretic aspects of Floer theory. Floer theory comes in many flavors, and it was the goal of that paper to understand the common algebraic topological properties that underly them. A Floer stable homotopy theory of three-manifolds in the Seiberg–Witten setting was defined and studied by Manolescu in [16, 17]. This in turn was related to the stable homotopy viewpoint of Seiberg–Witten invariants of closed four-manifolds studied by Bauer and Furuta in [2, 3]. A Floer stable homotopy type in the setting of the symplectic topology of the cotangent bundle, T^*M, was shown

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to exist by the author in [4], and it was calculated to be the stable homotopy type of the free loop space, LM. These were the results reported on by the author at the Abel conference in Oslo.

In this largely expository note, we discuss basic notions of Floer homotopy type, and generalize them to discuss obstructions to the existence of a Floer E_*-homology theory, when E^* is a generalized cohomology theory.

In a "Floer theory" one typically has a functional, $\mathcal{A} : \mathcal{Y} \to \mathbb{R}$, defined on an infinite dimensional manifold, $\mathcal{Y}$, whose critical points generate a "Floer chain complex", $(CF_*(\mathcal{A}), \partial_{\mathcal{A}})$. The boundary maps in this complex are determined by the zero dimensional moduli spaces of flow lines connecting the critical points, much the same as in classical Morse theory. In Morse theory, however, given a Morse function $f : M \to \mathbb{R}$, there is an associated handlebody decomposition of the manifold. This leads to a corresponding CW-complex X_f with one cell for each critical point of f, which is naturally homotopy equivalent to M. The associated cellular chain complex, (C_*^f, ∂_f), is equal to the Morse chain complex generated by critical points, with boundary homomorphisms computed by counting gradient flow lines. The relative attaching map between cells of X_f, in the case when the corresponding critical points α and β labeling these cells are "successive", was shown by Franks in [11] to be given by a map of spheres, which, under the Pontrjagin–Thom construction, is represented by the compact framed manifold of flow lines, $\mathcal{M}(\alpha, \beta)$, connecting α to β. In [5] this was generalized so as not to assume that α and β are successive. Namely it was shown that all the stable attaching maps in the complex X_f are represented by the framed cobordism classes of the compactified moduli spaces $\bar{\mathcal{M}}(\gamma, \delta)$ of flow lines. Here these moduli spaces are viewed as compact, framed, manifolds with corners.

A natural question, addressed in [5], is under what conditions, is there an underlying CW-complex or spectrum, or even "prospectrum" as described there, in which the underlying cellular chain complex is exactly the Floer complex. In particular, given a Floer functional, $\mathcal{A} : \mathcal{Y} \to \mathbb{R}$, one is looking for a CW-(pro)-spectrum that has one cell for each critical point, and whose relative attaching maps between critical points α and β of relative index one are determined by the zero dimensional, compact, oriented moduli space $\mathcal{M}(\alpha, \beta)$ of gradient flow lines connecting them. Moreover, one would like this "Floer (stable) homotopy type" to have its higher attaching maps, that is to say its entire homotopy type, determined by the higher dimensional compactified moduli spaces, $\bar{\mathcal{M}}(\gamma, \delta)$, in a natural way, as in the Morse theory setting. Roughly, the obstruction to the existence of this Floer homotopy type was shown in [5] to be the existence of the structures of smooth, framed manifolds with corners on these compact moduli spaces. This was expressed in terms of the notion of "compact, smooth, framed" categories.

The goal of the present paper is to review and generalize these ideas, in order to address the notion of when a "Floer E_*-theory" exists, where E^* is a multiplicative, generalized cohomology theory. That is, E^* is represented by a commutative ring spectrum E. (By a "commutative" ring spectrum, we mean an E_∞-ring spectrum in the setting of algebra spectra over operads as in [8], or a commutative symmetric ring spectrum as in [14].) We will show that if the compact moduli spaces $\bar{\mathcal{M}}(\gamma, \delta)$ are smooth manifolds with corners, and have a compatible family of E^*-orientations, then a Floer E_*-theory exists.

In order to state this result more precisely, we observe that this Floer theory realization question is a special case of the question of the realization of a chain complex by an E-module spectrum. When E is the sphere spectrum S^0, then this question becomes that of the realization of a chain complex by a stable homotopy type. To be more precise, consider a connective chain complex of finitely generated free abelian groups,

$$\to \cdots \to C_i \xrightarrow{\partial_i} C_{i-1} \xrightarrow{\partial_{i-1}} \cdots \to C_0$$

where we are given a basis, $\mathcal{B}_i$, for the chain group C_i. For example, if this is a Morse complex or a Floer complex, the bases $\mathcal{B}_i$ are given as the critical points of index i. Now consider the tensor product chain complex, $(C_* \otimes E_*, \partial \otimes 1)$, where $E_* = E^*(S^0)$ is the coefficient ring. Notice that for each i, one can take the free E-module spectrum generated by $\mathcal{B}_i$,

$$\mathcal{E}_i = \bigvee_{\alpha \in \mathcal{B}_i} E,$$

and there is a natural isomorphism of E_*-modules,

$$\pi_*(\mathcal{E}_i) \simeq C_i \otimes E_*.$$

Definition 1. We say that an E-module spectrum X realizes the complex $C_* \otimes E_*$, if there exists a filtration of E-module spectra converging to X,

$$X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_k \hookrightarrow \cdots X$$

satisfying the following properties:

1. There is an equivalence of E-module spectra of the subquotients

$$X_i/X_{i-1} \simeq \Sigma^i \mathcal{E}_i.$$

2. The induced composition map in homotopy groups,

$$\begin{aligned} C_i \otimes E_* &\cong \pi_{*+i}(X_i/X_{i-1}) \xrightarrow{\delta_i} \pi_{*+i}(\Sigma X_{i-1}) \xrightarrow{\rho_{i-1}} \pi_{*+i}(\Sigma(X_{i-1}/X_{i-2})) \\ &\cong C_{i-1} \otimes E_* \end{aligned}$$

is the boundary homomorphism, $\partial_i \otimes 1$.

Here the subquotient X_i/X_{i-1} refers to the homotopy cofiber of the map $X_{i-1} \to X_i$, the map $\rho_i : X_i \to X_i/X_{i-1}$ is the projection map, and the map $\delta_i : X_i/X_{i-1} \to \Sigma X_{i-1}$ is the Puppe extension of the homotopy cofibration sequence $X_{i-1} \to X_i \xrightarrow{\rho_i} X_i/X_{i-1}$.

In [5], the authors introduced a category $\mathcal{J}_0$, whose objects are the nonnegative integers $\mathbb{Z}^+$, and whose space of morphisms from i to j for $i > j+1$ is homeomorphic to the one point compactification of the manifold with corners,

$$Mor_{\mathcal{J}}(i,j) \cong (\mathbb{R}_+)^{i-j-1} \cup \infty$$

and $Mor_{\mathcal{J}}(j+1,j) = S^0$. Here $\mathbb{R}_+$ is the space of nonnegative real numbers. The following is a generalization of the realization theorem proved in [5].

Theorem 2. *Let $E-mod$ be the category of E-module spectra. Then realizations of the chain complex $C_* \otimes E_*$ by E-module spectra correspond to extensions of the association $j \to \mathcal{E}_j$ to functors $Z : \mathcal{J} \to E-mod$, with the property that for each j, the map*

$$\begin{aligned} Z_{j+1,j} : Mor_{\mathcal{J}}(j+1,j) \wedge \mathcal{E}_{j+1} &\to \mathcal{E}_j \\ S^0 \wedge \mathcal{E}_{j+1} &\to \mathcal{E}_j \end{aligned}$$

induces the boundary homomorphism $\partial_{j+1} \otimes 1$ on the level of homotopy groups.

This theorem was proved for $E = S^0$ in [5] by displaying an explicit geometric realization of such a functor. In this note we indicate how that construction can be extended to prove this more general theorem.

The next question that we address in this paper is how to give geometric conditions on the moduli spaces of flow lines to induce a natural E-module realization of a Floer complex. That is, we want to describe geometric conditions that will induce a functor $Z : \mathcal{J} \to E-mod$ as in Theorem 2. Roughly, the condition is that the moduli spaces are smooth and admit a compatible family of E^*-orientations.

The compatibility conditions of these orientations are described in terms of the "flow-category" of a Floer functional (see [5]). One of the conditions required is that this category be a "smooth, compact, category". This notion was defined in [5]. Such a category is a topological category $\mathcal{C}$ whose objects form a discrete set, whose morphism spaces, $Mor(a,b)$, are compact, smooth manifolds with corners, and such that the composition maps $\mu : Mor(a,b) \times Mor(b,c) \to Mor(a,c)$ are smooth codimension one embeddings whose images lie in the boundary. In [4] this was elaborated upon by describing a "Morse–Smale" condition on such a category, whereby the objects are equipped with a partial ordering and have the notion of an "index" assigned to them. The "flow category" of a Morse function $f : M \to \mathbb{R}$ satisfies this condition if the metric on M is chosen so that the Morse–Smale transversality condition holds. This is the condition which states that each unstable manifold and stable manifold intersect each other transversally. When the flow category of a Floer functional satisfies these properties, we call it a "smooth Floer theory" (see Definition 4 below).

In Sect. 4 we study the issue of orientations of the manifolds with corners comprising the morphism spaces of a smooth, compact category. This will involve a discussion of some basic properties of manifolds with corners, following the exposition given in [15]. In particular we will make use of the fact that such manifolds with corners define a diagram of spaces, via the corner structure. Our notion of an E^*-orientation of a smooth compact category will be given by a functorial collection of E^*-Thom classes of the Thom spectra of the stable normal bundles of the morphism manifolds. These Thom spectra themselves are corresponding diagrams of spectra, and the orientation maps representing the Thom classes are morphisms of diagrams of spectra. See Definition 10 below.

Our main result of this section is stated in Theorem 1 below. It says that if a smooth Floer theory has an E^*- orientation of its flow category, then the Floer complex has a natural realization by an E-module spectrum. The homotopy groups of this spectrum can then be viewed as the "Floer E_*-homology".

2 Floer Homotopy Theory

2.1 *Preliminaries from Morse Theory*

In classical Morse theory one begins with a smooth, closed n-manifold M^n, and a smooth function $f : M^n \to \mathbb{R}$ with only nondegenerate critical points. Given a Riemannian metric on M, one studies the flow of the gradient vector field ∇f. In particular a flow line is a curve $\gamma : \mathbb{R} \to M$ satisfying the ordinary differential equation,

$$\frac{d}{dt}\gamma(s) + \nabla f(\gamma(s)) = 0.$$

By the existence and uniqueness theorem for solutions to ODEs, one knows that if $x \in M$ is any point then there is a unique flow line γ_x satisfying $\gamma_x(0) = x$. One then studies unstable and stable manifolds of the critical points,

$$W^u(a) = \{x \in M : \lim_{t\to-\infty} \gamma_x(t) = a\}$$
$$W^s(a) = \{x \in M : \lim_{t\to+\infty} \gamma_x(t) = a\}.$$

The unstable manifold $W^u(a)$ is diffeomorphic to a disk $D^{\mu(a)}$, where $\mu(a)$ is the index of the critical point a. Similarly the stable manifold $W^s(a)$ is diffeomorphic to a disk $D^{n-\mu(a)}$.

For a generic choice of Riemannian metric, the unstable manifolds and stable manifolds intersect transversally, and their intersections,

$$W(a, b) = W^u(a) \cap W^s(b),$$

are smooth manifolds of dimension equal to the relative index, $\mu(a) - \mu(b)$. When the choice of metric satisfies these transversality properties, the metric is said to be

"Morse–Smale". The manifolds $W(a,b)$ have free $\mathbb{R}$-actions defined by "going with the flow". That is, for $t \in \mathbb{R}$, and $x \in M$,

$$t \cdot x = \gamma_x(t).$$

The "moduli space of flow lines" is the manifold

$$\mathcal{M}(a,b) = W(a,b)/\mathbb{R}$$

and has dimension $\mu(a)-\mu(b)-1$. These moduli spaces are not generally compact, but they have canonical compactifications in the following way.

In the case of a Morse–Smale metric, (which we assume throughout the rest of this section), there is a partial order on the finite set of critical points given by $a \geq b$ if $\mathcal{M}(a,b) \neq \emptyset$. We then define

$$\bar{\mathcal{M}}(a,b) = \bigcup_{a=a_1>a_2>\cdots>a_k=b} \mathcal{M}(a_1,a_2) \times \cdots \times \mathcal{M}(a_{k-1},a_k), \tag{1}$$

The topology of $\bar{\mathcal{M}}(a,b)$ can be described as follows. Since the Morse function $f : M \to \mathbb{R}$ is strictly decreasing along flow lines, a flow $\gamma : \mathbb{R} \to M$ with the property that $\gamma(0) \in W(a,b)$ determines a diffeomorphism $\mathbb{R} \cong (f(b), f(a))$ given by the composition,

$$\mathbb{R} \xrightarrow{\gamma} M \xrightarrow{f} \mathbb{R}.$$

This defines a parameterization of any $\gamma \in \mathcal{M}(a,b)$ as a map

$$\gamma : [f(b), f(a)] \to M$$

that satisfies the differential equation

$$\frac{d\gamma}{ds} = \frac{\nabla f(\gamma(s))}{|f(\gamma(s))|^2}, \tag{2}$$

as well as the boundary conditions

$$\gamma(f(b)) = b \quad \text{and} \quad \gamma(f(a)) = a. \tag{3}$$

From this viewpoint, the *compactification* $\bar{\mathcal{M}}(a,b)$ can be described as the space of all continuous maps $[f(b), f(a)] \to M$ that are piecewise smooth (and indeed smooth off of the critical values of f that lie between $f(b)$ and $f(a)$), and that satisfy the differential equation (2) subject to the boundary conditions (3). It is topologized with the compact-open topology.

These moduli spaces have natural framings on their stable normal bundles (or equivalently, their stable tangent bundles) in the following manner. Let $a > b$ be critical points. Let $\epsilon > 0$ be chosen so that there are no critical values in the half open interval $[f(a)-\epsilon, f(a))$. Define the *unstable sphere* to be the level set of the stable manifold,

$$S^u(a) = W^u(a) \cap f^{-1}(f(a)-\epsilon).$$

The sphere $S^u(a)$ has dimension $\mu(a)-1$. Notice there is a natural diffeomorphism,

$$\mathcal{M}(a,b) \cong S^u(a) \cap W^s(b).$$

This leads to the following diagram,

$$\begin{tikzcd} W^s(b) \arrow[r, "\hookrightarrow"] & M \\ \mathcal{M}(a,b) \arrow[u, "\cup"] \arrow[r, "\hookrightarrow"'] & S^u(a). \arrow[u, "\cup"'] \end{tikzcd} \tag{4}$$

From this diagram one sees that the normal bundle ν of the embedding $\mathcal{M}(a,b) \hookrightarrow S^u(a)$ is the restriction of the normal bundle of $W^s(b) \hookrightarrow M$. Since $W^s(b)$ is a disk, and therefore contractible, this bundle is trivial. Indeed an orientation of $W^s(b)$ determines a homotopy class of trivialization, or a framing. In fact this framing determines a diffeomorphism of the bundle to the product, $W^s(b) \times W^u(b)$. Thus these orientations give the moduli spaces $\mathcal{M}(a,b)$ canonical normal framings, $\nu \cong \mathcal{M}(a,b) \times W^u(b)$.

In the case when a and b are *successive* critical points, that is when there are no intermediate critical points c with $a > c > b$, then (1) tells us that $\mathcal{M}(a,b)$ is already compact. This means that $\mathcal{M}(a,b)$ is a closed, framed manifold, and its framed cobordism class is represented by the composition,

$$\tau_{a,b} : S^u(a) \xrightarrow{\tau} \mathcal{M}^\nu(a,b) \simeq \mathcal{M}(a,b)_+ \wedge (W^u(b) \cup \infty) \xrightarrow{proj} W^u(b) \cup \infty$$

Here $\tau : S^u(a) \to \mathcal{M}^\nu(a,b)$ is the "Thom collapse map" from the ambient sphere to the Thom space of the normal bundle. This map between spheres was shown by Franks in [11] to be the relative attaching map in the CW-structure of the complex X_f.

Notice that in the special case when $\mu(a) = \mu(b)+1$ the spheres $S^u(a)$ and $W^u(b) \cup \infty$ have the same dimension $(= \mu(b))$, and so the homotopy class of $\tau_{a,b}$ is given by its degree; by standard considerations this is the "count" $\#\mathcal{M}(a,b)$ of the number of points in the compact, oriented (framed) zero-manifold $\mathcal{M}(a,b)$ (counted with sign). Thus the "Morse–Smale chain complex", (C_*^f, ∂), which is the cellular chain complex of the CW-complex X_f, is given as follows:

$$\to \cdots \to C_i^f \xrightarrow{\partial_i} C_{i-1}^f \xrightarrow{\partial_{i-1}} \cdots \to C_0^f \tag{5}$$

where C_i^f is the free abelian group generated by the critical points of f of index i, and

$$\partial_i[a] = \sum_{\mu(b)=i-1} \#\mathcal{M}(a,b)\,[b]. \tag{6}$$

The framings on the higher dimensional moduli spaces, $\mathcal{M}(a,b)$, extend to their compactifications, $\bar{\mathcal{M}}(a,b)$, and they have the structure of compact, framed manifolds with corners. In [5] it was shown that these framed manifolds define the stable attaching maps of the CW-complex X_f, thus generalizing Franks' theorem.

The moduli spaces $\bar{\mathcal{M}}(a,b)$ fit together to define the *flow category*, $\mathcal{C}_f$, of the Morse function $f : M \to \mathbb{R}$. The objects of this category are the critical points of f and the morphisms are given by the spaces $\bar{\mathcal{M}}(a,b)$. The compositions are the inclusions into the boundary,

$$\bar{\mathcal{M}}(a,b) \times \bar{\mathcal{M}}(b,c) \hookrightarrow \bar{\mathcal{M}}(a,c).$$

What the above observations describe, is the structure on the flow category $\mathcal{C}_f$ of a *smooth, compact, Morse–Smale category* as defined in [5] and [4]:

Definition 3. A smooth, compact category is a topological category $\mathcal{C}$ whose objects form a discrete set, and whose morphism spaces, $Mor(a,b)$ are compact, smooth manifolds with corners, such that the composition maps, $\nu : Mor(a,b) \times Mor(b,c) \to Mor(a,c)$, are smooth codimension one embeddings (of manifolds with corners) whose images lie in the boundary.

A smooth, compact category $\mathcal{C}$ is said to be a "Morse–Smale" category if the following additional properties are satisfied:

1. The objects of $\mathcal{C}$ are partially ordered by the condition

$$a \geq b \quad \text{if} \quad Mor(a,b) \neq \emptyset.$$

2. $Mor(a,a) = \{\mathit{identity}\}$.
3. There is a set map, $\mu : Ob(\mathcal{C}) \to \mathbb{Z}$, which preserves the partial ordering, such that if $a > b$,

$$dim\, Mor(a,b) = \mu(a) - \mu(b) - 1.$$

The map μ is known as an "index" map. A Morse–Smale category such as this is said to have finite type, if there are only finitely many objects of any given index, and for each pair of objects $a > b$, there are only finitely many objects c with $a > c > b$.

As was described in [5] and [4], the Morse framings of the moduli spaces $\mathcal{M}(a,b)$ are compatible with the composition structure in the flow category $\mathcal{C}_f$ which gives the category the structure of a "smooth, compact, framed" category. We will recall this notion in Sect. 4 when we generalize it to the notion of a "smooth, compact, E^*-oriented" category, for E^* a multiplicative generalized cohomology theory. It was also shown in [5] that the framing on this category was precisely what was needed to (functorially) realize the Morse chain complex (5) by a stable homotopy type; which in this case was shown to be the suspension spectrum of the manifold, $\Sigma^\infty(M_+)$. It was shown also that if the framing was changed one produced a change in realization, typically to the Thom spectrum of a virtual bundle, M^ζ.

2.2 Smooth Floer Theories

In Floer theory, many, although not all of the above constructions go through. In such a theory, one typically starts with a smooth map

$$\mathcal{A} : \mathcal{L} \to \mathbb{R}$$

where often $\mathcal{L}$ is an infinite dimensional manifold, and the functional $\mathcal{A}$ is defined from geometric considerations. For example, in symplectic Floer theory $\mathcal{L}$ is the loop space of a symplectic manifold and $\mathcal{A}$ is the "symplectic action" of a loop. In "instanton" Floer theory, $\mathcal{L}$ is the space of gauge equivalence classes of $SU(n)$-connections on the trivial bundle over a fixed three-manifold and $\mathcal{A}$ is the Chern–Simons functional.

Oftentimes the functional $\mathcal{A}$ is then perturbed so that the critical points are isolated and nondegenerate. Also, a choice of metric on $\mathcal{L}$ is typically shown to exist so that the Morse–Smale transversality conditions hold. So even though it is often the case that the unstable and stable manifolds, $W^u(a)$ and $W^s(a)$ are infinite dimensional, in such a Floer theory the intersections

$$W(a, b) = W^u(a) \cap W^s(b)$$

are finite dimensional smooth manifolds. Thus even when one cannot make sense of the index, $\mu(a)$, one *can* make sense of the relative index,

$$\mu(a, b) = dim\, W(a, b).$$

One can then form the moduli spaces, $\mathcal{M}(a, b) = W(a, b)/\mathbb{R}$, with dimension $\mu(a, b) - 1$. One can also form the space of "piecewise flows", $\bar{\mathcal{M}}(a, b)$, using the same method as above (1). However in general these space *may not be compact*. This is typically due to bubbling phenomena. Moreover, the smoothness of $\bar{\mathcal{M}}(a, b)$ may be difficult to establish. In general this is a deep analytic question, which has been addressed in particular examples throughout the literature (e.g. [1, 7, 9, 20]). Furthermore a general analytical theory of "polyfolds" is being developed by Hofer (see for example [13]) to deal with these types of questions in general. However in Floer theory one generally knows that when the relative index $\mu(a, b)$ is 1, then the spaces $\mathcal{M}(a, b)$ are compact, oriented, zero dimensional manifolds. By picking a "base" critical point, say a_0, one can then can form a "Floer chain complex" (relative to a_0),

$$\to \cdots \to C_i^{\mathcal{A}} \xrightarrow{\partial_i} C_{i-1}^{\mathcal{A}} \xrightarrow{\partial_{i-1}} \cdots \to C_0^{\mathcal{A}} \tag{7}$$

where $C_i^{\mathcal{A}}$ is the free abelian group generated by the critical points a of $\mathcal{A}$ of index $\mu(a, a_0) = i$, and

$$\partial_i[a] = \sum_{\mu(b)=i-1} \#\mathcal{M}(a,b)\,[b]. \tag{8}$$

By allowing the choice of base critical point a_0 to vary one obtains an inverse system of Floer chain complexes as in [5]. Another way of handling the lack of well defined index is to use coefficients in a Novikov ring as in [18].

The specific question addressed in [5] is the realizability of the Floer complex by a stable homotopy type: a spectrum in the case of a fixed connective chain complex as (7), or a prospectrum in the case of an inverse system of such chain complexes. In this note we ask the following generalization of this question. Given a commutative ring spectrum E, when does the Floer complex $C_*^{\mathcal{A}} \otimes E_*$ have a realization by an E-module spectrum as in Definition 1 in the introduction?

Ultimately the criteria for such realizations rests on the properties of the flow category of the Floer functional, $\mathcal{C}_{\mathcal{A}}$. This is defined as in the Morse theory case, where the objects are critical points and the morphisms are the spaces of piecewise flow lines, $\bar{\mathcal{M}}(a,b)$. The first criterion is that these moduli spaces can be given be given the structure of smooth, compact, manifolds with corners. As mentioned above, this involves the analytic issues of transversality, compactness, and gluing:

Definition 4. We say that a Floer functional $\mathcal{A} : \mathcal{L} \to \mathbb{R}$ generates a "smooth Floer theory" if the flow category $\mathcal{C}_{\mathcal{A}}$ has the structure of a smooth, compact, Morse–Smale category of finite type, as in Definition 3.

Remark. Notice that in this definition we are assuming that a global notion of index can be defined, as opposed to only the relative index with respect to a base critical point. If this is not the case, one can often replace the flow category with an inverse system of categories, each of which has a global notion of index. See [5] for a more thorough discussion of this point.

Examples of "smooth Floer theories" were given in [5] and [4]. In particular, the symplectic Floer theory of the cotangent bundle T^*M was shown to have this property in [4].

In the next section we establish a general functorial description of realizations of chain complexes by E-module spectra, and in Sect. 4 we interpret these criteria geometrically, in terms of the flow category $\mathcal{C}_{\mathcal{A}}$ of a smooth Floer theory.

3 Realizing Chain Complexes by E-module Spectra

Suppose one is given a chain complex of finitely generated free abelian groups,

$$\to \cdots C_i \xrightarrow{\partial_i} C_{i-1} \to \cdots \to C_0 \tag{9}$$

where C_i has a given basis, $\mathcal{B}_i$. In the examples of a smooth Floer theory and Morse theory, $\mathcal{B}_i$ represents the set of critical points of index i. In the category of spectra, the wedge $Z_i = \bigvee_{\beta \in \mathcal{B}_i} S^0$, where S^o is the sphere spectrum, realizes the chains in the sense that

$$H_*(Z_i) \cong C_i.$$

Equivalently, if H represents the integral Eilenberg–MacLane spectrum, the wedge $\mathcal{H}_i = \bigvee_{\beta \in \mathcal{B}_i} H$ has the property that $\pi_*(\mathcal{H}_i) \cong C_i$. More generally, given a commutative ring spectrum E, the wedge $\mathcal{E}_i = \bigvee_{\beta \in \mathcal{B}_i} E$ is a free E-module spectrum with the property that its homotopy groups,

$$\pi_*(\mathcal{E}_i) \cong C_i \otimes E_*.$$

The key in the realization of the chain complex $(C_* \otimes E_*, \partial \otimes 1)$ by an E-module spectrum is to understand the role of the attaching maps. To do this, we "work backwards", in the sense that we study the attaching maps in an arbitrary E-module spectrum. For ease of exposition, we consider a finite E-module spectrum. That is, we assume X is an E-module that has a filtration by E-spectra,

$$X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_n = X,$$

where each $X_{i-1} \hookrightarrow X_i$ is a cofibration with cofiber, $K_i = X_i \cup c(X_{i-1})$, a free E-module in that there is an equivalence

$$K_i \simeq \bigvee_{\mathcal{D}_i} \Sigma^i E$$

where $\mathcal{D}_i$ is a finite indexing set.

Following the ideas of [5], then one can "rebuild" the homotopy type of the n-fold suspension, $\Sigma^n X$, as the union of iterated cones and suspensions of the K_i's,

$$\Sigma^n X \simeq \Sigma^n K_0 \cup c(\Sigma^{n-1} K_1) \cup \cdots \cup c^i(\Sigma^{n-i} K_i) \cup \cdots \cup c^n K_n. \tag{10}$$

This decomposition can be described as follows. Define a map $\delta_i : \Sigma^{n-i} K_i \to \Sigma^{n-i+1} K_{i-1}$ to be the iterated suspension of the composition,

$$\delta_i : K_i \to \Sigma K_{i-1} \to \Sigma K_{i-1}$$

where the two maps in this composition come from the cofibration sequence, $X_{i-1} \to X_i \to K_i \to \Sigma X_{i-1} \cdots$. Notice that δ_i is a map of E-module spectra. As was pointed out in [4], this induces a "homotopy chain complex",

$$K_n \xrightarrow{\delta_n} \Sigma K_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_{i+1}} \Sigma^{n-i} K_i \xrightarrow{\delta_i} \Sigma^{n-i+1} K_{i-1} \xrightarrow{\delta_{i-1}} \cdots \xrightarrow{\delta_1} \Sigma^n K_0 = \Sigma^n X_0. \tag{11}$$

We refer to this as a homotopy chain complex because examination of the defining cofibrations leads to canonical null homotopies of the compositions,

$$\delta_j \circ \delta_{j+1}.$$

This canonical null homotopy defines an extension of δ_j to the mapping cone of δ_{j+1}:

$$c(\Sigma^{n-j-1}K_{j+1}) \cup_{\delta_{j+1}} \Sigma^{n-j}K_j \longrightarrow \Sigma^{n-j+1}K_{j-1}.$$

More generally, for every q, using these null homotopies, we have an extension to the iterated mapping cone,

$$\begin{aligned} &c^q(\Sigma^{n-j-q}K_{j+q}) \cup c^{q-1}(\Sigma^{n-j-q+1}K_{j+q-1}) \cup \cdots \\ &\quad \cup c(\Sigma^{n-j-1}K_{j+1}) \cup_{\delta_{j+1}} \Sigma^{n-j}K_j \longrightarrow \Sigma^{n-j+1}K_{j-1}. \end{aligned} \tag{12}$$

In other words, for each $p > q$, these null homotopies define a map of free E-spectra,

$$\phi_{p,q} : c^{p-q-1}\Sigma^{n-p}K_p \to \Sigma^{n-q}K_q. \tag{13}$$

To keep track of the combinatorics of these attaching maps, a category $\mathcal{J}$ was introduced in [5] and used again in [4]. The objects of $\mathcal{J}$ are the integers, $\mathbb{Z}$. To describe the morphisms, we first introduced spaces $J(n,m)$ for any pair of integers $n > m$. These spaces were defined by:

$$\begin{aligned} J(n,m) = \{(t_i,\, i \in \mathbb{Z}), \quad &\text{where each } t_i \text{ is a nonnegative real number, and } t_i = 0, \\ &\text{unless } m < i < n.\} \end{aligned} \tag{14}$$

Notice that $J(n,m) \cong \mathbb{R}_+^{n-m-1}$, where $\mathbb{R}_+^q$ is the space of q-tuples of nonnegative real numbers. Furthermore there are natural inclusions,

$$\iota : J(n,m) \times J(m,p) \hookrightarrow J(n,p).$$

The image of ι consists of those sequences in $J(n,p)$ which have a 0 in the mth coordinate. This then allows the definition of the morphisms in $\mathcal{J}$ as follows. For integers $n < m$ there are no morphisms from n to m. The only morphism from an integer n to itself is the identity. If $n = m+1$, the space of morphisms is defined to be the two point space, $Mor(m+1,m) = S^0$. If $n > m+1$, $Mor(n,m)$ is the one point compactification,

$$Mor(n,m) = J(n,m)^+ = J(n,m) \cup \infty.$$

For consistency of notation we refer to all the morphism spaces $Mor(n,m)$ as $J(n,m)^+$. Composition in the category is given by addition of sequences,

$$J(n,m)^+ \wedge J(m,p)^+ \to J(n,p)^+.$$

In this smash product, the points at infinity are the basepoints, and this map is basepoint preserving. A key feature of this category is that for a based space or spectrum, Y, the smash product $J(n,m)^+ \wedge Y$ is homeomorphic iterated cone,

$$J(n,m)^+ \wedge Y \cong c^{n-m-1}(Y).$$

Given integers $p > q$, then there are subcategories $\mathcal{J}_q^p$ defined to be the full subcategory generated by integers $q \geq m \geq p$. The category $\mathcal{J}_q$ is the full subcategory of $\mathcal{J}$ generated by all integers $m \geq q$.

As described in [5], given a functor to the category of spaces, $Z : \mathcal{J}_q \to Spaces_*$, one can take its geometric realization,

$$|Z| = \coprod_{q \leq j} Z(j) \wedge J(j,q-1)^+ / \sim \tag{15}$$

where one identifies the image of $Z(j) \wedge J(j,i)^+ \wedge J(i,q-1)^+$ in $Z(j) \wedge J(j,q-1)^+$ with its image in $Z(i) \wedge J(i,q-1)$ under the map on morphisms.

For a functor whose value is in E-modules, we replace the above construction of the geometric realization $|Z|$ by a coequalizer, in the following way:

Let $Z : \mathcal{J}_q \to E - mod$. Define two maps of E-modules,

$$\iota, \mu : \bigvee_{q \leq j} Z(j) \wedge J(j,i)^+ \wedge J(i,q-1)^+ \longrightarrow \bigvee_{q \leq j} Z(j) \wedge J(j,q-1)^+. \tag{16}$$

The first map ι is induced by the composition of morphisms in $\mathcal{J}_q$, $J(j,i)^+ \wedge J(i,q-1)^+ \hookrightarrow J(j,q-1)^+$. The second map μ is the given by the wedge of maps,

$$Z(j) \wedge J(j,i)^+ \wedge J(i,q-1)^+ \xrightarrow{\mu_q \wedge 1} Z(i) \wedge J(i,q-1)^+$$

where $\mu_q : Z(j) \wedge J(j,i)^+ \to Z(i)$ is the action of the morphisms.

Definition 5. Given a functor $Z : \mathcal{J}_q \to E - mod$ we define its geometric realization to be the homotopy coequalizer (in the category $E - mod$) of the two maps,

$$\iota, \mu : \bigvee_{q \leq j} Z(j) \wedge J(j,i)^+ \wedge J(i,q-1)^+ \longrightarrow \bigvee_{q \leq j} Z(j) \wedge J(j,q-1)^+.$$

The following is a more precise version of Theorem 2 of the introduction.

Theorem 6. *Let E be a commutative ring spectrum, and let*

$$\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_i \xrightarrow{\partial_i} C_{i-1} \xrightarrow{\partial_{i-1}} \cdots \xrightarrow{\partial_1} C_0$$

be a chain complex of finitely generated free abelian groups where C_i has basis $\mathcal{B}_i$. Then each realization of $C_* \otimes E_*$ by an E-module spectrum X, occurs as the geometric realization $|Z_X|$ of a functor

$$Z_X : \mathcal{J}_0 \to E - mod$$

with the following properties:

1. $Z_X(i) = \bigvee_{\mathcal{B}_i} E.$
2. *On morphisms of the form* $J(i, i-1)^+ = S^0$*, the functor* $Z_X : \bigvee_{\mathcal{B}_i} E \to \bigvee_{\mathcal{B}_{i-1}} E$ *induces the boundary homomorphism on homotopy groups:*

$$\partial_i : C_i \otimes E_* \to C_{i-1} \otimes E_*.$$

Proof. The proof mirrors the arguments given in [5] and [4] which concern realizing chain complexes of abelian groups by stable homotopy types. We therefore present a sketch of the constructions and indicate the modifications needed to prove this theorem.

First suppose that $Z : \mathcal{J}_0 \to E - mod$ is a functor satisfying the properties (1) and (2) as stated in the theorem. The geometric realization $|Z|$ has a natural filtration by E-module spectra,

$$|Z|_0 \hookrightarrow |Z|_1 \hookrightarrow \cdots \hookrightarrow |Z|_{k-1} \hookrightarrow |Z|_k \hookrightarrow \cdots |Z|$$

where the kth-filtration, $|Z|_k$, is the homotopy coequalizer of the maps ι and μ as in Definition 5, but where the wedges only involve the first k-terms. Notice that the subquotients (homotopy cofibers) are homotopy equivalent to a wedge,

$$|Z|_k/|Z|_{k-1} \simeq \bigvee_{\mathcal{B}_k} \Sigma^k E \simeq \Sigma^k Z(i).$$

The second equivalence holds by property (1) in the theorem. Furthermore, by property 2 the composition defining the attaching map,

$$\begin{aligned} C_k \otimes E_* &\cong \pi_{*+k}(|Z|_k/|Z|_{k-1}) \to \pi_{*+k}(\Sigma|Z|_{k-1}) \to \pi_{*+k}(\Sigma(|Z|_{k-1}/|Z|_{k-2}) \\ &\cong C_{k-1} \otimes E_* \end{aligned}$$

is the boundary homomorphism, $\partial_k \otimes 1$. Thus the geometric realization of such a functor $Z : \mathcal{J}_0 \to E - mod$ gives a realization of the chain complex $C_* \otimes E_*$ according to Definition 1.

Conversely, suppose that the filtered $E - module$

$$X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_k \hookrightarrow \cdots X$$

realizes $C_* \otimes E_*$ as in Definition 1. We define a functor

$$Z_X : \mathcal{J}_0 \to E - mod$$

by letting $Z_X(i) = \bigvee_{\mathcal{B}_i} E = \Sigma^{-i} X_i/X_{i-1}$. To define the value of Z_X on morphisms, we need to define compatible maps of $E - modules$,

$$J(i,j)^+ \wedge Z_X(i) \to Z_X(j)$$
$$c^{i-j-1}\Sigma^{-i} X_i/X_{i-1} \to \Sigma^{-j} X_j/X_{j-1}$$

These are defined to be the maps $\phi_{i,j}$ given by the attaching maps in a filtered E-module as described above in (13). □

4 Manifolds with Corners, E^*-orientations of Flow Categories, and Floer E_*-homology

Our goal in this section is to apply Theorem 6 to a Floer chain complex in order to determine when a smooth Floer theory (Definition 4) can be realized by an E-module to produce a "Floer E_*-homology theory". To do this we want to identify the appropriate properties of the moduli spaces $\bar{\mathcal{M}}(a, b)$ defining the morphisms in the flow category $\mathcal{C}_{\mathcal{A}}$ so that they induce a functor $Z_{\mathcal{A}} : \mathcal{J}_0 \to E - mod$ that satisfies Theorem 6. The basic structure these moduli spaces need to possess is an E^*-orientation on their stable normal bundles. The functor will then be defined via a Pontrjagin–Thom construction.

These moduli spaces have a natural corner structure and these E^*-orientations must be compatible with this corner structure. Therefore we begin this section with a general discussion of manifolds with corners, their embeddings, and induced normal structures. This general discussion follows that of [15].

Recall that an n-dimensional manifold with corners, M, has charts which are local homeomorphisms with $\mathbb{R}^n_+$. Let $\psi : U \to \mathbb{R}^n_+$ be a chart of a manifold with corners M. For $x \in U$, the number of zeros of this chart, $c(x)$ is independent of the chart. We define a *face* of M to be a connected component of the space $\{m \in M \text{ such that } c(m) = 1\}$.

Given an integer k, there is a notion of a manifold with corners having "codimension k", or a $\langle k \rangle$-manifold. We recall the definition from [15].

Definition 7. A $\langle k \rangle$-manifold is a manifold with corners, M, together with an ordered k-tuple $(\partial_1 M, \ldots, \partial_k M)$ of unions of faces of M satisfying the following properties:

1. Each $m \in M$ belongs to $c(m)$ faces.
2. $\partial_1 M \cup \cdots \cup \partial_k M = \partial M$.
3. For all $1 \le i \ne j \le k$, $\partial_i M \cap \partial_j M$ is a face of both $\partial_i M$ and $\partial_j M$.

The archetypical example of a $\langle k \rangle$-manifold is $\mathbb{R}^k_+$. In this case the face $F_j \subset \mathbb{R}^k_+$ consists of those k-tuples with the jth-coordinate equal to zero.

As described in [15], the data of a $\langle k\rangle$-manifold can be encoded in a categorical way as follows. Let $\underline{2}$ be the partially ordered set with two objects, $\{0,1\}$, generated by a single nonidentity morphism $0 \to 1$. Let $\underline{2}^k$ be the product of k-copies of the category $\underline{2}$. A $\langle k\rangle$-manifold M then defines a functor from $\underline{2}^k$ to the category of topological spaces, where for an object $a = (a_1, \cdots, a_k) \in \underline{2}^k$, $M(a)$ is the intersection of the faces $\partial_i(M)$ with $a_i = 0$. Such a functor is a k-dimensional cubical diagram of spaces, which, following Laures' terminology, we refer to as a $\langle k\rangle$-diagram, or a $\langle k\rangle$-space. Notice that $\mathbb{R}^k_+(a) \subset \mathbb{R}^k_+$ consists of those k-tuples of nonnegative real numbers so that the ith-coordinate is zero for every i with $a_i = 0$. More generally, consider the $\langle k\rangle$-Euclidean space, $\mathbb{R}^k_+ \times \mathbb{R}^n$, where the value on $a \in \underline{2}^k$ is $\mathbb{R}^k_+(a) \times \mathbb{R}^n$. In general we refer to a functor $\phi : \underline{2}^k \to \mathcal{C}$ as a $\langle k\rangle$-object in the category $\mathcal{C}$.

In this section we will consider embeddings of manifolds with corners into Euclidean spaces $M \hookrightarrow \mathbb{R}^k_+ \times \mathbb{R}^n$ of the form given by the following definition.

Definition 8. A "neat embedding" of a $\langle k\rangle$-manifold M into $\mathbb{R}^k_+ \times \mathbb{R}^m$ is a natural transformation of $\langle k\rangle$-diagrams

$$e : M \hookrightarrow \mathbb{R}^k_+ \times \mathbb{R}^m$$

that satisfies the following properties:

1. For each $a \in \underline{2}^k$, $e(a)$ is an embedding.
2. For all $b < a$, the intersection $M(a) \cap \left(\mathbb{R}^k_+(b) \times \mathbb{R}^m\right) = M(b)$, and this intersection is perpendicular. That is, there is some $\epsilon > 0$ such that

$$M(a) \cap \left(\mathbb{R}^k_+(b) \times [0,\epsilon)^k(a-b) \times \mathbb{R}^m\right) = M(b) \times [0,\epsilon)^k(a-b).$$

Here $a - b$ denotes the object of $\underline{2}^k$ obtained by subtracting the k-vector b from the k-vector a.

In [15] it was proved that every $\langle k\rangle$-manifold neatly embeds in $\mathbb{R}^k_+ \times \mathbb{R}^N$ for N sufficiently large. In fact it was proved there that a manifold with corners, M, admits a neat embedding into $\mathbb{R}^k_+ \times \mathbb{R}^N$ *if and only if* M has the structure of a $\langle k\rangle$-manifold. Furthermore in [12] it is shown that the connectivity of the space of neat embeddings, $Emb_{\langle k\rangle}(M; \mathbb{R}^k_+ \times \mathbb{R}^N)$ increases with the dimension N.

Notice that an embedding of manifolds with corners, $e : M \hookrightarrow \mathbb{R}^k_+ \times \mathbb{R}^m$, has a well defined normal bundle. In particular, for any pair of objects in $\underline{2}^k$, $a > b$, the normal bundle of $e(a) : M(a) \hookrightarrow \mathbb{R}^k_+(a) \times \mathbb{R}^m$, when restricted to $M(b)$, is the normal bundle of $e(b) : M(b) \hookrightarrow \mathbb{R}^k_+(b) \times \mathbb{R}^m$.

Now let $\mathcal{C}$ be the flow category of a smooth Floer theory. Recall from Definition 4 that this means that $\mathcal{C}$ is a smooth, compact, Morse–Smale category of finite type. By assumption, the morphism spaces, $Mor(a,b) = \bar{\mathcal{M}}(a,b)$, are compact, smooth, manifolds with corners. Notice that in fact they are $\langle k(a,b)\rangle$-manifolds, where $k(a,b) = \mu(a) - \mu(b) - 1$. This structure is given by the faces

$$\partial_j(\bar{\mathcal{M}}(a,b)) = \bigcup_{\mu(c)=\mu(a)-j} \bar{\mathcal{M}}(a,c) \times \bar{\mathcal{M}}(c,b).$$

These faces clearly satisfy the intersection property necessary for being a $\langle k(a,b)\rangle$-manifold (Definition 7). The condition on the flow category $\mathcal{C}$ necessary for it to induce a geometric realization of the Floer complex by E-modules will be that the moduli spaces, $\bar{\mathcal{M}}(a,b)$, admit a compatible family of E^*-orientations. To define this notion carefully, we first observe that a commutative ring spectrum E induces a $\langle k\rangle$-diagram in the category of spectra ("$\langle k\rangle$-spectrum"), $E\langle k\rangle$, defined in the following manner.

For $k = 1$, we let $E\langle 1\rangle : \underline{2} \to Spectra$ be defined by $E\langle 1\rangle(0) = S^0$, the sphere spectrum, and $E\langle 1\rangle(1) = E$. The image of the morphism $0 \to 1$ is the unit of the ring spectrum $S^0 \to E$.

To define $E\langle k\rangle$ for general k, let a be an object of $\underline{2}^k$. We view a as a vector of length k, whose coordinates are either zero or one. Define $E\langle k\rangle(a)$ to be the multiple smash product of spectra, with a copy of S^0 for every zero coordinate, and a copy of E for every string of successive ones. For example, if $k = 6$, and $a = (1,0,1,1,0,1)$, then $E\langle k\rangle(a) = E \wedge S^0 \wedge E \wedge S^0 \wedge E$.

Given a morphism $a \to a'$ in $\underline{2}^k$, one has a map $E\langle k\rangle(a) \to E\langle k\rangle(a')$ defined by combining the unit $S^0 \to E$ with the ring multiplication $E \wedge E \to E$.

Said another way, the functor $E\langle k\rangle : \underline{2}^k \to Spectra$ is defined by taking the k-fold product functor $E\langle 1\rangle : \underline{2} \to Spectra$ which sends $(0 \to 1)$ to $S^0 \to E$, and then using the ring multiplication in E to "collapse" successive strings of Es.

This structure allows us to define one more construction. Suppose $\mathcal{C}$ is a smooth, compact, Morse–Smale category of finite type as in Definition 3. We can then define an associated category, $E_{\mathcal{C}}$, whose objects are the same as the objects of $\mathcal{C}$ and whose morphisms are given by the spectra,

$$Mor_{E_{\mathcal{C}}}(a,b) = E\langle k(a,b)\rangle$$

where $k(a,b) = \mu(a) - \mu(b) - 1$. Here $\mu(a)$ is the index of the object a as in Definition 3. The composition law is the pairing,

$$\begin{aligned} E\langle k(a,b)\rangle \wedge E\langle k(b,c)\rangle &= E\langle k(a,b)\rangle \wedge S^0 \wedge E\langle k(b,c)\rangle \\ &\xrightarrow{1\wedge u\wedge 1} E\langle k(a,b)\rangle \wedge E\langle 1\rangle \wedge E\langle k(b,c)\rangle \\ &\xrightarrow{\mu} E\langle k(a,c)\rangle. \end{aligned}$$

Here $u : S^0 \to E = E\langle 1\rangle$ is the unit. This category encodes the multiplication in the ring spectrum E.

We now make precise what it means to say that the moduli spaces in a Floer flow category, $\bar{\mathcal{M}}(a,b)$, or more generally, the morphisms in a smooth, compact, Morse–Smale category, have an E^*-orientation.

Let M be a $\langle k\rangle$-manifold, and let $e : M \hookrightarrow \mathbb{R}_+^k \times \mathbb{R}^N$ be a neat embedding. The Thom space, $Th(M,e)$, has the structure of an $\langle k\rangle$-space, where for $a \in \underline{2}^k$,

$Th(M,e)(a)$ is the Thom space of the normal bundle of the associated embedding, $M(a) \hookrightarrow \mathbb{R}^k_+(a) \times \mathbb{R}^N$. We can then desuspend and define the Thom spectrum, $M^\nu_e = \Sigma^{-N} Th(M,e)$, to be the associated $\langle k \rangle$-spectrum. The Pontrjagin–Thom construction defines a map of $\langle k \rangle$-spaces,

$$\tau_e : \left(\mathbb{R}^k_+ \times \mathbb{R}^N\right) \cup \infty = ((\mathbb{R}^k_+) \cup \infty) \wedge S^N \to Th(M,e).$$

Desuspending we get a map of $\langle k \rangle$-spectra, $\Sigma^\infty((\mathbb{R}^k_+) \cup \infty) \to M^\nu_e$. Notice that the homotopy type (as $\langle k \rangle$-spectra) of M^ν_e is independent of the embedding e. We denote the homotopy type of this normal Thom spectrum as M^ν, and the homotopy type of the Pontrjagin–Thom map, $\tau : \Sigma^\infty((\mathbb{R}^k_+) \cup \infty) \to M^\nu$.

We define an E^*-normal orientation to be a cohomology class (Thom class) represented by a map $u : M^\nu \to E$ such that cup product defines an isomorphism,

$$\cup u : E^*(M) \xrightarrow{\cong} E^*(M^\nu).$$

In order to solve the realization problem, we need to have coherent E^*-orientations of all the moduli spaces, $\bar{\mathcal{M}}(a,b)$, making up a smooth Floer theory. In order to make this precise, we make the following definition.

Definition 9. Let $\mathcal{C}$ be the flow category of a smooth Floer theory. Then a "normal Thom spectrum" of the category $\mathcal{C}$ is a category, $\mathcal{C}^\nu$, enriched over spectra, that satisfies the following properties:

1. The objects of $\mathcal{C}^\nu$ are the same as the objects of $\mathcal{C}$.
2. The morphism spectra $Mor_{\mathcal{C}^\nu}(a,b)$ are $\langle k(a,b) \rangle$-spectra, having the homotopy type of the normal Thom spectra $\bar{\mathcal{M}}(a,b)^\nu$, as $\langle k(a,b) \rangle$-spectra. The composition maps,

$$\circ : Mor_{\mathcal{C}^\nu}(a,b) \wedge Mor_{\mathcal{C}^\nu}(b,c) \to Mor_{\mathcal{C}^\nu}(a,c)$$

have the homotopy type of the maps,

$$\bar{\mathcal{M}}(a,b)^\nu \wedge \bar{\mathcal{M}}(b,c)^\nu \to \bar{\mathcal{M}}(a,c)^\nu$$

of the normal bundles corresponding to the composition maps in $\mathcal{C}$, $\bar{\mathcal{M}}(a,b) \times \bar{\mathcal{M}}(b,c) \to \bar{\mathcal{M}}(a,c)$. Recall that these are maps of $\langle k(a,c) \rangle$-spaces induced by the inclusion of a component of the boundary.
3. The morphism spectra are equipped with maps $\tau_{a,b} : \Sigma^\infty(J(\mu(a),\mu(b))^+) = \Sigma^\infty((\mathbb{R}^{k(a,b)}_+) \cup \infty)) \to Mor_{\mathcal{C}^\nu}(a,b)$ that are of the homotopy type of the Pontrjagin–Thom map $\tau : \Sigma^\infty((\mathbb{R}^{k(a,b)}_+) \cup \infty)) \to \bar{\mathcal{M}}(a,b)^\nu$, and such that the following diagram commutes:

$$\begin{tikzcd} \Sigma^\infty(J(\mu(a),\mu(b))^+) \wedge \Sigma^\infty(J(\mu(b),\mu(c))^+) \arrow[r] \arrow[d, "\tau_{a,b}\wedge\tau_{b,c}"'] & \Sigma^\infty(J(\mu(a),\mu(c))^+) \arrow[d, "\tau_{a,c}"] \\ Mor_{\mathcal{C}^\nu}(a,b) \wedge Mor_{\mathcal{C}^\nu}(b,c) \arrow[r] & Mor_{\mathcal{C}^\nu}(a,c). \end{tikzcd}$$

Here the top horizontal map is defined via the composition maps in the category $\mathcal{J}$, and the bottom horizontal map is defined via the composition maps in $\mathcal{C}^\nu$.

Notice that according to this definition, a normal Thom spectrum is simply a functorial collection of Thom spectra of stable normal bundles. Finite, smooth, compact categories have such normal Thom spectra, defined via embeddings into Euclidean spaces (see [4]).

We can now define what it means to have coherent E^*-orientations on the moduli spaces $\bar{\mathcal{M}}(a,b)$.

Definition 10. An E^*-orientation of a flow category of a smooth Floer theory, $\mathcal{C}$, is a functor, $u : \mathcal{C}^\nu \to E_{\mathcal{C}}$, where $\mathcal{C}^\nu$ is a normal Thom spectrum of $\mathcal{C}$, such that on morphism spaces, the induced map

$$Mor_{\mathcal{C}^\nu}(a,b) \to E\langle k(a,b)\rangle$$

is a map of $\langle k(a,b)\rangle$-spectra that defines an E^* orientation of $Mor_{\mathcal{C}^\nu}(a,b) \simeq \bar{\mathcal{M}}(a,b)^\nu$.

Notice that when $E = S^0$, the sphere spectrum, then an E^*-orientation, as defined here, is equivalent to a *framing* of the category $\mathcal{C}$, as defined in [5] and [4].

We can now indicate the proof of the main theorem of this section.

Theorem 11. *Let $\mathcal{C}$ be the flow category of a smooth Floer theory defined by a functional $\mathcal{A} : \mathcal{L} \to \mathbb{R}$, and let E be a commutative ring spectrum. Let $u : \mathcal{C}^\nu \to E_{\mathcal{C}}$ be an E^*-orientation of the category $\mathcal{C}$. Then the induced Floer complex $(\mathcal{C}_*^{\mathcal{A}}, \partial)$ has a natural realization by an E-module, $|Z_E^{\mathcal{A}}|$.*

Remarks.

1. In the case when $E = S^0$, this is a formulation of the result in [5] that says that a Floer theory that defines a *framed*, smooth, compact flow category defines a "Floer stable homotopy type".
2. When $E = H\mathbb{Z}$, the integral Eilenberg–MacLane spectrum, then the notion of an $H\mathbb{Z}$-orientation of a flow category $\mathcal{C}$ is equivalent to the existence of ordinary "coherent orientations" of the moduli spaces, $\bar{\mathcal{M}}(a,b)$, as was studied in [10].
3. Given a flow category $\mathcal{C}$ satisfying the hypotheses of the theorem with respect to a ring spectrum E, then one defines the "Floer E_*-homology groups of the functional $\mathcal{A}$" to be the homotopy groups of the corresponding realization, $\pi_*(|Z_E^{\mathcal{A}}|)$.

Proof. Let $\mathcal{C}$ be such a category and $u : \mathcal{C}^\nu \to E_{\mathcal{C}}$ an E^*-orientation. Consider the corresponding Floer complex [see (7)]:

$$\to \cdots \to C_i^{\mathcal{A}} \xrightarrow{\partial_i} C_{i-1}^{\mathcal{A}} \xrightarrow{\partial_{i-1}} \cdots \to C_0^{\mathcal{A}}$$

Recall that the chain group $C_i^{\mathcal{A}}$ has a basis, $\mathcal{B}_i$, consisting of the objects (critical points) of index i, and the boundary maps are defined by counting (with sign) the elements of the oriented moduli spaces $\mathcal{M}(a,b)$, where a and b have relative index one ($\mu(a) = \mu(b) + 1$.)

We define a functor $Z_E^{\mathcal{A}} : \mathcal{J}_0 \to E - mod$ that satisfies the conditions of Theorem 6. As required, on objects we define

$$Z_E^{\mathcal{A}}(i) = \bigvee_{\mathcal{B}_i} E. \tag{17}$$

To define $Z_E^{\mathcal{A}}$ on the morphisms in $\mathcal{J}_0$ we need to define, for every $i > j$, E-module maps

$$\begin{aligned} Z_E^{\mathcal{A}}(i,j) : J(i,j)^+ \wedge Z_E^{\mathcal{A}}(i) &\to Z_E^{\mathcal{A}}(j) \\ \bigvee_{\mathcal{B}_i} J(i,j)^+ \wedge E &\to \bigvee_{\mathcal{B}_j} E. \end{aligned}$$

Since finite coproducts (wedges) and products coincide in the category of spectra, it suffices to define, for each $a \in \mathcal{B}_i$ and $b \in \mathcal{B}_j$, an E-module map

$$Z_E^{\mathcal{A}}(a,b) : J(i,j)^+ \wedge E \to E.$$

This is equivalent to defining an ordinary map of spectra,

$$Z_E^{\mathcal{A}}(a,b) : \Sigma^\infty(J(i,j)^+) \to E.$$

This map is defined by the E^*-orientation of the moduli space $\bar{\mathcal{M}}(a,b)$ as the composition,

$$\Sigma^\infty(J(i,j)^+) \xrightarrow{\tau_{a,b}} Mor_{\mathcal{C}^\nu}(a,b) \xrightarrow{u} E\langle i-j-1\rangle$$

where $\tau_{a,b}$ is the Pontrjagin–Thom collapse map of the normal Thom spectrum $Mor_{\mathcal{C}^\nu}(a,b) \simeq \bar{\mathcal{M}}(a,b)^\nu$ (see Definition 9), and u is the map of $\langle i-j-1\rangle$-spectra coming from the E^*-orientation (Definition 10). The functorial properties of these maps is precisely what is required to show that they fit together to define a functor, $Z_E^{\mathcal{A}} : \mathcal{J}_0 \to E - mod$. It is straightforward to check that this functor satisfies the conditions of Theorem 6 and so, by that theorem, $|Z_E^{\mathcal{A}}|$ is an E-module spectrum that realizes the Floer complex $C_*^{\mathcal{A}} \otimes E_*$. $\square$

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Relative Chern Characters for Nilpotent Ideals

G. Cortiñas and C. Weibel

1 Introduction

When A is a unital ring, the *absolute Chern character* is a group homomorphism $ch_* : K_*(A) \to HN_*(A)$, going from algebraic K-theory to negative cyclic homology (see [7, 11.4]). There is also a relative version, defined for any ideal I of A:

$$ch_* : K_*(A, I) \to HN_*(A, I). \tag{1.1}$$

Now suppose that A is a $\mathbb{Q}$-algebra and that I is nilpotent. In this case, Goodwillie proved in [4] that (1.1) is an isomorphism. His proof uses another character

$$ch'_* : K_*(A, I) \to HN_*(A, I) \tag{1.2}$$

which is defined only when I is nilpotent and $\mathbb{Q} \subset A$. Goodwillie showed that (1.2) is an isomorphism whenever it is defined, and that it coincides with (1.1) when $I^2 = 0$. Using this and a 5-Lemma argument, he deduced that (1.1) is an isomorphism for any nilpotent I. The question of whether ch_* and ch'_* agree for general nilpotent ideals I was left open in [4], and announced without proof in [7, 11.4.11]. This paper answers the question, proving in Theorem 6.5.1 that

$$ch_* = ch'_* \text{ for all nilpotent ideals } I. \tag{1.3}$$

Here are some applications of (1.3). Let I a nilpotent ideal in a commutative $\mathbb{Q}$-algebra A. Cathelineau proved in [1] that (1.2) preserves the direct sum decomposition coming from the eigenspaces of λ-operations and/or Adams operations. By (1.3), the relative Chern character (1.1) preserves the direct sum decomposition for nilpotent ideals. Next, Cathelineau's result and (1.3) are used in [2] to prove that

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the absolute Chern character ch_* also preserves the direct sum decomposition. In addition, our result (1.3) can be used to strengthen Ginot's results in [3].

This paper is laid out according to the following plan. In Sect. 2 we show that the bar complex $\mathrm{B}(\mathsf{H})$ of a cocommutative Hopf algebra H has a natural cyclic module structure. The case $\mathsf{H} = k[G]$ is well known, and the case of enveloping algebras is implicit in [6]. In Sect. 3, we relate this construction to the usual cyclic module of the algebra underlying H. In Sect. 4 we consider a Lie algebra $\mathfrak{g}$ and factor the Loday–Quillen map $\wedge^n\mathfrak{g} \to C^{\lambda}_{n-1}(U\mathfrak{g})$ through our construction. In Sect. 5 we consider a nilpotent Lie algebra $\mathfrak{g}$ and its associated nilpotent group G and relate the constructions for $U\mathfrak{g}$ and $\mathbb{Q}[G]$ using the Mal'cev theory of [10, Appendix A]. In Sect. 6, we review the definitions of ch_* and ch'_* and prove our main theorem, that (1.3) holds.

1.1 Notation

If $M = (M_*, b, B)$ is a mixed complex [7, 2.5.13], we will write $HH(M)$ for the chain complex (M_*, b), and $HN(M)$ for the total complex of Connes' left half-plane (b, B)-complex (written as $\mathcal{B}M^-$ in [7, 5.1.7]). By definition, the homology of $HH(M)$ is the Hochschild homology $HH_*(M)$ of M, the homology of $HN(M)$ is the negative cyclic homology $HN_*(M)$ of M, and the projection $\pi : HN(M) \to HH(M)$ induces the canonical map $HN_*(M) \to HH_*(M)$.

We refer the reader to [7, 2.5.1] for the notion of a cyclic module. There is a canonical cyclic k-module $C(A)$ associated to any algebra A, with $C_n(A) = A^{\otimes n+1}$, whose underlying simplicial module (C, b) is the Hochschild complex (see [7, 2.5.4]). We will write $HH(A)$ and $HN(A)$ in place of the more awkward expressions $HH(C(A))$ and $HN(C(A))$.

We write $C_{\geq n}$ for the good truncation $\tau_{\geq n}C$ of a chain complex C [13, 1.2.7]. If $n \geq 0$, then $C_{\geq n}$ can and will be regarded as a simplicial module via the Dold–Kan correspondence [13, 8.4].

2 Cyclic Homology of Cocommutative Hopf Algebras

If A is the group algebra of a group G, then the bar resolution $\mathrm{E}(A) = k[\mathrm{E}G]$ admits a cyclic G-module structure and the bar complex $\mathrm{B}(A) = k[\mathrm{B}G]$ also admits a cyclic k-module structure [7]. In this section, we show that the cyclic modules $\mathrm{E}(A)$ and $\mathrm{B}(A)$ can be defined for any cocommutative Hopf algebra A.

2.1 Bar Resolution and Bar Complex of an Augmented Algebra

Let k be a commutative ring, and A an augmented unital k-algebra, with augmentation $\epsilon : A \to k$. We write $\mathrm{E}(A)$ for the *bar resolution* of k as a left A-module [13, 8.6.12]; this is the simplicial A-module $\mathrm{E}_n(A) = A^{\otimes n+1}$, whose face and degeneracy operators are given by

$$\begin{aligned} \mu_i : \mathrm{E}_n(A) &\to \mathrm{E}_{n-1}(A) \qquad (i = 0, \dots, n) \\ \mu_i(a_0 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \qquad (i < n) \\ \mu_n(a_0 \otimes \cdots \otimes a_n) &= \varepsilon(a_n) a_0 \otimes \cdots \otimes a_{n-1} \\ s_j : \mathrm{E}_n(A) &\to \mathrm{E}_{n+1}(A) \qquad (j = 0, \dots, n) \\ s_j(a_0 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots \otimes a_j \otimes 1 \otimes a_{j+1} \otimes \cdots \otimes a_n \end{aligned} \tag{2.1.1}$$

We write ∂' for the usual boundary map $\sum_{i=0}^{n}(-1)^i \mu_i : \mathrm{E}_n(A) \to \mathrm{E}_{n-1}(A)$. The augmentation induces a quasi-isomorphism $\epsilon : \mathrm{E}(A) \to k$. The unit $\eta : k \to A$ is a k-linear homotopy inverse of ϵ; we have $\epsilon\eta = 1$ and the extra degeneracy $s : \mathrm{E}(A)_n \to \mathrm{E}(A)_{n+1}$,

$$s(x) = 1 \otimes x \tag{2.1.2}$$

satisfies $1 - \eta\epsilon = [\partial', s]$. The *bar complex* of A is $\mathrm{B}(A) = k \otimes_A \mathrm{E}(A)$; $\mathrm{B}_n(A) = A^{\otimes n}$. We write $\partial = 1 \otimes \partial' : \mathrm{B}(A)_n \to \mathrm{B}(A)_{n-1}$ for the induced boundary map, and $\mathrm{E}(A)_{\mathrm{norm}}$ and $\mathrm{B}(A)_{\mathrm{norm}}$ for the normalized complexes.

2.2 The Cyclic Module of a Cocommutative Coalgebra

If C is a k-coalgebra, with counit $\epsilon : C \to k$ and coproduct Δ, we have a simplicial k-module $R(C)$, with $R_n(C) = C^{\otimes n+1}$, and face and degeneracy operators given by

$$\begin{aligned} \varepsilon_i : R_n(C) &\to R_{n-1}(C) \qquad (i = 0, \dots, n) \\ \varepsilon_i(c_0 \otimes \cdots \otimes c_n) &= \epsilon(c_i) c_0 \otimes \cdots \otimes c_{i-1} \otimes c_{i+1} \otimes \dots c_n \\ \Delta_i : R_n(C) &\to R_{n+1}(C) \qquad (i = 0, \dots, n) \\ \Delta_i(c_0 \otimes \cdots \otimes c_n) &= c_0 \otimes \dots c_{i-1} \otimes c_i^{(0)} \otimes c_i^{(1)} \otimes \cdots \otimes c_n \end{aligned}$$

Here and elsewhere we use the (summationless) Sweedler notation $\Delta(c) = c^{(0)} \otimes c^{(1)}$ of [12].

We remark that each $R_n(C)$ has a coalgebra structure, and that the face maps ε_i are coalgebra homomorphisms. If in addition C is cocommutative, then the degeneracies Δ_j are also coalgebra homomorphisms, and $R(C)$ is a simplicial coalgebra. In fact $R_n(C)$ is the product of $n + 1$ copies of C in the category of cocommutative coalgebras, and $R(C)$ is a particular case of the usual product simplicial resolution of an object in a category with finite products, which is a functor not only on the simplicial category of finite ordinals and monotone maps, but also on the larger category $\mathfrak{Fin}$ with the same objects, but where a homomorphism is just any set theoretic map, not necessarily order preserving. By [7, 6.4.5] we have:

Lemma 2.2.1. *For a cocommutative coalgebra C, the simplicial module $R(C)$ has the structure of a cyclic k-module, with cyclic operator*

$$\lambda(c_0 \otimes \cdots \otimes c_n) = (-1)^n c_n \otimes c_0 \otimes \cdots \otimes c_{n-1}.$$

Remark 2.2.2. Let G be a group; write $k[G]$ for the group algebra. Note $k[G]$ is a Hopf algebra, and in particular, a coalgebra, with coproduct determined by $\Delta(g) = g \otimes g$. The cyclic module $R(k[G])$ thus defined is precisely the cyclic module whose associated cyclic bicomplex was considered by Karoubi in [5, 2.21], where it is written $\tilde{C}_{**}(G)$.

2.3 The Case of Hopf Algebras

Let H be a Hopf algebra with unit η, counit ϵ and antipode S. We shall assume that $S^2 = 1$, which is the case for all cocommutative Hopf algebras. We may view $R(\mathsf{H})$ as a simplicial left H-module via the diagonal action:

$$a \cdot (h_0 \otimes \cdots \otimes h_n) = a^{(0)} h_0 \otimes \cdots \otimes a^{(n)} h_n.$$

Consider the maps (defined using summationless Sweedler notation):

$$\alpha : \mathrm{E}_n(\mathsf{H}) \to R_n(\mathsf{H}),$$
$$\alpha(h_0 \otimes \cdots \otimes h_n) = h_0^{(0)} \otimes h_0^{(1)} h_1^{(0)} \otimes \cdots \otimes h_0^{(n)} h_1^{(n-1)} \cdots h_{n-1}^{(1)} h_n \tag{2.3.1}$$
$$\beta : R_n(\mathsf{H}) \to \mathrm{E}_n(\mathsf{H}),$$
$$\beta(h_0 \otimes \cdots \otimes h_n) = h_0^{(0)} \otimes (S h_0^{(1)}) h_1^{(0)} \otimes \cdots \otimes (S h_{n-1}^{(1)}) h_n \tag{2.3.2}$$

A straightforward computation reveals:

Lemma 2.3.3. *The maps* (2.3.1) *and* (2.3.2) *are inverse isomorphisms of simplicial* H*-modules:* $\mathrm{E}_n(\mathsf{H}) \cong R_n(\mathsf{H})$.

2.4 Cyclic Complexes of Cocommutative Hopf Algebras

From now on, we shall assume that H is a cocommutative Hopf algebra. In this case the cyclic operator $\lambda : R(\mathsf{H}) \to R(\mathsf{H})$ of Lemma 2.2.1 is a homomorphism of H-modules. Thus $R(\mathsf{H})$ is a cyclic H-module, and we can use the isomorphisms

α and β of Lemma 2.3.3 to translate this structure to the bar resolution $\mathrm{E}(\mathsf{H})$. We record this as a corollary:

Corollary 2.4.1. *When* H *is a cocommutative Hopf algebra,* $\mathrm{E}(\mathsf{H})$ *is a cyclic* H-*module, and* $\mathrm{B}(\mathsf{H}) = k \otimes_{\mathsf{H}} \mathrm{E}(\mathsf{H})$ *is a cyclic* k*-module. The cyclic operator* $t := \beta\lambda\alpha$ *on* $\mathrm{E}(\mathsf{H})$ *is given by the formulas* $t(h) = h$, $t(h_0 \otimes h_1) = -h_0 h_1^{(0)} \otimes S h_1^{(1)}$ *and:*

$$t(h_0 \otimes \cdots \otimes h_n) = (-1)^n h_0 h_1^{(0)} \ldots h_n^{(0)} \otimes S(h_1^{(1)} \ldots h_n^{(1)}) \otimes h_1^{(2)} \otimes \cdots \otimes h_{n-1}^{(2)}.$$

Remark 2.4.2. If $g_0, \ldots, g_n \in \mathsf{H}$ are grouplike elements then

$$t(g_0 \otimes \cdots \otimes g_n) = (-1)^n g_0 \ldots g_n \otimes (g_1 \ldots g_n)^{-1} \otimes g_1 \otimes \cdots \otimes g_{n-1}.$$

In particular, for $\mathsf{H} = k[G]$, $\mathrm{B}(k[G])$ with the cyclic structure of Corollary 2.4.1 is the cyclic module associated to the cyclic set $\mathrm{B}(G, 1)$ of [7, 7.3.3].

For the extra degeneracy of [7, 2.5.7],

$$s' = (-1)^{n+1} t s_n : \mathrm{E}_n(\mathsf{H}) \to \mathrm{E}_{n+1}(\mathsf{H}), \tag{2.4.3}$$

it is immediate from (2.1.1) and Corollary 2.4.1 that s' is signfree:

$$s'(h_0 \otimes \cdots \otimes h_n) = h_0 h_1^{(0)} \ldots h_n^{(0)} \otimes S(h_1^{(1)} \ldots h_n^{(1)}) \otimes h_1^{(2)} \otimes \cdots \otimes h_n^{(2)}.$$

By Corollary 2.4.1, the Connes' operator B' is the H-module homomorphism:

$$B' = (1 - t) s' \sum_{i=0}^{n} t^i : \mathrm{E}_n(\mathsf{H}) \to \mathrm{E}_{n+1}(\mathsf{H}). \tag{2.4.4}$$

(See [8, p. 569].) We write $B : \mathrm{B}(\mathsf{H})_n \to \mathrm{B}(\mathsf{H})_{n+1}$ for the induced k-module map.

Definition 2.4.5. We define the mixed H-module complex $\mathrm{M}' = \mathrm{M}'(\mathsf{H})$, and the mixed k-module complex $\mathrm{M} = \mathrm{M}(\mathsf{H})$ to be the normalized mixed complexes associated to the cyclic modules $\mathrm{E}_*(\mathsf{H})$ and $\mathrm{B}_*(\mathsf{H})$ of Corollary 2.4.1:

$$\mathrm{M}'(\mathsf{H}) := (\mathrm{E}_*(\mathsf{H})_{\mathrm{norm}}, \partial', B')$$
$$\mathrm{M}(\mathsf{H}) := k \otimes_{\mathsf{H}} \mathrm{M}'(\mathsf{H}) = (\mathrm{B}_*(\mathsf{H})_{\mathrm{norm}}, \partial, B).$$

Remark 2.4.6. Consider the map $s'' = (-1)^n s_n : \mathrm{E}_n(\mathsf{H}) \to \mathrm{E}_{n+1}(\mathsf{H})$, and set $B'' = -t s'' N$, where as usual $N = \sum t^i$ is the norm map. Then $B' = B''$ on $\mathrm{E}(\mathsf{H})_{\mathrm{norm}}$ because the relations $s_0 t_n = (-1)^n t_{n+1}^2 s_n$ and $tN = N$ yield for all $x \in \mathrm{E}_n(\mathsf{H})$:

$$\begin{aligned}(B''-B')(x) &= ((-1)^{n+1}ts_nN + (-1)^n(1-t)ts_nN)(x)\\ &= (-1)^n(-ts_nN(x)+ts_nN(x)-t^2s_nN(x))\\ &= -s_0tN(x) = -s_0N(x) \equiv 0 \quad \text{in } \mathrm{E}(\mathsf{H})_{\mathrm{norm}}.\end{aligned}$$

Lemma 2.4.7. *The map* $B' : \mathrm{E}(\mathsf{H})_{\mathrm{norm}} \to \mathrm{E}(\mathsf{H})_{\mathrm{norm}}[1]$ *induced by* (2.4.4) *is given by the explicit formula:*

$$\begin{aligned}B'(h_0\otimes\cdots\otimes h_n) = \sum_{i=0}^{n}(-1)^{ni}h_0h_1^{(0)}\cdots h_{n-i}^{(0)}\otimes h_{n-i+1}^{(0)}\otimes\cdots\otimes h_n^{(0)}\otimes S(h_1^{(1)}\cdots h_n^{(1)})\\ \otimes h_1^{(2)}\otimes\cdots\otimes h_{n-i}^{(2)}.\end{aligned}$$

Proof. For convenience, let us write $Sh^{(1)}$ for $S(h_1^{(1)}\cdots h_n^{(1)})$. It follows from Corollary 2.4.1, cocomutativity and induction on i that $t^i(h_0) = h_0$, and if $i \le n$ then:

$$t^i(h_0\otimes\cdots\otimes h_n) =$$
$$(-1)^{ni}h_0h_1^{(0)}\dots h_{n-i+1}^{(0)}\otimes h_{n-i+2}^{(0)}\otimes\cdots\otimes h_n^{(0)}\otimes Sh^{(1)}\otimes h_1^{(2)}\otimes\cdots\otimes h_{n-i}^{(2)}.$$

Let $s'' = (-1)^n s_n$ be as in Remark 2.4.6 and set $m = n(i+1)+1$. We have

$$\begin{aligned}-ts''t^i(h_0\otimes\cdots\otimes h_n) &= (-1)^{n+1}ts_nt^i(h_0\otimes\cdots\otimes h_n)\\ = (-1)^m t(h_0h_1^0\dots h_{n-i+1}^{(0)}&\otimes h_{n-i+2}^{(0)}\otimes\cdots\otimes h_n^{(0)}\otimes Sh^{(1)}\otimes h_1^{(2)}\otimes\cdots\otimes h_{n-i}^{(2)}\otimes 1)\\ = (-1)^{ni}h_0h_1^{(0)}\dots h_{n-i}^{(0)}&\otimes h_{n-i+1}^{(0)}\otimes\cdots\otimes h_n^{(0)}\otimes Sh^{(1)}\otimes h_1^{(2)}\otimes\cdots\otimes h_{n-i}^{(2)}.\end{aligned} \tag{2.4.5}$$

Now sum up over i to get B'' and use Remark 2.4.6. □

Corollary 2.4.8. *Suppose that $x_1,\dots,x_n$ are primitive elements of* H*, and $h\in\mathsf{H}$. Then $B'(h\otimes x_1\otimes\cdots\otimes x_n) = 0$ in* $\mathrm{E}(\mathsf{H})_{\mathrm{norm}}$.

Proof. When x is primitive, $x^{(0)}\otimes x^{(1)}\otimes x^{(2)}$ is $1\otimes 1\otimes x + 2\otimes x\otimes 1 + x\otimes 1\otimes 1$. By Lemma 2.4.7, $B'(h\otimes x_1\otimes\cdots\otimes x_n)$ is a sum of terms of the form

$$\pm h'\otimes x_{j+1}^{(0)}\otimes\cdots\otimes x_n^{(0)}\otimes S(x_1^{(1)}\cdots x_n^{(1)})\otimes x_1^{(2)}\otimes\cdots\otimes x_j^{(2)}.$$

By inspection, each such term is degenerate, and vanishes in $\mathrm{E}(\mathsf{H})_{\mathrm{norm}}$. □

2.5 Adic Filtrations and Completion

As usual, we can use an adic topology on a Hopf algebra to define complete Hopf algebras, and topological versions of the above complexes.

First we recall some generalities about filtrations and completions of k-modules, following [10]. There is a category of filtered k-modules and filtration-preserving maps; a filtered module V is a module equipped with a decreasing filtration $V = \mathcal{F}_0(V) \supseteq \mathcal{F}_1(V) \supseteq \cdots$. The completion of V is $\hat{V} = \lim_n V/\mathcal{F}_n V$; it is a filtered module in the evident way. If W is another filtered k-module, then the tensor product $V \otimes W$ is a filtered module with filtration

$$\mathcal{F}_n(V \otimes W) = \sum_{p+q=n} \mathrm{image}(\mathcal{F}_p V \otimes \mathcal{F}_q W \to V \otimes W). \tag{2.5.1}$$

We define $\hat{V}\hat{\otimes}\hat{W}$ to be $\widehat{\hat{V} \otimes \hat{W}}$. Note that

$$\mathcal{F}_n(V \otimes W) \supseteq \mathrm{image}\Big(\mathcal{F}_n V \otimes W + V \otimes \mathcal{F}_n W \to V \otimes W\Big) \supseteq \mathcal{F}_{2n}(V \otimes W). \tag{2.5.2}$$

Hence the topology defined by $\{\mathcal{F}_n(V \otimes W)\}$ is the same as that defined by $\{\ker(V \otimes W \to V/\mathcal{F}_n V \otimes W/\mathcal{F}_n W)\}$. It follows that $\hat{V}\hat{\otimes}\hat{W}$ satisfies

$$\hat{V}\hat{\otimes}\hat{W} = \widehat{V \otimes W}. \tag{2.5.3}$$

All this has an obvious extension to tensor products of finitely many factors.

If A is a filtered algebra (an algebra which is filtered as a k-module by ideals), then $\hat{A}$ is an algebra. If I is an ideal then $\{I^n\}$ is called the I-adic filtration on A.

Now suppose that H is a cocommutative Hopf algebra, equipped with the I-adic filtration, where I is both a (2-sided) ideal and a (2-sided) coideal of H, closed under the antipode S and satisfying $\epsilon(I) = 0$. The coideal condition on I means that

$$\Delta(I) \subset \mathsf{H} \otimes I + I \otimes \mathsf{H}.$$

This implies that $\Delta : \mathsf{H} \to \mathsf{H} \otimes \mathsf{H}$ is filtration-preserving; by (2.5.3) it induces a map $\hat{\Delta} : \hat{\mathsf{H}} \to \hat{\mathsf{H}}\hat{\otimes}\hat{\mathsf{H}}$, making $\hat{\mathsf{H}}$ into a *complete Hopf algebra* in the sense of [10].

Consider the induced filtrations (2.5.1) in $\mathrm{E}_n(\mathsf{H})$ and $\mathrm{B}_n(\mathsf{H})$ ($n \geq 0$). It is clear, from the formulas (2.1.1) and our assumption that $\epsilon(I) = 0$, that the simplicial structures of $\mathrm{E}(\mathsf{H})$ and $\mathrm{B}(\mathsf{H})$ are compatible with the filtration; i.e., $\mathrm{E}(\mathsf{H})$ and $\mathrm{B}(\mathsf{H})$ are simplicial filtered objects. Since I is a coideal closed under S, the formula in Corollary 2.4.1 implies that t preserves the filtration. Thus $\mathrm{E}(\mathsf{H})$ and $\mathrm{B}(\mathsf{H})$ are cyclic filtered modules. It follows that $\mathrm{M}'(\mathsf{H})$ and $\mathrm{M}(\mathsf{H})$ are filtered mixed complexes.

The identities (2.5.3) show that the completed objects $\hat{\mathrm{E}}(\mathsf{H})$, $\hat{\mathrm{B}}(\mathsf{H})$, etc., depend only on the topological Hopf algebra $\hat{\mathsf{H}}$, and can be regarded as its topological bar resolution, bar complex, etc., which are defined similarly to their algebraic counterparts, but substituting $\hat{\otimes}$ for $\otimes$ everywhere. In this spirit, we shall write $\mathrm{E}^{\mathrm{top}}(\hat{\mathsf{H}})$, $\mathrm{B}^{\mathrm{top}}(\hat{\mathsf{H}})$, etc., for $\hat{\mathrm{E}}(\mathsf{H})$, $\hat{\mathrm{B}}(\mathsf{H})$, etc.

3 Comparison with the Cyclic Module of the Algebra H

Let H be a cocommutative Hopf algebra. In this section, we construct an injective cyclic module map $\tau : \mathrm{B}(\mathsf{H}) \to C(\mathsf{H})$, from the cyclic bar complex $\mathrm{B}(\mathsf{H})$ of Corollary 2.4.1 to the canonical cyclic k-module $C(\mathsf{H})$ of the algebra underlying H [7, 2.5.4], and a lift c of τ to the negative cyclic complex $HN(\mathsf{H})$ of H.

In the group algebra case, these constructions are well-understood. The cyclic module inclusion $\tau : \mathrm{B}(k[G]) \subset C(k[G])$ is given in [7, 7.4]; see Example 16 below. Goodwillie proved in [4] that τ admits a natural lifting to a chain map $c : \mathrm{B}(k[G]) \to HN(k[G])$ to the negative cyclic complex, and that c is unique up to natural homotopy. An explicit formula for such a lifting was given by Ginot [3], in the normalized, mixed complex setting.

3.1 A Natural Section of the Projection $\mathrm{H}N(\mathrm{M}'(\mathsf{H})) \to HH(\mathrm{M}'(\mathsf{H}))$

Recall from Definition 2.4.5 that $\mathrm{M}' = \mathrm{M}'(\mathsf{H})$ is a mixed complex whose underlying chain complex is $HH(\mathrm{M}') = (\mathrm{E}(\mathsf{H})_{\mathrm{norm}}, \partial')$, and write π' for the projection from the negative cyclic complex $HN(\mathrm{M}')$ to $HH(\mathrm{M}')$. Following the method of Ginot [3], we shall define a natural H-linear chain homomorphism $\Upsilon' : HH(\mathrm{M}') \to HN(\mathrm{M}')$ such that $\pi' \, \Upsilon' = 1$.

We shall use a technical lemma about maps between chain complexes of modules over a k-algebra A. Assume given all of the following:

1. A homomorphism of chain A-modules $f : C \to D$, with $C_n = 0$ for $n < n_0$
2. A decomposition $C_n \cong A \otimes V_n$ for each n, where V_n is a k-module
3. A k-linear chain contraction s for D

Lemma 3.1.1. *Given (1)–(3), there is an A-linear chain contraction κ^f of f, defined by $\kappa^f_{n_0}(av) = asf(v)$ and the inductive formula:*

$$\kappa^f_n : C_n = A \otimes V_n \to D_{n+1}, \quad \kappa^f_n(av) = a\, s\left(f - \kappa^f_{n-1} d\right)(v).$$

Proof. We have to verify the formula $f(av) = \kappa^f d(av) + d\kappa^f(av)$ for $a \in A$ and $v \in V_n$. When $n = n_0$, this is easy as $d(av) = 0$ and $f(v) = dsf(v)$:

$$d\kappa^f_{n_0}(av) = adsf(v) = af(v) = f(av).$$

Inductively, suppose that the formula holds for $n-1$. Since $dv \in C_{n-1}$, we have

$$d(f - \kappa^f d)(v) = f(dv) - d\kappa^f(dv) = (\kappa^f d)(dv) = 0.$$

Using this, and the definition of κ_n^f, we compute:

$$(d\kappa_n^f)(v) = ds(f - \kappa^f d)(v) = (1 - sd)(f - \kappa^f d)(v) = (f - \kappa^f d)(v).$$

Since $\kappa^f d(av) = \kappa^f (a\, dv) = a\kappa^f (dv)$ by construction,

$$d\kappa^f (av) + \kappa^f d(av) = ad\kappa^f (v) + a\kappa^f (dv) = af(v) = f(av). \qquad \square$$

Lemma 3.1.2. *There is a sequence of* H*-linear maps* $\Upsilon'^n : \mathrm{E}(\mathsf{H}) \to \mathrm{E}(\mathsf{H})[2n]$, *starting with* $\Upsilon'^0 = 1$, *such that* $B'(\Upsilon'^n \partial' - \partial' \Upsilon'^n) = 0$.

They induce maps on the normalized complexes $\Upsilon'^n : HH(\mathrm{M}') \to HH(\mathrm{M}')[2n]$.

Proof. Inductively, we suppose we have constructed Υ'^n satisfying $B'[\Upsilon'^n, \partial'] = 0$. Now any chain map from $C = \mathrm{E}(\mathsf{H})$ to $\mathrm{E}(\mathsf{H})[2n+1]$ must land in the good truncation $D = \mathrm{E}(\mathsf{H})[2n+1]_{\geq 0}$, and the k-linear chain contraction $-s$ of (2.1.2) is also a contraction of D. We claim that the H-linear map $f = -B'\Upsilon'^n : C \to D$ is a chain map. Since the differential on D is $-\partial'$, the claim follows from:

$$f(\partial') - (-\partial')f = -B'\Upsilon'^n\partial' - \partial' B'\Upsilon'^n = B'\left(\partial'\Upsilon'^n - \Upsilon'^n\partial'\right) = 0.$$

We define Υ'^{n+1} to be the H-linear chain contraction of $f = -B'\Upsilon'^n$ given by the formulas in Lemma 10. That is,

$$\Upsilon'^{n+1} := \kappa^f = \kappa^{-B'\Upsilon'^n} \qquad (n \geq 0). \tag{3.1.3}$$

The chain contraction condition $[\Upsilon'^{n+1}, \partial'] = f$ for (3.1.3) implies that the inductive hypothesis $B'[\Upsilon'^{n+1}, \partial'] = B'f = 0$ holds.

Finally, note that the normalized mixed complex $HH(M')$ is a quotient of $\mathrm{E}(\mathsf{H})$, and its terms have the form $HH(M')_n = \mathsf{H} \otimes W_n$ for a quotient module W_n of V_n. By naturality of κ^f in f, the above construction also goes through with $\mathrm{E}(\mathsf{H})$ replaced by $M'(\mathsf{H})$, and the maps Υ' on $\mathrm{E}(\mathsf{H})$ and $HH(M')$ are compatible. $\square$

We define maps $\Upsilon' \colon HH(\mathrm{E}(\mathsf{H})) \to HN(\mathrm{E}(\mathsf{H}))$ and $\Upsilon' \colon HH(\mathrm{M}') \to HN(\mathrm{M}')$ by

$$\Upsilon' = \sum\nolimits_{n=0}^{\infty} \Upsilon'^n. \tag{3.1.4}$$

That is, $\Upsilon'(x)$ is $(\cdots, \Upsilon'^n(x), \cdots, \Upsilon'^1(x), x)$.

Lemma 3.1.5. *The maps* Υ' *in* (3.1.4) *are morphisms of chain* H*-modules, and* $\pi'\Upsilon' = 1$. *Here* π' *is the appropriate canonical projection, either* $HN(\mathrm{E}(\mathsf{H})) \to HH(\mathrm{E}(\mathsf{H}))$ *or* $HN(\mathrm{M}') \to HH(\mathrm{M}')$.

Proof. It is clear that Υ' is H-linear and that $\pi'\Upsilon' = \Upsilon'^0 = 1$. To see that it is a chain map, we observe that the nth coordinate of $(B' + \partial')\Upsilon' - \Upsilon'\partial'$ is $B'\Upsilon'^{n-1} + \partial'\Upsilon'^n - \Upsilon'^n\partial'$. This is zero by the chain contraction condition for (3.1.3). $\square$

Remark 3.1.6. Write $[1]^n$ for the element $1 \otimes \cdots \otimes 1$ of $k^{\otimes n}$. By induction, we may check that $\Upsilon'^n(1) = (-1)^n (2n)!/n!\,[1]^{2n+1}$. Thus $\Upsilon'(1) = (0, \ldots, 0, 1)$ in $HN(M')$.

Recall from Definition 2.4.5 that $M = k \otimes_{\mathsf{H}} M'$, and that $\mathrm{B}(A) = k \otimes_A \mathrm{E}(A)$.

Corollary 3.1.7. *There are morphisms of chain k-modules, $\Upsilon : HH(\mathrm{B}(\mathsf{H})) \to HN(\mathrm{B}(\mathsf{H}))$ and $\Upsilon : HH(\mathrm{M}) \to HN(\mathrm{M})$, defined by*

$$\Upsilon = \sum\nolimits_{n=0}^{\infty} 1_k \otimes_{\mathsf{H}} \Upsilon'^n,$$

and $\pi\Upsilon = 1$. Here π is the appropriate projection $\pi : HN \to HH$.

3.2 The Lift $HH(\mathrm{B}(\mathsf{H})) \xrightarrow{c} HN(\mathsf{H})$

Recall that $C(\mathsf{H})$ denotes the canonical cyclic complex of the algebra underlying H [7, 2.5.4]. We set $\tau_0 = \eta : k \to \mathsf{H}$.

Lemma 3.2.1. *Let H be a cocommutative Hopf algebra. Then the k-linear map*

$$\tau : \mathrm{B}(\mathsf{H}) \to C(\mathsf{H})$$
$$\tau(h_1 \otimes \cdots \otimes h_n) = S(h_1^{(0)} \ldots h_n^{(0)}) \otimes h_1^{(1)} \otimes \cdots \otimes h_n^{(1)}, \quad n > 0,$$

is an injective homomorphism of cyclic k-modules. It induces an injection of the associated mixed complexes, $M(\mathsf{H}) \hookrightarrow C(\mathsf{H})_{\mathrm{norm}}$.

Proof. One has to check that τ commutes with the face, degeneracy and cyclic operators; these are all straightforward, short calculations. The fact that the maps are injective follows from the antipode identity $(Sh^{(0)})h^{(1)} = \eta\epsilon(h)$. $\square$

Remark 3.2.2. If $g_1, \ldots, g_n \in \mathsf{H}$ are grouplike, then

$$\tau(g_1 \otimes \cdots \otimes g_n) = (g_1 \ldots g_n)^{-1} \otimes g_1 \otimes \cdots \otimes g_n.$$

Thus for $\mathsf{H} = k[G]$, the τ of Lemma 3.2.1 is the map $k[\mathrm{B}(G,1)] \hookrightarrow HH(k[G])$ of [7, 7.4.5].

We define $c : \mathrm{B}(\mathsf{H}) \to HN(\mathsf{H})$ to be the natural chain map

$$c : \mathrm{B}(\mathsf{H}) \xrightarrow{\Upsilon} HN(\mathrm{B}(\mathsf{H})) \xrightarrow{\tau} HN(\mathsf{H}). \tag{3.2.3}$$

We will also write c for the normalized version $HH(M) \to HN(\mathsf{H})_{\mathrm{norm}}$ of this map.

Theorem 3.2.4. *The following diagram commutes*

$$\begin{tikzcd} & HN(\mathsf{H}) \arrow[d, "\pi"] \\ \mathrm{B}(\mathsf{H}) \arrow[ur, "c"] \arrow[r, "\tau"'] & HH(\mathsf{H}). \end{tikzcd}$$

Proof. By (3.2.3), Lemma 3.2.1 and Corollary 3.1.7, $\pi c = \pi\tau\Upsilon = \tau\pi\Upsilon = \tau$. $\square$

Remark 3.2.5. Goodwillie proved in [4, II.3.2] that, up to chain homotopy, there is a unique chain map $\mathrm{B}(k[G]) \to HN(k[G])$ lifting τ, natural in the group G. Ginot [3] has given explicit formulas for one such map; it follows that Ginot's map is naturally chain homotopic to the map c constructed in (3.2.3) for $\mathsf{H} = k[G]$.

3.3 Passage to Completion

If A is a filtered algebra, the induced filtration (2.5.1) on the canonical cyclic module $C(A)$ makes it a cyclic filtered module. Passing to completion we obtain a cyclic module $C^{\mathrm{top}}(\hat{A})$ with $C_n^{\mathrm{top}}(\hat{A}) = \hat{A}^{\hat{\otimes} n+1}$. In the spirit of Sect. 2.5, we write $HH^{\mathrm{top}}(\hat{A})$, $HN^{\mathrm{top}}(\hat{A})$, etc., for the Hochschild and cyclic complexes, etc., of the mixed complex associated to $C^{\mathrm{top}}(\hat{A})$.

In particular this applies if $A = \mathsf{H}$ is a cocommutative Hopf algebra, equipped with an I-adic filtration, where I is an ideal and coideal of H with $\epsilon(I) = 0$, closed under the antipode S. Write $\hat{\mathsf{H}}$ for the associated complete Hopf algebra.

It is clear from the formula in Lemma 3.2.1 that τ is a morphism of cyclic filtered modules. Hence it induces continuous maps $\hat{\tau}$ between the corresponding complexes for HH, HN, etc.

Proposition 3.3.1. *The map c of* (3.2.3) *induces a continuous map $\hat{c}$ which fits into a commutative diagram*

$$\begin{tikzcd} & HN^{\mathrm{top}}(\hat{\mathsf{H}}) \arrow[d, "\pi"] \\ \mathrm{B}^{\mathrm{top}}(\hat{\mathsf{H}}) \arrow[ur, "\hat{c}"] \arrow[r, "\hat{\tau}"'] & HH^{\mathrm{top}}(\hat{\mathsf{H}}). \end{tikzcd}$$

Proof. It suffices to show that Υ (and hence c) is a filtered morphism. We observed in Sect. 2.5 that $\mathrm{E}(\mathsf{H})$ is a cyclic filtered module. Similarly, it is clear that s, s' and B' are filtered morphisms from their definitions in (2.1.2), (2.4.3) and (2.4.4). The recursion formulas in Lemma 3.1.1 and (3.1.3) show that each Υ'^n is filtered, whence so are Υ' and Υ, as required. $\square$

4 The Case of Universal Enveloping Algebras of Lie Algebras

Let $\mathfrak{g}$ be a Lie algebra over a commutative ring k. Then the enveloping algebra $U\mathfrak{g}$ is a cocommutative Hopf algebra, so the constructions of the previous sections apply to $U\mathfrak{g}$. In particular a natural map $c : \mathrm{B}(U\mathfrak{g}) \to HN(U\mathfrak{g})$ was constructed in (3.2.3). In this section, we show that the Loday–Quillen map

$$\wedge\mathfrak{g} \xrightarrow{\theta} C^{\lambda}(U\mathfrak{g})[-1] \xrightarrow{B} HN(U\mathfrak{g})$$

factors through c up to chain homotopy (see Theorem 4.2.2).

4.1 Chevalley–Eilenberg Complex

The Chevalley–Eilenberg resolution of k as a $U\mathfrak{g}$-module has the form $(U\mathfrak{g} \otimes \wedge\mathfrak{g}, d')$, and is given in [13, 7.7]. Tensoring it over $U\mathfrak{g}$ with k, we obtain a complex $(\wedge\mathfrak{g}, d)$. Kassel showed in [6, 8.1] that the anti-symmetrization map

$$e : \wedge^n\mathfrak{g} \to (U\mathfrak{g})^{\otimes n} \tag{4.1.1}$$
$$e(x_1 \wedge \cdots \wedge x_n) = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sg}(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$$

induces chain maps $e : \wedge\mathfrak{g} \to \mathrm{B}(U\mathfrak{g})$ and $1 \otimes e : U\mathfrak{g} \otimes \wedge\mathfrak{g} \to \mathrm{E}(U\mathfrak{g})$, because $ed = \partial e$ and $(1 \otimes e)d' = \partial'(1 \otimes e)$. Moreover, e and $1 \otimes e$ are quasi-isomorphisms; see [6, 8.2].

Lemma 4.1.2. *The map $\psi' : U\mathfrak{g} \otimes \wedge\mathfrak{g} \to HN(\mathrm{M}'(U\mathfrak{g}))$ defined by*

$$\psi'(x) = (\ldots, 0, 0, 0, 1 \otimes e(x))$$

is a morphism of chain $U\mathfrak{g}$-modules, and the map $\psi : \wedge\mathfrak{g} \to HN(\mathrm{M}(U\mathfrak{g}))$ defined by

$$\psi(x) = (\ldots, 0, 0, 0, e(x))$$

is a morphism of chain k-modules.

Proof. Consider $U\mathfrak{g}\otimes\wedge\mathfrak{g}$ and $\wedge\mathfrak{g}$ as mixed complexes with trivial Connes' operator. By Corollary 2.4.8, $B'(1 \otimes e) = 0$ in $M'(U\mathfrak{g})$. Thus both $1 \otimes e$ and e induce morphisms of mixed complexes $U\mathfrak{g} \otimes \wedge\mathfrak{g} \to \mathrm{M}'(U\mathfrak{g})$ and $\wedge\mathfrak{g} \to \mathrm{M}'(U\mathfrak{g})$. □

Lemma 4.1.3. *The diagrams*

$$\begin{tikzcd} U\mathfrak{g}\otimes\wedge\mathfrak{g} \arrow[r, "\psi'"] \arrow[d, "1\otimes e"'] & HN(\mathrm{M}'(U\mathfrak{g})) \\ \mathrm{E}(U\mathfrak{g}) \arrow[ur, "\Upsilon'"'] & \end{tikzcd} \qquad \begin{tikzcd} \wedge\mathfrak{g} \arrow[r, "\psi"] \arrow[d, "e"'] & HN(\mathrm{M}(U\mathfrak{g})) \arrow[d, "\tau"] \\ \mathrm{B}(U\mathfrak{g}) \arrow[ur, "\Upsilon"] \arrow[r, "c"'] & HN(U\mathfrak{g})_{\mathrm{norm}} \end{tikzcd}$$

commute up to natural $U\mathfrak{g}$-linear (resp., natural k-linear) chain homotopy.

Proof. By Corollary 3.1.7, Lemma 4.1.2 and (3.2.3), it suffices to consider the left diagram.
Consider the mixed subcomplex $N\subset \mathrm{M}'(U\mathfrak{g})$ given by $N_0=0$, $N_1=\ker\partial'$ and $N_n=\mathrm{E}_n(U)_{\mathrm{norm}}$. Because $\psi'(1)=\Upsilon'(1)$ by Example 3.1.6, the difference $f=\Upsilon'(1\otimes e)-\psi'$ factors through $HN(N)$. Put $\phi^n=(-1)^n(sB')^n s: N\to N[2n+1]$. One checks that $\phi:=\sum_{i=0}^{\infty}\phi^n$ is a natural, k-linear contracting homotopy for $HN(N)$. Now apply Lemma 3.1.1. □

There are simple formulas for τ and $\tau\psi$ in the normalized complexes.

Lemma 4.1.4. *Let $x_1,\dots,x_n\in\mathfrak{g}$. Then in $C(U\mathfrak{g})_{\mathrm{norm}}$ we have:*

$$\tau(x_1\otimes\cdots\otimes x_n)=1\otimes x_1\otimes\cdots\otimes x_n$$

Proof. Let $\nabla^{(n)}: U\mathfrak{g}^{\otimes n}\to U\mathfrak{g}$ be the multiplication map and $\sigma\in\mathfrak{S}_{2n}$ the (bad) riffle shuffle $\sigma(2i-1)=i$, $\sigma(2i)=n+i$. By definition (see Lemma 3.2.1),

$$\tau=(S\otimes 1^{\otimes n})\circ(\nabla^{(n)}\otimes 1^{\otimes n})\circ\sigma\circ\Delta^{\otimes n} \tag{4.1.5}$$

in $C(U\mathfrak{g})$. Since the x_i are primitive,

$$\Delta^{\otimes n}(x_1\otimes\cdots\otimes x_n)=(x_1\otimes 1+1\otimes x_1)\otimes\cdots\otimes(x_n\otimes 1+1\otimes x_n)$$

Expanding this product gives a sum in which

$$x=1\otimes x_1\otimes 1\otimes x_2\otimes\cdots\otimes 1\otimes x_n$$

is the only term not mapped to a degenerate element of $C(U\mathfrak{g})$ under the composition (4.1.5). Thus in $C(U\mathfrak{g})_{\mathrm{norm}}$ we have:

$$\begin{aligned}\tau(x_1\otimes\cdots\otimes x_n)&=(S\otimes 1^{\otimes n})(\nabla^{(n)}\otimes 1^{\otimes n})\sigma(x)\\ &=(S\otimes 1^{\otimes n})(\nabla^{(n)}\otimes 1^{\otimes n})(1\otimes\cdots\otimes 1\otimes x_1\otimes\cdots\otimes x_n)\\ &=(S\otimes 1^{\otimes n})(1\otimes x_1\otimes\cdots\otimes x_n)=1\otimes x_1\otimes\cdots\otimes x_n.\end{aligned}$$ □

Corollary 4.1.6. *We have: $\tau\psi(x_1\wedge\cdots\wedge x_n)=(\dots,0,0,0,1\otimes e(x_1\wedge\cdots\wedge x_n))$.*

Proof. Combine Lemmas 4.1.2 and 4.1.4. □

4.2 The Loday–Quillen Map

We can now show that $\tau\psi$ factors through the Connes' complex $C^\lambda(U\mathfrak{g}) = \operatorname{coker}(1-t : U^{\otimes *} \to HH(U\mathfrak{g}))$. We have a homomorphism θ which lifts the Loday–Quillen map of [7, 10.2.3.1, 11.3.12] to $U\mathfrak{g}$:

$$\theta : \wedge^{n+1}\mathfrak{g} \to C_n^\lambda(U\mathfrak{g})$$
$$\theta(x_0 \wedge x_1 \wedge \cdots \wedge x_n) = x_0 \otimes e(x_1 \wedge \cdots \wedge x_n).$$

Because we are working modulo the image of $1-t$, θ is well defined. The following result is implicit in the proof of [7, 10.2.4] for $r = 1$, $A = U\mathfrak{g}$.

Lemma 4.2.1. *θ is a chain homomorphism $\wedge\mathfrak{g} \to C^\lambda(U\mathfrak{g})[-1]$.*

Proof. To show that $b\theta = -\theta d$, we fix a monomial $x_1 \wedge \cdots \wedge x_n$ and compute that $b\theta(x_0 \wedge \cdots \wedge x_n) = \sum_{\sigma\in\mathfrak{S}_n} b(x_0 \otimes x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}) = A + B$, where A equals:

$$\begin{aligned}
&\sum_{\sigma\in\mathfrak{S}_n} \operatorname{sg}(\sigma)\Big(x_0x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(n)} + (-1)^n x_{\sigma(n)}x_0 \otimes x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n-1)}\Big)\\
&= \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sg}(\sigma)\Big(x_0x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(n)} - x_{\sigma(1)}x_0 \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(n)}\Big)\\
&= \sum_{i=1}^{n} \sum_{\substack{\sigma\in\mathfrak{S}\\ \sigma(i)=i}} (-1)^{i-1}\operatorname{sg}(\sigma)[x_0, x_i] \otimes x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(i-1)} \otimes x_{\sigma(i+1)} \otimes \cdots \otimes x_{\sigma(n)}\\
&= \sum_{i=1}^{n} (-1)^{i-1}[x_0, x_i] \otimes e(x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_n),
\end{aligned}$$

and B equals

$$\begin{aligned}
&\sum_{\sigma\in\mathfrak{S}_n} \sum_{i=1}^{n-1} (-1)^i \operatorname{sg}(\sigma) x_0 \otimes x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(i)}x_{\sigma(i+1)} \otimes \cdots \otimes x_{\sigma(n)}\\
&\qquad = x_0 \otimes e(d(x_1 \wedge \cdots \wedge x_n)).
\end{aligned}$$

Similarly, $\theta\, d(x_0 \wedge \cdots \wedge x_n)$ is the sum of $-A$ and

$$\sum_{0<i<j\le n} (-1)^{i+j+1} x_0 \otimes e([x_i, x_j] \wedge x_1 \wedge \cdots \wedge x_n) = -x_0 \otimes e(d(x_1 \wedge \cdots \wedge x_n))$$

which equals $-B$. Therefore $b\theta(x_0 \wedge \cdots \wedge x_n) = -\theta d(x_0 \wedge \cdots \wedge x_n)$. $\square$

Warning: the sign convention used for d in [7, 10.1.3.3] differs by -1 from the usual convention, used here and in [13, 7.7] and [6].

It is well known and easy to see that, because Connes' operator B vanishes on the image of $1-t$, it induces a chain map for every algebra A:

$$B : C^{\lambda}(A)[-1] \to HN(A)$$
$$B[x] = (\ldots, 0, 0, 0, Bx).$$

Let $\wedge^{+}\mathfrak{g}$ denote the positive degree part of $\wedge\mathfrak{g}$.

Theorem 4.2.2. *We have $\tau\psi = B\theta$ as chain maps $\wedge^{+}\mathfrak{g} \to HN(U\mathfrak{g})_{\mathrm{norm}}$. Hence the following diagram commutes up to natural chain homotopy.*

$$\begin{tikzcd}
\wedge^{+}\mathfrak{g} \arrow[d, "\theta"'] \arrow[dr, "ce"] & \\
C^{\lambda}(U\mathfrak{g})[-1] \arrow[r, "B"] & HN(U\mathfrak{g})_{\mathrm{norm}}.
\end{tikzcd}$$

Theorem 4.2.2 fails for the degree 0 part k of $\wedge\mathfrak{g}$. Indeed, $\theta(1) = 0$ for degree reasons, while from Example 3.1.6 we see that $c\,e(1) = \tau\psi(1) = (\ldots, 0, 1)$ is nonzero.

Proof. By Lemma 4.1.3, it suffices to check that $\tau\psi = B\theta$. By Corollary 4.1.6 and (4.1.1), we have

$$\tau\psi(x_1 \wedge \cdots \wedge x_n) = (\ldots, 0, 0, \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sg}(\sigma) 1 \otimes x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}). \tag{4.2.3}$$

Note the expression above contains no products in $U\mathfrak{g}$; neither do the formulas for $\theta(x_0 \wedge x_1 \wedge \cdots \wedge x_n)$ and the Connes' operator in $C(U\mathfrak{g})$. Thus we may assume that $\mathfrak{g}$ is abelian, and that $U\mathfrak{g} = S\mathfrak{g}$ is a (commutative) symmetric algebra. We may interpret $\theta(x_1 \wedge \cdots \wedge x_n)$ as the shuffle product $x_1 \star B(x_2) \star \cdots \star B(x_n)$ (see [7, 4.2.6]), and the nonzero coordinate of (4.2.3) as the shuffle product

$$\begin{aligned}
(1 \otimes x_1) \star \cdots \star (1 \otimes x_n) &= B(x_1) \star \cdots \star B(x_n) \\
&= B\big(x_1 \star B(x_2) \star \cdots \star B(x_n)\big) \qquad \text{by [8, 3.1] or [7, 4.3.5]} \\
&= B(\theta(x_1 \wedge \cdots \wedge x_n)).
\end{aligned}$$

□

5 Nilpotent Lie Algebras and Nilpotent Groups

In this and the remaining sections we shall fix the ground ring $k = \mathbb{Q}$. Let $\mathfrak{g}$ be a nilpotent Lie algebra; consider the completion $\hat{U}\mathfrak{g}$ of its enveloping algebra with respect to the augmentation ideal, and set

$$G = \exp \mathfrak{g} \subset \hat{U}\mathfrak{g}.$$

This is a nilpotent group, and $\mathsf{H} = \mathbb{Q}[G]$ is a Hopf algebra. The inclusion $G \subset \hat{U}\mathfrak{g}$ induces a homomorphism $\mathsf{H} = \mathbb{Q}[G] \hookrightarrow \hat{U}\mathfrak{g}$, and $\hat{\mathsf{H}} \cong \hat{U}\mathfrak{g}$ by [10, A.3].

On the other hand, Suslin and Wodzicki showed in [11, 5.10] that there is a natural quasi-isomorphism $sw : \mathrm{B}(\mathbb{Q}[G]) \to \wedge\mathfrak{g}$. Putting these maps together with those considered in the previous sections, we get a diagram

$$\begin{tikzcd}
\mathrm{B}(\mathbb{Q}[G]) \arrow[rr, "c"] \arrow[d, "sw"'] \arrow[dr] & & HN(\mathbb{Q}[G]) \arrow[dr] & \\
\wedge\mathfrak{g} \arrow[d, "e"'] & \mathrm{B}^{\mathrm{top}}(\hat{\mathsf{H}}) \arrow[rr, "\hat{c}"] & & HN^{\mathrm{top}}(\hat{\mathsf{H}}). \\
B(U\mathfrak{g}) \arrow[rr, "c"'] \arrow[ur] & & HN(U\mathfrak{g}) \arrow[ur] &
\end{tikzcd} \tag{5.1}$$

Proposition 5.2. *Diagram* (5.1) *commutes up to natural chain homotopy.*

Proof. The two parallelograms commute by naturality. The triangle on the left of (5.1) commutes up to natural homotopy by Lemma 5.3 below. □

Lemma 5.3. *The following diagram commutes up to natural chain homotopy.*

$$\begin{tikzcd}
\mathrm{B}(\mathbb{Q}[G]) \arrow[r] \arrow[d, "sw"'] & \mathrm{B}^{\mathrm{top}}(\hat{\mathsf{H}}) \\
\wedge\mathfrak{g} \arrow[r, "e"] & \mathrm{B}(U\mathfrak{g}) \arrow[u, "\iota"']
\end{tikzcd}$$

Proof. By construction (see [11, 5.10]), the map sw is induced by a map $\mathrm{E}(\mathbb{Q}[G]) \to \hat{\mathsf{H}}\hat{\otimes} \wedge \mathfrak{g}$. Let ι be the upward vertical map; ιe is induced by $1 \otimes \iota e : \hat{\mathsf{H}}\hat{\otimes} \wedge \mathfrak{g} \to \mathrm{E}^{\mathrm{top}}(\hat{\mathsf{H}})$. Thus it suffices to show that the composite $\mathrm{E}(\mathbb{Q}[G]) \to \mathrm{E}^{\mathrm{top}}(\hat{\mathsf{H}})$ is naturally homotopic to the map induced by the homomorphism $\mathbb{Q}[G] \to \hat{\mathsf{H}}$. By definition, these maps agree on $\mathrm{E}_0(\mathbb{Q}[G])$; thus their difference goes to the subcomplex $\ker(\hat{\epsilon} : \mathrm{E}^{\mathrm{top}}(\hat{\mathsf{H}}) \to \mathbb{Q})$ which is contractible, with contracting homotopy induced by (2.1.2). Hence we may apply Lemma 3.1.1; this finishes the proof. □

Remark 5.4. Let $\mathfrak{g} = J_{\mathrm{Lie}}$ be the Lie algebra associated to an ideal J in a $\mathbb{Q}$-algebra A ($\mathfrak{g}$ is J with the commutator bracket); there is a canonical algebra map $U\mathfrak{g} \to A$ sending $\mathfrak{g}$ onto J. If J is a nilpotent ideal then $\mathfrak{g}$ is a nilpotent Lie algebra and the induced algebra map $\hat{U}\mathfrak{g} \to A$ restricts to an isomorphism $G = \exp(\mathfrak{g}) \xrightarrow{\cong} (1+J)^{\times}$, as is proven in [11, 5.2].

Let $C(A, J)$ denote the kernel of $C(A) \to C(A/J)$, and let $C^{\lambda}(U\mathfrak{g}, \mathcal{I})$ denote the kernel of $C^{\lambda}(U\mathfrak{g}) \to C^{\lambda}(k)$. The composite of θ with the map induced by $U\mathfrak{g} \to A$ factors through $C^{\lambda}(A, J)$, giving rise to a commutative diagram

$$
\begin{tikzcd}
\wedge\mathfrak{g} \arrow[r, "\theta"] \arrow[d, "\rho"'] & C^{\lambda}(U\mathfrak{g}, \mathcal{I})[-1] \arrow[dl] \arrow[d] \\
C^{\lambda}(A, J)[-1] \arrow[r, hook] & C^{\lambda}(A)[-1].
\end{tikzcd}
$$

The composite $\mathbb{Q}[G] \to \hat{U}\mathfrak{g} \to A$ sends the augmentation ideal $\mathcal{I}_G$ of $\mathbb{Q}[G]$ to J. Consider the resulting map

$$
j : HN(\mathbb{Q}[G], \mathcal{I}_G) \to HN(\hat{U}\mathfrak{g}, \hat{\mathcal{I}}) \to HN(A, J).
$$

Putting together Theorem 4.2.2 with Proposition 5.2, we get a naturally homotopy commutative diagram (with $G = (1 + J)^{\times}$):

$$
\begin{tikzcd}
\mathrm{B}(\mathbb{Q}[G], \mathcal{I}_G) \arrow[r, "c"] \arrow[d, "sw"'] & HN(\mathbb{Q}[G], \mathcal{I}_G) \arrow[dr, "j"] & \\
\wedge^{+}\mathfrak{g} \arrow[r, "coe"] \arrow[dr, "\theta"'] & HN(U\mathfrak{g}, \mathcal{I})_{\text{norm}} \arrow[r] & HN(A, J)_{\text{norm}} \\
& C^{\lambda}(U\mathfrak{g}, \mathcal{I})[-1] \arrow[r] \arrow[u, "B"'] & C^{\lambda}(A, J)[-1]. \arrow[u, "B"']
\end{tikzcd}
$$

6 The Relative Chern Character of a Nilpotent Ideal

In this section we establish Theorem 36, promised in (1.3), that the two definitions (1.1) and (1.2) of the Chern character $K_*(A, I) \to HN_*(A, I)$ agree for a nilpotent ideal I in a unital $\mathbb{Q}$-algebra A. The actual proof is quite short, and most of this section is devoted to the construction of the maps (1.1) and (1.2).

For this it is appropriate to regard a non-negative chain complex C as a simplicial abelian group via Dold–Kan, identifying the homology of the complex with the homotopy groups $\pi_*(C)$ by abuse of notation. If X is a simplicial set, we write $\mathbb{Z}[X]$ for its singular complex, so that $H_*(X; \mathbb{Z})$ is $\pi_*\mathbb{Z}[X]$, and the Hurewicz map is induced by the simplicial map $h : X \to \mathbb{Z}[X]$.

6.1 The Absolute Chern Character

Let A be a unital $\mathbb{Q}$-algebra, and $\mathrm{BGL}(A)$ the classifying space of $GL(A)$. Now the plus construction $\mathrm{BGL}(A) \to \mathrm{BGL}(A)^+$ is a homology isomorphism, and $K_n(A) = \pi_n \mathrm{BGL}(A)^+$ for $n \geq 1$. In particular, the singular chain complex map

$\mathbb{Z}[\mathrm{BGL}(A)] \to \mathbb{Z}[\mathrm{BGL}(A)^+]$ is a quasi-isomorphism. As described in [7, 11.4.1], the absolute Chern character $ch_n : K_n(A) \to HN_n(A)$ (of Goodwillie, Jones et al.) is the composite $ch = ch_A^- \circ h$ of the Hurewicz map $h : \mathrm{B}GL(A)^+ \to \mathbb{Z}[\mathrm{B}GL(A)^+] \simeq \mathbb{Z}[\mathrm{BGL}(A)]$, the identification $\mathbb{Z}[\mathrm{B}G] = \mathrm{B}(\mathbb{Z}[G])$ with the bar complex, and the chain complex map ch_A^-, which is defined as the stabilization (for $GL_n \subset GL_{n+1}$) of the composites:

$$\begin{tikzcd}
\mathrm{B}(\mathbb{Z}[\mathrm{GL}_n(A)]) \arrow[r, "c"] \arrow[d] & HN(\mathbb{Z}[\mathrm{GL}_n(A)]) \arrow[r] \arrow[d] & HN(M_n(A)) \arrow[r, "\mathrm{tr}"] & HN(A)_{\mathrm{norm}}. \\
\mathrm{B}(\mathbb{Q}[\mathrm{GL}_n(A)]) \arrow[r, "c"] & HN(\mathbb{Q}[\mathrm{GL}_n(A)]) \arrow[ur] & &
\end{tikzcd} \tag{6.1.1}$$

Here c is the natural map defined in (3.2.3) for $k = \mathbb{Z}$ and $\mathbb{Q}$; the middle maps in the diagram are induced by the fusion maps $\mathbb{Z}[\mathrm{GL}_n(A)] \subset \mathbb{Q}[\mathrm{GL}_n(A)] \to M_n(A)$, and tr is the trace map. The maps $HN(\mathbb{Q}[\mathrm{GL}_n(A)]) \to HN(A)_{\mathrm{norm}}$ are independent of n by [7, 8.4.5], even though the fusion maps are not.

Remark 6.1.2. If A is commutative and connected, the composition of the rank map $K_0(A) \to \mathbb{Z}$ sending $[A^r]$ to r with the map $H_0(ch_A^-)$ yields the Chern character $ch_0^- : K_0(A) \to HN_0(A)$ of [7, 8.3]. Composing this with the maps $HN(A) \to HC(A)[2n]$ yields the map $ch_{0,n} : K_0(A) \to HC_{2n}(A)$ of [7, 8.3.4]. From Example 3.1.6 above, with $A = k$, we see that $ch([k]) = c(1)$, and $ch_{0,n}([k]) = (-1)^n(2n)!/n!$ in $HC_{2n}(k) \cong k$, in accordance with [7, 8.3.7].

6.2 Volodin Models for the Relative Chern Character of Nilpotent Ideals

In order to define the relative version ch_* of the absolute Chern character, we need to recall a chunk of notation about Volodin models. For expositional simplicity, we shall assume that I is a nilpotent ideal in a unital $\mathbb{Q}$-algebra A.

Definition 6.2.1. Let I a nilpotent ideal in a $\mathbb{Q}$-algebra A, and σ a partial order of $\{1, \ldots, n\}$. We let $\mathcal{T}_n^\sigma(A, I)$ be the nilpotent subalgebra of $M_n(A)$ defined by:

$$\mathcal{T}_n^\sigma(A, I) := \{a \in M_n(A) : a_{ij} \in I \text{ if } i \not<_\sigma j\}.$$

We write $\mathfrak{t}_n^\sigma(A, I)$ for the associated Lie algebra, and $T_n^\sigma(A, I)$ for the group

$$T_n^\sigma(A, I) = \exp \mathfrak{t}_n^\sigma(A, I) = 1 + \mathcal{T}_n^\sigma(A, I) \subset GL_n(A).$$

The *Volodin space* $X(A) \subset BGL(A)$ is defined to be the union of the spaces $X_n(A) = \bigcup_\sigma BT_n^\sigma(A, 0)$; see [7, 11.2.13]. The *relative Volodin space* $X(A, I)$ is defined to be the union of the spaces $\bigcup_\sigma BT_n^\sigma(A, I)$; see [7, 11.3.3].

The morphism $ch_A^- : \mathbb{Q}[B\mathrm{GL}(A)] \to HN(A)_{\mathrm{norm}}$ of (6.1.1) sends the subcomplex $\mathbb{Q}[X(A,I)]$ to $HN(A,I)_{\mathrm{norm}}$, which is the kernel of $HN(A)_{\mathrm{norm}} \to HN(A/I)_{\mathrm{norm}}$; see [7, 11.4.6]. The chain map ch^- is defined to be the restriction of ch_A^-:

$$ch^- : \mathbb{Q}[X(A,I)] \to HN(A,I)_{\mathrm{norm}}. \tag{6.2.2}$$

It will be useful to have a more detailed description of the restriction of (6.2.2) to $\mathbb{Q}[BT_n^\sigma(A,I)]$. Recall that if Λ is a non-unital subalgebra of R then $\mathbb{Q}+\Lambda$ is a unital subalgebra of R; we write $C(\Lambda)$ for the cyclic submodule $C(\mathbb{Q}+\Lambda,\Lambda)$ of $C(\mathbb{Q}+\Lambda)$ and hence of $C(R)$, following [7, 2.2.16]. When $\Lambda = \mathcal{T}_n^\sigma(A,I)$ and $R = M_n(A)$, we obtain the cyclic submodule $C(\mathcal{T}_n^\sigma(A,I))$ of $C(M_n(A))$.

The trace map $C(M_n(A)) \to C(A)$ is a morphism of cyclic modules, sending $C(\mathcal{T}_n^\sigma(A,I))$ to the submodule $C(A,I)$. On the other hand, by Example 5.4, we also have a map

$$C(\mathbb{Q}[G],\mathcal{I}_G) \xrightarrow{j} C(G), \quad \text{for } G = \mathcal{T}_n^\sigma(A,I).$$

From the definition of ch_A^- in (6.1.1) and the naturality of c, we obtain the promised description of ch^-, which we record.

Lemma 6.2.3. *Set $G = T_n^\sigma(A,I)$. The restriction of ch^- to $\mathrm{B}(\mathbb{Q}[G],\mathcal{I}_G)$ is the composition*

$$\mathrm{B}(\mathbb{Q}[G],\mathcal{I}_G) \xrightarrow{c} HN(\mathbb{Q}[G],\mathcal{I}_G) \xrightarrow{j} HN(\mathcal{T}_n^\sigma(A,I)) \xrightarrow{tr} HN(A,I)_{\mathrm{norm}}.$$

6.3 The Relative Chern Character for Rational Nilpotent Ideals

When I is a nilpotent ideal in an algebra A, we define $K(A,I)$ to be the homotopy fiber of $B\mathrm{GL}(A)^+ \to B\mathrm{GL}(A/I)^+$; $K(A,I)$ is a connected space whose homotopy groups are the relative K-groups $K_n(A,I)$ for all n. We now cite Theorem 6.1 of [9] for nilpotent I; the proof in [9] is reproduced on page 361 of [7].

Theorem 6.3.1. *If I is a nilpotent ideal in A, there are homotopy fibrations*

$$X(A,I) \to B\mathrm{GL}(A) \to B\mathrm{GL}(A/I)^+,$$
$$X(A) \to X(A,I) \to K(A,I).$$

Moreover, $X(A,I)^+ \xrightarrow{\sim} K(A,I)$ and $\mathbb{Z}[X(A,I)] \xrightarrow{\sim} \mathbb{Z}[K(A,I)]$ are homotopy equivalences (i.e., $X(A,I) \to K(A,I)$ is a homology isomorphism).

Definition 6.3.2. (see [7, 11.4.7]) The *relative Chern character* for the ideal I of a $\mathbb{Q}$-algebra A is the composite of the Hurewicz map, the inverse of the homotopy equivalence of Theorem 6.3.1 and the map ch^- of (6.2.2):

$$ch : K(A, I) \xrightarrow{h} \mathbb{Q}[K(A, I)] \xleftarrow[6.4]{\sim} \mathbb{Q}[X(A, I)] \xrightarrow{ch^-} HN(A, I)_{\mathrm{norm}}.$$

6.4 The Rational Homotopy Theory Character for Nilpotent Ideals

For a nilpotent ideal I, consider the chain subcomplex of the Chevalley–Eilenberg complex $\wedge\mathfrak{gl}(A)$,

$$x(A, I) = \sum_{n,\sigma} \wedge\mathfrak{t}_n^{\sigma}(A, I).$$

Because sw is natural in G, the family of maps $\mathrm{B}(\mathbb{Q}[T_n^{\sigma}(A, I)]) \xrightarrow{sw} \wedge\, \mathfrak{t}_n^{\sigma}(A, I)$ induces a morphism of complexes

$$sw_X : \mathbb{Q}[X(A, I)] \to x(A, I).$$

On the other hand, for each n and σ, the composite of the map $B\rho : \wedge\mathfrak{t}_n^{\sigma}(A, I) \to HN(\mathcal{T}_n^{\sigma}(A, I))$ of Example 5.4 with the inclusion and the trace, i.e., with

$$HN(\mathcal{T}_n^{\sigma}(A, I)) \subset HN(M_n(A)) \xrightarrow{tr} HN(A),$$

sends $\wedge\mathfrak{t}_n^{\sigma}(A, I)$ into the subcomplex $HN(A, I)$. All of these maps are natural in n and σ; by abuse of notation, we write $\mathrm{tr}(B\,\rho)$ for the resulting map:

$$\mathrm{tr}(B\,\rho) : x(A, I) \to HN(A, I).$$

Definition 6.4.1. The map $ch^-_{\mathrm{rht}} : \mathbb{Q}[X(A, I)] \to x(A, I) \to HN(A, I)_{\mathrm{norm}}$ is defined to be $\mathrm{tr}(B\,\rho) \circ sw_X$, followed by $HN(A, I) \to HN(A, I)_{\mathrm{norm}}$. *The rational homotopy theory character* of [7, 11.3.1], cited in (1.2), is the composite $ch' : K(A, I) \to HN(A, I)_{\mathrm{norm}}$ defined by:

$$K(A, I) \xrightarrow{h} \mathbb{Q}(K(A, I)) \xleftarrow{\simeq} \mathbb{Q}[X(A, I)] \xrightarrow{ch^-_{\mathrm{rht}}} HN(A, I)_{\mathrm{norm}}.$$

Remark 6.4.2. We will not need the unnormalized version of ch'. By construction, ch' is the map $K(A, I) \to C^{\lambda}(A, I)[-1]$ of [2, A.13], followed by Connes' operator B.

6.5 Main Theorem

Let I be a nilpotent ideal in a $\mathbb{Q}$-algebra A. The relative Chern character ch of Definition 6.3.2 induces the relative Chern character ch_* of (1.1) on homotopy groups, and the rational homotopy character ch' of Definition 6.4.1 induces the character ch'_* of (1.2) on homotopy groups. Therefore the equality $ch_* = ch'_*$ of (1.3) is an immediate consequence of our main theorem.

Theorem 6.5.1. *The maps ch^- and ch^-_{rht} are naturally chain homotopic. Hence the maps ch and ch' are homotopic for each A and I.*

Proof. We first consider the restriction of ch^- and ch^-_{rht} to $\mathrm{B}(\mathbb{Q}[T_n^\sigma(A,I)])$ for some fixed n and σ. By Lemma 2.2, the restriction of ch^- to $\mathrm{B}(\mathbb{Q}[T_n^\sigma(A,I)])$ is the map $\mathrm{tr}(jc)$; by Example 5.4, there is a natural chain homotopy from jc to $B\,\rho \circ sw$. Since $\mathrm{tr}(B\,\rho)\circ sw$ is the restriction of ch^-_{rht} to $\mathrm{B}(\mathbb{Q}[T_n^\sigma(A,I)])$, the chain homotopies glue together by naturality to give the desired chain homotopy from ch^-_{rht} to ch^-. □

6.6 Naturality

In order to formulate a naturality result for the homotopy between ch and ch', it is necessary to give definitions for the maps ch and ch' which are natural in A and I. Contemplation of Definitions 6.3.2 and 6.4.1 shows that we need to find a natural inverse for the backwards quasi-isomorphism of Theorem 6.3.1. One standard way is to fix a small category of pairs (A, I) (say all pairs with a fixed cardinality bound on A) and consider the global model structure on the category of covariant functors from this category to Simplicial Sets; this will yield naturality with respect to all morphisms in the small category.

Let $K(A,I)'$ be the cofibrant replacement of $K(A,I)$, and factor the backwards as map $\mathbb{Q}[K(A,I)] \overset{\simeq}{\twoheadleftarrow} C \overset{\simeq}{\leftarrowtail} \mathbb{Q}[X(A,I)]$. Then h lifts to a map $h' : K(A,I)' \to C$ and, since $HN(A,I)$ is fibrant, ch^- and ch^-_{rht} both lift to maps $C \to HN(A,I)$. We can then define ch to be the composite $K(A,I)' \xrightarrow{h'} C \xrightarrow{ch^-} HN(A,I)$; ch' is defined similarly using ch^-_{rht}. By Theorem 6.5.1, there is a homotopy between the two maps $C \to HN(A,I)$, and hence between the two maps $K(A,I)' \to HN(A,I)$. Since this is a homotopy of functors from the small category of pairs (A,I) to Simplicial Sets, it provides a natural homotopy between ch and ch' as maps $K(A,I)' \to HN(A,I)$.

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Algebraic Differential Characters of Flat Connections with Nilpotent Residues*

Hélène Esnault

Abstract We construct unramified algebraic differential characters for flat connections with nilpotent residues along a strict normal crossings divisor.

1 Introduction

In [6], Chern and Simons defined classes $\hat{c}_n((E,\nabla)) \in H^{2n-1}(X, \mathbb{R}/\mathbb{Z}(n))$ for $n \geq 1$ and a flat bundle (E,∇) on a $\mathcal{C}^\infty$ manifold X, where $\mathbb{Z}(n) := \mathbb{Z} \cdot (2\pi\sqrt{-1})^n$. Cheeger and Simons defined in [5] the group of real $\mathcal{C}^\infty$ differential characters $\hat{H}^{2n-1}(X, \mathbb{R}/\mathbb{Z})$, which is an extension of global $\mathbb{R}$-valued $2n$-closed forms with $\mathbb{Z}(n)$-periods by $H^{2n-1}(X, \mathbb{R}/\mathbb{Z}(n))$. They show that the Chern–Simons classes extend to classes $\hat{c}_n((E,\nabla)) \in \hat{H}^{2n-1}(X, \mathbb{R}/\mathbb{Z})$, if ∇ is a (not necessarily flat) connection, such that the associated differential form is the Chern form computing the nth Chern class associated to the curvature of ∇.

If X now is a complex manifold, and (E,∇) is a bundle with an algebraic connection, Chern–Simons and Cheeger–Simons invariants give classes $\hat{c}_n((E,\nabla)) \in \hat{H}^{2n-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z})$ with a similar definition of complex $\mathcal{C}^\infty$ differential characters. Those classes have been studied by various authors, and most remarquably, it was shown by A. Reznikov that if X is projective and (E,∇) is flat, then the classes $\hat{c}_n((E,\nabla))$ are torsion, for $n \geq 2$. This answered positively a conjecture by S. Bloch [2], which echoed a similar conjecture by Cheeger–Simons in the $\mathcal{C}^\infty$ category [4, 5].

On the other hand, for X a smooth complex algebraic variety, we defined in [11] the group $AD^n(X)$ of algebraic differential characters. It is easily written as the

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hypercohomology group $\mathbb{H}^n(X, \mathcal{K}_n \xrightarrow{d\log} \Omega^n_X \xrightarrow{d} \Omega^{n+1}_X \to \ldots \xrightarrow{d} \Omega^{2n-1}_X)$, where $\mathcal{K}_n$ is the Zariski sheaf of Milnor K-theory which is unramified in codimension 1. It has the property that it maps to the Chow group $CH^n(X)$, to algebraic closed $2n$-forms which have $\mathbb{Z}(n)$-periods, and to the complex $\mathcal{C}^\infty$ differential characters $\hat{H}^{2n-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z})$. If (E, ∇) is a bundle with an algebraic connection, it has classes $c_n((E,\nabla)) \in AD^n(X)$ which lift both the Chern classes of E in $CH^n(X)$ and $\hat{c}_n((E,\nabla))$. All those constructions are contravariant in $(X, (E, \nabla))$, the differential characters have an algebra structure, and the classes fulfill the Whitney product formula. They admit a logarithmic version: if $j : U \to X$ is a (partial) smooth compactification of U such that $D := X \setminus U$ is a strict normal crossings divisor, one defines the group $AD^n(X, D) = \mathbb{H}^n(X, \mathcal{K}_n \xrightarrow{d\log} \Omega^n_X(\log D) \xrightarrow{d} \Omega^{n+1}_X(\log D) \to \ldots \xrightarrow{d} \Omega^{2n-1}_X(\log D))$. Obviously one has maps $AD^n(X) \to AD^i(X, D) \to AD^n(U)$. The point is that if (E, ∇) extends a pole free connection $(E, \nabla)|_U$ to a connection on X with logarithmic poles along D, then $c_n((E, \nabla)|_U) \in AD^n(U)$ lifts to well defined classes $c_n((E,\nabla)) \in AD^n(X, D)$ with the same functoriality and additivity properties.

If X is a smooth algebraic variety defined over a characteristic 0 field, and $X \supset U$ is a smooth (partial) compactification of U, it is computed in [8, Appendix B] that one can express the Atiyah class [1] of a bundle extension E of $E|_U$ in terms the residues of the extension ∇ of $\nabla|_U$ along $D = X \setminus U$. In particular, if X is projective, ∇ has logarithmic poles along D and has nilpotent residues, one obtains that the de Rham Chern classes of E are zero. If $k = \mathbb{C}$, this implies that the (analytic) Chern classes of E in Deligne–Beilinson cohomology $H^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n))$ lie in the continuous part $H^{2n-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(n))/F^n \subset H^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n))$.

The purpose of this note is to show that this lifting property is in fact stronger.

Theorem 1.1. *Let $X \supset U$ be a smooth (partial) compactification of a complex variety U, such that $D = \sum_j D_j = X \setminus U$ is a strict normal crossings divisor. Let (E, ∇) be a flat connection with logarithmic poles along D such that its residues Γ_j along D_j are all nilpotent. Then the classes $c_n((E,\nabla)) \in AD^n(X, D)$ lift to well defined classes $c_n((E, \nabla, \Gamma)) \in AD^n(X)$, which satisfy the Whitney product formula. More precisely, the classes $c_n((E,\nabla,\Gamma))$ lie in the subgroup $AD^n_\infty(X) = \mathbb{H}^n(X, \mathcal{K}_n \xrightarrow{d\log} \Omega^n_X \xrightarrow{d} \Omega^{n+1}_X \to \ldots \xrightarrow{d} \Omega^{\dim(X)}_X) \subset AD^n(X)$ of classes mapping to 0 in $H^0(X, \Omega^{2n}_X)$.*

They also fulfill some functoriality property, and one can express what their restriction to the various strata of D precisely are.

Let us denote by $\hat{c}_n((E, \nabla, \Gamma))$ the image of $c_n((E, \nabla, \Gamma))$ via the regulator map $AD^n(X) \to \hat{H}^{2n-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z})$ defined in [11] and [10], which restricts to a regulator map $AD^n_\infty(X) \to H^{2n-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(n))$. As an immediate consequence, one obtains the following:

Corollary 1.2. *Let $(X, (E, \nabla, \Gamma))$ be as in the theorem. Then the Cheeger–Chern–Simons classes $\hat{c}_n((E,\nabla)|_U) \in H^{2n-1}(U_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(n)) \subset \hat{H}^{2n-1}(U_{\mathrm{an}}, \mathbb{C}/\mathbb{Z})$ lift to well defined classes $\hat{c}_n((E,\nabla,\Gamma)) \in H^{2n-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(n)) \subset \hat{H}^{2n-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z})$, with the same properties.*

A direct $\mathcal{C}^\infty$ construction of $\hat{c}_n((E,\nabla,\Gamma)) \in H^{2n-1}(X_{\rm an}, \mathbb{C}/\mathbb{Z}(n))$ in the spirit of Cheeger–Chern–Simons has been performed by Deligne and is written in a letter of Deligne to the authors of [13]. It consists in modifying the given connection ∇ by a $\mathcal{C}^\infty$ one form with values in $\mathcal{E}nd(E)$, so as to obtain a (possibly non-flat) connection without residues along D. This modified connection admits classes in $H^{2n-1}(X_{\rm an}, \mathbb{C}/\mathbb{Z}(n)) \subset \hat{H}^{2n-1}(X_{\rm an}, \mathbb{C}/\mathbb{Z})$. That they do not depend on the choice of the one form relies essentially on the argument showing that if ∇ is flat with logarithmic poles along D (and without further conditions on the residues), for $n \geq 2$, the image of $c_n((E,\nabla))$ in $H^0(U, \mathcal{H}^{2n-1}_{DR})$, where $\mathcal{H}^j_{DR}$ is the Zariksi sheaf of j-th de Rham cohomology, in fact lies in the unramified cohomology $H^0(X, \mathcal{H}^{2n-1}_{DR}) \subset H^0(U, \mathcal{H}^{2n-1}_{DR})$. For this, see [3, Theorem 6.1.1]. In the case when D is smooth, Iyer and Simpson constructed the $\mathcal{C}^\infty$ classes $\hat{c}_n((E,\nabla,\Gamma)) \in H^{2n-1}(X_{\rm an}, \mathbb{C}/\mathbb{Z}(n))$ using the existence of the $\mathcal{C}^\infty$ trivialization of the canonical extension after an étale cover, a fact written by Deligne in a letter, together with Deligne's suggestion of considering patched connections. They then show that Reznikov's argument and theorem [14] adapt to those classes. Our note is motivated by the question raised in [13] on the construction in the general case.

Our algebraic construction in Theorem 1.1 relies on the modified splitting principle developed in [9–11] in order to define the classes in $AD^n(X,D)$. Let $q : Q \to X$ be the complete flag bundle of E. A flat connection on E with logarithmic poles along D defines a map of differential graded algebras $\tau : \Omega^\bullet_Q(\log q^{-1}(D)) \to \mathcal{K}^\bullet$ where $\mathcal{K}^i = q^*\Omega^i_X(\log D)$ and $Rq_*\mathcal{K}^\bullet = \Omega^\bullet_X(\log D)$. This defines a partial flat connection $\tau \circ q^*\nabla : q^*E \to q^*\Omega^1_X(\log D) \otimes_{\mathcal{O}_Q} q^*E$ which has the property that it stabilizes all the rank one subquotients of q^*E. On the other hand, the nilpotency of Γ allows to filter the restriction $E|_\Sigma$ to the different strata Σ of D, in such a way that the restriction $\nabla|_\Sigma : E|_\Sigma \to \Omega^1_X(\log D)|_\Sigma \otimes E|_\Sigma$ of the connection stabilizes the filtration $F^\bullet_\Sigma$, and has the following important extra property: the induced flat connection $\nabla|_\Sigma$ on $gr(F^\bullet_\Sigma)$ has values in $\Omega^1_\Sigma(\log \text{rest})$, where rest is the intersection with Σ of the part of D which is transversal to Σ. This fact translates into a sort of stratification of the flag bundle Q, where τ is refined on this stratification and has values in the pull back of $\Omega^1_\Sigma(\text{rest})$. Modulo some geometry in Q, the next observation consists in expressing the sections $\alpha \in \Omega^i_X$ of forms without poles as pairs $\alpha = (\beta \oplus \gamma) \in \Omega^1_X(\log D) \oplus \Omega^1_D$ such that $\beta|_D = \gamma$, where $\Omega^i_D = \Omega^i_X/\Omega^i_X(\log D)(-D) \subset \Omega^i_X(\log D)|_D$. This yields a complex receiving quasi-isomorphically $\Omega^{\geq i}_X$, which is convenient to define the wished classes.

2 Filtrations

Let X be a smooth variety defined over a characteristic 0 field k. Let $D \subset X$ be a strict normal crossings divisor (i.e., the irreducible components are smooth over k), and let (E,∇) be a connection $\nabla : E \to \Omega^1_X(\log D) \otimes E$ with residue Γ defined by the composition

$$\begin{tikzcd} E \arrow[r, "\nabla"] \arrow[dr, "\Gamma"'] & \Omega^1_X(\log D) \otimes_{\mathcal{O}_X} E \arrow[d, "1\otimes \mathrm{res}"] \\ & \nu_* \mathcal{O}_{D^{(1)}} \otimes_{\mathcal{O}_X} E \end{tikzcd} \tag{2.1}$$

where $D^{(1)} = \sqcup_j D_j$, where $\nu\colon D^{(1)} \to D$ is the nomalization of the divisor D. The composition of Γ with the projection $\nu_*\mathcal{O}_{D^{(1)}} \to \mathcal{O}_{D_j}$ defines $\Gamma_j\colon E \to \mathcal{O}_{D_j} \otimes E$ which factors through $\Gamma_j \in \mathrm{End}(\mathcal{O}_{D_j} \otimes E)$. We write

$$\Gamma \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_D \otimes_{\mathcal{O}_X} E, \nu_*\mathcal{O}_{D^{(1)}} \otimes_{\mathcal{O}_X} E). \tag{2.2}$$

Recall that if ∇ is integrable, then

$$[\Gamma_i|_{D_{ij}}, \Gamma_j|_{D_{ij}}] = 0. \tag{2.3}$$

We use the notation $D_I = D_{i_1} \cap \ldots \cap D_{i_r}$ if $I = \{i_1, \ldots, i_r\}$, $D = D^I + \sum_{s\in I} D_s$ with $D^I = \sum_{\ell \notin I} D_\ell$. The connection $\nabla : E \to \Omega^1_X(\log D) \otimes E$ stabilizes $E(-D_j)$, but also $E \otimes \mathcal{I}_{D_I}$, as the Kähler differential on $\mathcal{O}_X$ restricts to a flat $\Omega^1_X(\log(\sum_{s\in I} D_s))$-connection on $\mathcal{I}_{D_I}$, where $\mathcal{I}_{D_I}$ is the ideal sheaf of D_I. Thus ∇ induces a flat connection

$$\nabla_I : E|_{D_I} \to \Omega^1_X(\log D)|_{D_I} \otimes E|_{D_I}. \tag{2.4}$$

One has the diagram

$$\begin{tikzcd} & \Omega^1_{D_j}(\log(D^j \cap D_j)) \otimes E \arrow[d] \arrow[ddr, "1\otimes\Gamma_j"] & \Omega^2_{D_j}(\log(D^j \cap D_j)) \otimes E \arrow[d] \\ E|_{D_j} \arrow[r, "\nabla_j"] \arrow[dr, "\Gamma_j"'] & \Omega^1_X(\log D)|_{D_j} \otimes E \arrow[r, "\nabla_j"] \arrow[d, "\mathrm{res}"] & \Omega^2_X(\log D)|_{D_j} \otimes E \arrow[d, "\mathrm{res}"] \\ & E|_{D_j} & \Omega^1_{D_j}(\log(D^j \cap D_j)) \otimes E \end{tikzcd} \tag{2.5}$$

We define $F^1_j = \mathrm{Ker}(\Gamma_j) \subset E|_{D_j}$. It is a coherent subsheaf. ∇_j sends F^1_j to $\Omega^1_{D_j}(\log D^j \cap D_j) \otimes E$, but because of integrability, the diagram (3) shows that ∇_{D_j} induces a flat connection $F^1_j \to \Omega^1_{D_j}(\log D^j \cap D_j) \otimes F^1_j$.

Claim 2.1. $F^1_j \subset E|_{D_j}$ is a subbundle.

Proof. We use Deligne's Riemann–Hilbert correspondence [7]: the data are defined over a field of finite type k_0 over $\mathbb{Q}$, so embeddable in $\mathbb{C}$, and the question is compatible with the base changes $\otimes_{k_0} k$ and $\otimes_k \mathbb{C}$. So it is enough to consider the question for the underlying analytic connection on a polydisk $(\Delta^*)^r \times \Delta^s$ with coordinates

x_j, where D_j is defined by $x_j = 0$ for $1 \le j \le r$. By the Riemann–Hilbert correspondence, the argument given in [7, p. 86] shows that the analytic connection is isomorphic to $(V \otimes \mathcal{O}, \sum_1^r \Gamma_j^0 \frac{dx_i}{x_i})$, where the matrices Γ_j^0 are constant nilpotent. Thus F_j^1 is isomorphic to $F_j^1(V) \otimes \mathcal{O}_{D_j}$ on the polydisk, with $F_j^1(V) := \mathrm{Ker}(\Gamma_j^0)$, thus is a subbundle. □

We can replace $E|_{D_j}$ by $E|_{D_j}/F_j^1$ in (2.4) and redo the construction. This defines by pull back $F_j^2 \subset E|_{D_j} \twoheadrightarrow \mathrm{Ker}(\Gamma_j : E|_{D_j}/F_j^1 \to E|_{D_j}/F_j^1)$ with $F_j^2 \supset F_j^1$ etc.

Claim 2.2. $F_j^\bullet : F_j^0 = 0 \subset F_j^1 \subset \ldots \subset F_j^i \subset \ldots \subset F_j^{r_j} = E|_{D_j}$ is a filtration by subbundles with a flat $\Omega_X^1(\log D)|_{D_j}$-valued connection, such that the induced connection ∇_j on $gr(F_j^\bullet)$ is flat and $\Omega_{D_j}^1(\log D^j \cap D_j)$-valued. (One can also tautologically say that $F_j^\bullet$ refines the (trivial) filtration on $E|_{D_j}$).

Proof. By construction, the flat $\Omega_X^1(\log D)|_{D_j}$-valued connection ∇_j on $E|_{D_j}$ respects the filtration and induces a flat $\Omega_{D_j}^1(\log D^j \cap D_j)$-connection on $gr(F_j^\bullet)$. We use the transcendental argument to show that this is a filtration by subbundles. With the notations as in the proof of the Claim 2.1, F_j^s is analytically isomorphic to $F_j^s(V) \otimes \mathcal{O}_{D_j}$, where $F_j^1(V) \subset F_j^2(V) \subset \ldots \subset V$ is the filtration on V defined by the successive kernels of Γ_j^0, so $F_j^2(V)$ is the inverse image of $\mathrm{Ker}(\Gamma_j^0)$ on $V/F_j^1(V)$, etc. □

The argument which allows us to construct $F_j^\bullet$ can be used to define successive refinements on all $E|_{D_I}$. We consider now the case $|I| = r \ge 2$. We refine the filtrations $F_J^\bullet|_{D_I}$, which have been constructed inductively, where $J \subset I, |J| < r$. In fact, we do the construction directly on $E|_{D_I}$. We have r linear maps induced by Γ_j

$$\Gamma_j|_{D_I} : E|_{D_I} \xrightarrow{\nabla_I} \Omega_X^1(\log(D))|_{D_I} \otimes E|_{D_I} \to \mathcal{O}_{D_j} \otimes E|_{D_I} = E_{D_I} \tag{2.6}$$

We define

$$F_I^1 = \cap_{j \in I} \mathrm{Ker}(\Gamma_j|_{D_I}) = \cap_{j \in I} F_j^1|_{D_I}. \tag{2.7}$$

Claim 2.3. $F_I^1 \subset E|_{D_I}$ is a subbundle, stabilized by the connection ∇_I, and more precisely one has $\nabla_I : F_I^1 \to \Omega_{D_I}^1(\log(D^I \cap D_I)) \otimes F_I^1$.

Proof. We argue analytically as in the proof of Claim 2.1. With notations as there, the analytic F_I^1 is isomorphic to $F_I^1(V) \otimes \mathcal{O}_{D_I}$. □

Thus ∇_I induces a flat $\Omega_X^1(\log D)|_{D_I}$-valued connection on the quotient $E|_{D_I}/F_I^1$. We define $F_I^2 \supset F_I^1$ in $E|_{D_I}$ to be the inverse image via the projection $E|_{D_I} \to E|_{D_I}/F_I^1$ of $\cap_{j \in I} \mathrm{Ker}(\Gamma_j|_{D_I})$, etc.

Claim 2.4. The filtration $F_I^\bullet : F_I^0 = 0 \subset F_I^1 \subset F_I^2 \subset \ldots \subset F_I^{r_I} = E|_{D_I}$ is a filtration by subbundles, stabilized by ∇_I, such that ∇_I on $gr(F_I^\bullet)$ is a flat

$\Omega^1_{D_I}(\log(D^I \cap D_I))$-valued connection. Furthermore, $F_I^\bullet$ refines all $F_J^\bullet|_{D_I}$ for all $J \subset I, |J| < r$ and one has compatibility of the refinements in the sense that if $K \subset J \subset I$, then the refinement $F_I^\bullet$ of $F_K^\bullet|_{D_I}$ is the composition of the refinements $F_I^\bullet$ of $F_J^\bullet|_{D_J}$ and $F_J^\bullet$ of $F_K^\bullet|_{D_J}$.

Proof. We argue again analytically. Then F_I^s is isomorphic to $F_I^s(V) \otimes \mathcal{O}_{D_I}$ with the same definition. The filtration terminates as finitely many mutually commuting nilpotent endomorphisms on a finite dimensional vector space always have a common eigenvector. □

Definition 2.5. We call $F_I^\bullet$ the canonical filtration of $E|_{D_I}$ associated to ∇, which defines $(gr(F_I^\bullet), \nabla_I, \Gamma_I)$ where ∇_I is the flat $\Omega^1_I(\log(D^I \cap D_I)$-valued connection on $gr(F_I^\bullet)$, and Γ_I is its nilpotent residue along the normalization of $D^I \cap D_I$.

3 τ-Splittings

We first define flag bundles. We set $q_I : Q_I \to D_I$ to be the total flag bundle associated to $E|_{D_I}$. So the pull back of $E|_{D_I}$ to Q_I has a filtration by subbundles such that the associated graded bundle is a sum of rank one bundles ξ_I^s for $s = 1, \ldots, N = \operatorname{rank}(E)$. (It is here understood that $D_\emptyset = X$, and to simplify, we set $q = q_\emptyset : Q \to X, Q_\emptyset = Q$). For $J \subset I$, the inclusion $D_I \to D_J$ defines inclusions $i(J \subset I) : Q_I \to Q_J$. The canonical filtrations associated to ∇ allow one to define partial sections of the q_I. As an illustration, let us assume that $I = \{1\}$, thus D is smooth, and that $F_1^\bullet$ is a total flag, i.e., the $gr(F_1^\bullet)$ is a sum of rank one bundles. Then $F_1^\bullet$ defines a section $D \xrightarrow{\lambda_1^F} Q$.

More generally, let us define $G_I^s = F_I^s / F_I^{s-1}$. We define

$$\begin{tikzcd} Q_I \arrow[d, "q_I"'] & Q_I^F \arrow[l, "\lambda_I^F"'] \arrow[dl, "q_I^F"] \\ D_I & \end{tikzcd} \tag{3.1}$$

using the filtration: recall that $Q_I \to D_I$ is the composition of $\mathbb{P}(E|_{D_I}) \to D_I$ with $\mathbb{P}(E') \to \mathbb{P}(E|_{D_I})$ etc., where $E' \to \mathcal{O}_{\mathbb{P}(E)} \otimes E$ is the rank $(N-1)$ subbundle defined as the kernel to the rank 1 canonical bundle $\xi_I^N(\mathbb{P}(E|_{D_I}))$, the pull back of which to Q_I defines the last graded rank 1 quotient. Then the quotient $E|_{D_I} \to G_I^{r_I}$ defines a map $\mathbb{P}(E|_{D_I}) \leftarrow \mathbb{P}(G_I^{r_I})$ such that the pull back of $\xi_I^N(\mathbb{P}(E|_{D_I}))$ is ξ, where ξ is the canonical rank 1 bundle. Writing $G' \to G_I^{r_I}$ for the kernel, we redo the same construction for E', G' replacing $E|_{D_I}, G_I^{r_I}$ etc. We find this way that the flag bundle of $G_I^{r_I}$ maps to the intermediate step between D_I and Q_I which splits the first M rank 1 bundles, where M is the rank of $G_I^{r_I}$. Then we continue with the pull back of $G_I^{r_I-1}$ to the flag bundle of $G_I^{r_I}$, replacing $G_I^{r_I}$, and E'' replacing E, where E'' on this intermediate step is the rank $N-M$ bundle which is not yet split. All this is very classical.

We have extra closed embeddings $\lambda^F(I \subset J)$ which come from the refinements of the canonical filtrations, which are described in the same way: for $J \subset I$, one has commutative squares

$$\begin{tikzcd} Q_J^F \arrow[d, "q_J^F"'] & Q_I^F \arrow[l, "\lambda^F(I\subset J)"'] \arrow[d, "q_I^F"] \\ D_J & D_I \arrow[l, "i(I\subset J)"] \end{tikzcd} \qquad \begin{tikzcd} Q \arrow[d, "q"'] & Q_I^F \arrow[l, "\mu_I"'] \arrow[d, "q_I^F"] \\ X & D_I \arrow[l, "i_I"] \end{tikzcd} \tag{3.2}$$

where $i_I = i(\emptyset \subset I),\ \mu_I = \lambda(\emptyset \subset I)$.

Recall from [9–11] that ∇ yields a splitting $\tau{:}\Omega^1_Q(\log q^{-1}(D)) \to q^*\Omega^1_X(\log D)$, and that flatness of ∇ implies flatness of τ in the sense that it induces a map of differential graded algebras $(\Omega^\bullet_Q(\log q^{-1}(D)), d) \to (q^*\Omega^\bullet_X(\log D), d_\tau)$ so in particular, $(Rq_*\Omega^{\geq n}_X(\log D), d) = (\Omega^{\geq n}_X(\log D), d)$. Furthermore, the filtration on $q^*(E)$ which defines the rank one subquotient ξ^s has the property that it is stabilized by $\tau \circ q^*\nabla$, and this defines a τ-flat connection $\xi^s \to q^*\Omega^1_X(\log D) \otimes \xi^s$.

The τ-splitting is constructed first on $\mathbb{P}(E)$, with $p : \mathbb{P}(E) \to X$. Then $\tau \circ \nabla$ stabilizes the beginning of the flag $E' \subset$pull-back of E etc. Concretely, the composition $\Omega^1_{\mathbb{P}(E)/X}(1) \xrightarrow{\nabla} \Omega^1_{\mathbb{P}(E)} \otimes E \xrightarrow{\text{projection}} \Omega^1_{\mathbb{P}(E)} \otimes \mathcal{O}_{\mathbb{P}(E)}(1)$ defines the splitting. On the other hand, the flat $\Omega^1_X(\log D)|_{D_I}$-valued connection on $G_I^{r_I}$ has values in $\Omega^1_{D_I}(\log(D_I \cap D^I))$.

When we restrict to $\mathbb{P}(G_I^{r_I})$, then one has a factorization

$$\begin{tikzcd} \Omega^1_{\mathbb{P}(E)}(\log p^{-1}(D)) \otimes \mathcal{O}_{\mathbb{P}(G_I^{r_I})} \arrow[r, "\tau(G_I^{r_I})"] \arrow[dr, "\tau"'] & \Omega^1_{D_I}(\log(D_I \cap D^I)) \otimes \mathcal{O}_{\mathbb{P}(G_I^{r_I})} \arrow[d, "\text{inj}"] \\ & \Omega^1_X(\log D) \otimes \mathcal{O}_{\mathbb{P}(G_I^{r_I})} \end{tikzcd} \tag{3.3}$$

which defines a differential graded algebra $(\Omega^\bullet_{D_I}(\log(D_I \cap D^I)) \otimes \mathcal{O}_{\mathbb{P}(G_I^{r_I})}, d_\tau)$ with total direct image on D_I being $(\Omega^\bullet_{D_I}(\log(D_I \cap D^I)), d)$ and with the property that ξ has a flat connection with values in $\Omega^1_{D_I}(\log(D_I \cap D^I))$, which is compatible with the flat $p^*\Omega^1_X(\log D)$-connection on ξ^N. We can repeat the construction with $D_I \to X$ replaced by $\mathbb{P}(G_I^{r_I}) \to \mathbb{P}(E|_{D_I})$, with $E|_{D_I} \to G_I^{r_I}$ replaced by $E' \to G'$ where $E' = \mathrm{Ker}(E|_{D_I} \otimes \mathcal{O}_{\mathbb{P}(E|_{D_I})} \to \mathcal{O}(1))$ and $G' = \mathrm{Ker}(G_I^{r_I} \to \mathcal{O}(1))$. This splits the next rank 1 piece, one still has the splitting as in (3.3), and we go on till we reach the total flag bundle to $G_I^{r_I}$. Then we continue with the flag bundle to $G_I^{r_I-1}$ etc. We conclude

Claim 3.1. One has a factorization

$$\begin{tikzcd} \mu_I^*\Omega^1_Q(\log q^{-1}(D)) \arrow[r, "\tau_I"] \arrow[dr, "\tau"'] & (q_I^F)^*\Omega^1_{D_I}(\log(D_I \cap D^I)) \arrow[d, "\mathrm{inj}"] \\ & (q_I^F)^*\Omega^1_X(\log D)|_{D_I} \end{tikzcd} \tag{3.4}$$

τ_I defines a differential graded algebra $((q_I^F)^*\Omega^\bullet_{D_I}(\log(D_I \cap D^I)), d_\tau)$ which is a quotient of $\mu_I^*(\Omega^\bullet_Q(\log q^{-1}(D)), d)$. The flat $q^*\Omega^1_X(\log D)$-valued τ-connection on ξ^s, $s = 1, \ldots, N$, restricts via the splitting τ_I, to a flat $(q_I^F)^*\Omega^1_{D_I}(\log(D^I \cap D_I))$-valued τ-connection on $(\xi_I^F)^s = \mu_I^*\xi^s$.

Definition 3.2. On Q we define the complex of sheaves

$$A(n) = A^n \to A^{n+1} \to \ldots$$

with

$$\begin{aligned} A^i &= B^i \oplus C^i \\ B^i &= \oplus_I (\mu_I)_*(q_I^F)^*\Omega^i_{D_I}(\log(D^I \cap D_I)), \\ C^i &= \oplus_{I \neq \emptyset}(\mu_I)_*(q_I^F)^*\Omega^{i-1}_X(\log D)|_{D_I}, \end{aligned}$$

where $C^i = 0$ for $i = n$. The differentials D_τ are defined as follows: $(\oplus_I \beta_I, \oplus_I \gamma_I)$, where $\beta_I \in (\mu_I)_*(q_I^F)^*\Omega^i_{D_I}(\log(D^I \cap D_I))$, $\gamma_I \in (\mu_I)_*(q_I^F)^*\Omega^{i-1}_X(\log D)|_{D_I}$ is sent to

$$\begin{aligned} &\oplus_I d_\tau \beta_I \in (\mu_I)_*(q_I^F)^*\Omega^{i+1}_{D_I}(\log(D^I \cap D_I)), \\ &\qquad \oplus_I d_\tau \gamma_I + (-1)^i(\mu_I^*\beta - \beta_I) \in (\mu_I)_*(q_I^F)^*\Omega^i_X(\log D)|_{D_I}. \end{aligned}$$

Let $\mathcal{K}_n$ be the image of the Zariski sheaf of Milnor K-theory into Milnor K-theory $K_n(k(X))$ of the function field (which is the same as $\mathrm{Ker}(K_n(k(X)) \to \oplus K_{n-1}(\kappa(x)))$ on all codimension 1 points $x \in X$). The τ-differential defines $d_\tau \log : \mathcal{K}_n \to A^n = B^n$ ($C^n = 0$). The image in A^n is D_τ-flat. Thus this defines $d_\tau \log : \mathcal{K}_n \to A(n)[-1]$.

Definition 3.3. We define $\mathcal{K}_n\Omega^\infty_Q$ to be the complex $\mathcal{K}_n \xrightarrow{d_\tau \log} A(n)[-1]$ and $\mathcal{K}_n\Omega^\infty_Q \supset (\mathcal{K}_n\Omega^\infty_Q)_0$ to be the subcomplex $\mathcal{K}_n \xrightarrow{d_\tau \log} A^n_{D_\tau}$, where $A^n_{D_\tau}$ means the subsheaf of D_τ-closed sections.

Lemma 3.4. *The τ-connections on $(\xi_I^F)^s$ define a class $\xi^s(\nabla) \in \mathbb{H}^1(Q, (\mathcal{K}_1\Omega^\infty_Q)_0)$ with the property that the image of $\xi^s(\nabla)$ in $H^1(Q, \mathcal{K}_1)$ is $c_1(\xi^s)$.*

Proof. The cocycle of the class $\xi^s(\nabla)$ results from the Claim 3.1. Write $g^s_{\alpha\beta}$ for a $\mathcal{K}_1$-valued 1-cocyle for ξ^s. Then the flat τ-connection on ξ^s is defined by local sections ω^s_α in $q^*\Omega^1_X(\log D)$ which are d_τ flat for $d_\tau : q^*\Omega^1_X(\log D) \to q^*\Omega^2_X(\log D)$. So the cocyle condition reads $d_\tau \log g^s_{\alpha\beta} = \delta(\omega^s)_{\alpha\beta}$ where δ is the Cech differential. The Claim 3.1 implies then that $\mu_I^*(\omega^s_\alpha) \in (q_I^F)^*\Omega^1_{D_I}(\log(D_I \cap D^I))$, is τ-flat and one has $d_\tau \log \mu_I^*(g^s_{\alpha\beta}) = \delta\mu_I^*(\omega^s)_{\alpha\beta}$. So the class $(\xi_I^F)^s$ is defined by the Cech cocycle $(g^s_{\alpha\beta}, \mu_I^*\omega^s \oplus 0)$, with $\mu_I^*\omega^s \in B^1, 0 \in C^1$. $\square$

We define a product

$$(\mathcal{K}_m\Omega^\infty_Q)_0 \times (\mathcal{K}_n\Omega^\infty_Q)_0 \xrightarrow{\cup} (\mathcal{K}_{m+n}\Omega^\infty_Q)_0 \tag{3.5}$$

by using the formulae defined in [11, Definition 2.1.1], that is

$$x \cup y = \begin{cases} \{x, y\} & x \in \mathcal{K}_m, y \in \mathcal{K}_n \\ d_\tau \log x \wedge y \oplus d_\tau \log x \wedge y & x \in \mathcal{K}_m, y \in (B^n \oplus C^n)_{D_\tau} \\ 0 & \text{else.} \end{cases} \tag{3.6}$$

The product is well defined.

Definition 3.5. We define $c_n(q^*(E, \nabla, \Gamma)) \in \mathbb{H}^n(Q, \mathcal{K}_n\Omega^\infty_Q))$ to be the image via the map $\mathbb{H}^n(Q, (\mathcal{K}_n\Omega^\infty_Q)_0) \to \mathbb{H}^n(Q, \mathcal{K}_n\Omega^\infty_Q)$ of

$$\sum_{s_1 < s_2 \ldots < s_n} \xi^{s_1}(\nabla) \cup \cdots \cup \xi^{s_n}(\nabla).$$

Definition 3.6. On X we define the complex of sheaves

$$A_X(n) = A^n_X \to A^{n+1}_X \to \ldots$$

with

$$\begin{aligned} A^i_X &= B^i_X \oplus C^i_X \\ B^i_X &= \oplus_I (i_I)_*\Omega^i_{D_I}(\log(D^I \cap D_I)), \\ C^i_X &= \oplus_{I \neq \emptyset}(i_I)_*\Omega^{i-1}_X(\log D)|_{D_I}, \end{aligned}$$

where $C^i_X = 0$ for $i = n$. The differentials D_X are defined as follows: $(\oplus_I \beta_I, \oplus_I \gamma_I)$, where $\beta_I \in (i_I)_*\Omega^i_{D_I}(\log(D^I \cap D_I)), \gamma_I \in (i_I)_*\Omega^{i-1}_X(\log D)|_{D_I}$ is sent to

$$\begin{aligned} &\oplus_I d\beta_I \in (i_I)_*\Omega^{i+1}_{D_I}(\log(D^I \cap D_I)), \\ &\qquad \oplus_I d\gamma_I + (-1)^i(i_I^*\beta - \beta_I) \in (i_I)_*\Omega^i_X(\log D)|_{D_I}, \end{aligned}$$

where the differentials d_τ are the τ differentials in the various differential graded algebras $\Omega^\bullet_{D_I}(\log(D^I \cap D_I))$.

One has an injective morphism of complexes

$$\iota : \Omega_X^{\geq n} \to A_X^{\geq n} \tag{3.7}$$

sending $\alpha \in \Omega_X^i$ to $i_I^*\alpha \oplus 0$.

Proposition 3.7. *The morphism ι is a quasi-isomorphism. Furthermore, one has $Rq_*A(n) = A_X(n)$.*

Proof. We start with the second assertion: since μ_I is a closed embedding, one has $R(\mu_I)_* = (\mu_I)_*$ on coherent sheaves. Thus by the commutativity of the diagram (3.2), and the fact that $\mathcal{O}$ on the flag varieties is relatively acyclic, one has $Rq_*(R\mu_I)_*(q_I^F)^*\mathcal{E} = (i_I)_*\mathcal{E}$ for a locally free sheaf $\mathcal{E}$ on D_I. This shows the second statement. We show the first assertion. We first show that the 0th cohomology sheaf of $A_X(n)$ is $(\Omega_X^n)_d$. The condition $D(\beta, \beta_I) = 0$ means $d\beta = d\beta_I = 0$ and $i_I^*\beta = \beta_I$. Thus $\beta \in \Omega_X^n$ and $d\beta = 0$. Assume now $i \geq n+1$. Then modulo $DA^{i-1}(n)$, $((\beta, \beta_I), \gamma_I)$ is equivalent to $((\beta, \beta_I + (-1)^{i-1}d\gamma_I), 0)$. So we are back to the computation as in the case $i = n$ and $\mathrm{Ker}(D)$ on $B^i \oplus 0$ is $\mathrm{Ker}(d)$ on Ω_X^i. On the other hand, by the same reason, $D(B^{i-1} \oplus C^{i-1}) = D(B^{i-1} \oplus 0)$, and $D(B^{i-1} \oplus 0) \cap (B^i \oplus 0) = d(\Omega_X^i)$. This finishes the proof. □

Proposition 3.8. *The map $q^*: AD^n(X)_\infty = \mathbb{H}^n(X, \mathcal{K}_n \xrightarrow{d\log} \Omega_X^n \xrightarrow{d} \ldots \xrightarrow{d} \Omega^{\dim X})$ $\to \mathbb{H}^n(Q, \mathcal{K}_n\Omega^\infty)$ is injective. The classes $c_n((q^*(E, \nabla, \Gamma)) \in \mathbb{H}^n(Q, \mathcal{K}_n\Omega^\infty)$ in Definition 3.5 are of the shape $q^*c_n((E, \nabla, \Gamma))$ for uniquely defined classes $c_n((E, \nabla, \Gamma)) \in \mathbb{H}^n(X, \mathcal{K}_n \xrightarrow{d\log} \Omega_X^n \xrightarrow{d} \ldots \to \Omega_X^{\dim X})$.*

Proof. One has a commutative diagram of long exact sequences

$$\begin{tikzcd}
H^{n-1}(Q, \mathcal{K}_n) \arrow[r] & \mathbb{H}^{n-1}(A(n)) \arrow[r] & \mathbb{H}^n(\mathcal{K}_n\Omega_Q^\infty) \arrow[r] & H^n(Q, \mathcal{K}_n) \\
H^{n-1}(X, \mathcal{K}_n) \arrow[r] \arrow[u, "\mathrm{inj}"] & \mathbb{H}^{n-1}(A(n)_X) \arrow[r] \arrow[u, "="] & \mathbb{H}^n(\mathcal{K}_n\Omega_X^\infty) \arrow[r] \arrow[u] & H^n(X, \mathcal{K}_n) \arrow[u, "\mathrm{inj}"]
\end{tikzcd} \tag{3.8}$$

where $\mathcal{K}_n\Omega_X^\infty = \mathcal{K}_n \xrightarrow{d\log} \Omega_X^n \xrightarrow{d} \ldots \xrightarrow{d} \Omega_X^{\dim X}$. We write $H^i(Q, \mathcal{K}_j) = H^i(X, \mathcal{K}_j) \oplus$ rest, where the rest is divisible by the classes of powers of the $[\xi^s] \in H^1(Q, \mathcal{K}_1)$, with coefficients in some $H^a(X, \mathcal{K}_b)$. But $[\xi^s]$ comes by Lemma 3.4 from a class $\xi^s(\nabla) \in \mathbb{H}^1(Q, (\mathcal{K}_1\Omega_Q^\infty)_0)$. Consequently, the image of rest in $\mathbb{H}^i(A(n))$ dies. We conclude that one has an exact sequence $0 \to \mathbb{H}^n(\mathcal{K}_n\Omega_X^\infty) \to \mathbb{H}^n(\mathcal{K}_n\Omega_Q^\infty) \to \mathbb{H}^n(X, R^\bullet q_*\mathcal{K}_n/q_*\mathcal{K}_n)$. By the standard splitting principle for Chow groups, one has $H^n(Q, \mathcal{K}_n)/H^n(X, \mathcal{K}_n) = \mathbb{H}^n(X, R^\bullet q_*\mathcal{K}_n/q_*\mathcal{K}_n)$, and

$$\sum_{s_1 < s_2 \ldots < s_n} c_1(\xi^{s_1}) \cup \cdots \cup c_1(\xi^{s_n}) \in \mathrm{Im}(CH^n(X) \subset CH^n(Q)).$$

By Lemma 3.4, $\xi^s(\nabla) \in \mathbb{H}^1(Q, (\mathcal{K}_1\Omega^\infty)_0)$ maps to $c_1(\xi^s) \in H^1(Q, \mathcal{K}_1)$. Thus we conclude that $c_n(q^*(E, \nabla, \Gamma)) \in \mathrm{Im}(\mathbb{H}^n(\mathcal{K}_n\Omega^\infty_X)) \subset \mathbb{H}^n(\mathcal{K}_n\Omega^\infty_Q)$. This finishes the proof. □

Theorem 3.9. *Let $X \supset U$ be a smooth (partial) compactification of a variety U defined over a characteristic 0 field, such that $D = \sum_j D_j = X \setminus U$ is a strict normal crossings divisor. Let (E, ∇) be a flat connection with logarithmic poles along D such that its residues Γ_j along D_j are all nilpotent. Then the classes $c_n((E, \nabla)) \in AD^n(X, D)$ lift to well defined classes $c_n((E, \nabla, \Gamma)) \in AD^n(X)$. They are functorial: if $f : Y \to X$ with Y smooth, such that $f^{-1}(D)$ is a normal crossings divisor, étale over its image $\subset D$, then $f^*c_n((E, \nabla, \Gamma)) = c_n(f^*(E, \nabla, \Gamma))$ in $AD^n(Y)$. If $D' \supset D$ is a normal crossings divisor and ∇' is the connection ∇, but considered with logarithmic poles along D', thus with trivial residues along the components of $D' \setminus D$, then $c_n((E, \nabla, \Gamma)) = c_n((E, \nabla', \Gamma'))$. The classes $c_n((E, \nabla, \Gamma))$ satisfy the Whitney product formula. In addition, $c_n((E, \nabla, \Gamma))$ lies in the subgroup $AD^n_\infty(X) = \mathbb{H}^n(X, \mathcal{K}_n \xrightarrow{d\log} \Omega^n_X \xrightarrow{d} \Omega^{n+1}_X \to \ldots \xrightarrow{d} \Omega^{\dim(X)}_X) \subset AD^n(X)$ of classes mapping to 0 in $H^0(X, \Omega^{2n}_X)$. The restriction to $AD^n_\infty(D_I)$ of $c_n((E, \nabla, \Gamma))$ is $c_n((gr(F^\bullet_I), \nabla_I, \Gamma_I))$ where $(gr(F^\bullet_I), \nabla_I, \Gamma_I)$ is the canonical filtration (see Claim 2.4 and Definition 2.5).*

Proof. The construction is the Proposition 3.8. We discuss functoriality. If f is as in the theorem, then the filtrations $F^\bullet_I$ for (E, ∇) restrict to the filtration for $f^*(E, \nabla)$. Whitney product formula is proven exactly as in [9, 2.17, 2.18] and [10, Theorem 1.7], even if this is more cumbersome, as we have in addition to follow the whole tower of $F^\bullet_I$. Finally, the last property follows immediately from the definition of $\xi^s(\nabla)$ in Lemma 3.4. □

Theorem 3.10. *Assume given $k \subset \mathbb{C}$ and Γ is nilpotent. Then the classes $\hat{c}_n((E, \nabla)) \in H^{2n}((X \setminus D)_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(n))$ defined in [9], come from well defined classes $\hat{c}_n((E, \nabla, \Gamma)) \in H^{2n-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(n))$. Furthermore $\hat{c}_n((E, \nabla, \Gamma))$ fulfill the same functoriality, additivity, restriction, and enlargement of ∇ properties as $c_n((E, \nabla, \Gamma)) \in AD^n_\infty(X)$.*

Proof. We just have to use the regulator map $AD^n(X) \to H^{2n-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(n))$, which is an algebra homomorphism, and which defined in [10, Theorem 1.7]. Of course we can also follow the same construction directly in the analytic category. □

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Norm Varieties and the Chain Lemma (After Markus Rost)

Christian Haesemeyer and Chuck Weibel

Abstract This paper presents Markus Rost's proofs of two of his theorems, the Chain Lemma and the Norm Principle. This completes the published verification of the Bloch–Kato conjecture.

The goal of this paper is to present proofs of two results of Markus Rost, the Chain Lemma 0.1 and the Norm Principle 0.3. These are the steps needed to complete the published verification of the *Bloch–Kato conjecture*, that the norm residue maps are isomorphisms $K_n^M(k)/p \xrightarrow{\simeq} H_{et}^n(k,\mathbb{Z}/p)$ for every prime p, every n and every field k containing $1/p$. Throughout this paper, p is a fixed prime, and k is a field of characteristic 0, containing the p-th roots of unity. We fix an integer $n \geq 2$ and an n-tuple $(a_1,\ldots,a_n)$ of units in k, such that the symbol $\{\underline{a}\} = \{a_1,\ldots,a_n\}$ is nontrivial in the Milnor K-group $K_n^M(k)/p$.

Associated to this data are several notions. A field F over k is a *splitting field* for $\{\underline{a}\}$ if $\{\underline{a}\}_F = 0$ in $K_n^M(F)/p$. A variety X over k is called a *splitting variety* if its function field is a splitting field; X is *p-generic* if any splitting field F has a finite extension E/F of degree prime to p with $X(E) \neq \emptyset$. A *norm variety* for $\{\underline{a}\}$ is a smooth projective p-generic splitting variety for $\{\underline{a}\}$ of dimension $p^{n-1}-1$.

The following sequence of theorems reduces the Bloch–Kato conjecture to the Chain Lemma 0.1 and the Norm Principle 0.3; the notion of a *Rost variety* is defined in 0.5; the definition of a *Rost motive* is given in [16] and [15], and will not be needed in this paper.

(0) The Chain Lemma 0.1 and the Norm Principle 0.3 hold; these assertions are due to Rost, and are proven in this article.
(1) Given (0), Rost varieties exist; this is proven in [12, p. 253]. A careful statement is given in Theorem 0.7.
(2) If Rost varieties exist then Rost motives exist; this is proven in [15].
(3) If Rost motives exist then Bloch–Kato is true; this is proven in [14] and again in [16].

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Here is the statement of the Chain Lemma, which we quote from [12, 5.1] and prove in Sect. 5.

Theorem 0.1 (Rost's Chain Lemma). *Let $\{\underline{a}\} \in K_n^M(k)/p$ be a nontrivial symbol, where k is a field. Then there exists a smooth projective cellular variety S/k and a collection of invertible sheaves $J = J_1, J_1', \dots, J_{n-1}, J_{n-1}'$ equipped with nonzero p-forms $\gamma = \gamma_1, \gamma_1' \dots, \gamma_{n-1}, \gamma_{n-1}'$ satisfying the following conditions:*

1. $\dim S = p(p^{n-1} - 1) = p^n - p$;
2. $\{a_1, \dots, a_n\} = \{a_1, \dots, a_{n-2}, \gamma_{n-1}, \gamma_{n-1}'\} \in K_n^M(k(S))/p$ $\{a_1, \dots, a_{i-1}, \gamma_i\} = \{a_1, \dots, a_{i-2}, \gamma_{i-1}, \gamma_{i-1}'\} \in K_i^M(k(S))/p$ *for* $2 \le i < n$. *In particular,* $\{a_1, \dots, a_n\} = \{\gamma, \gamma_1', \dots, \gamma_{n-1}'\} \in K_n^M(k(S))/p$;
3. $\gamma \notin \Gamma(S, J)^{\otimes(-p)}$, *as is evident from (2);*
4. *For any* $s \in V(\gamma_i) \cup V(\gamma_i')$, *the field* $k(s)$ *splits* $\{a_1, \dots, a_n\}$;
5. $I(V(\gamma_i)) + I(V(\gamma_i')) \subseteq p\mathbb{Z}$ *for all* i, *as follows from (4);*
6. $\deg(c_1(J)^{\dim S})$ *is relatively prime to* p.

Without loss of generality, we may and shall assume that k contains the p-th roots of unity. (We explain p-forms in Sect. 1 and show in Definition 1.6 how they may be used to define elements of $K_n^M(k(S))/p$. For any V of finite type over k, $I(V)$ is defined to be the subgroup of $\mathbb{Z}$ generated by the $[k(v) : k]$ for all closed $v \in V$.)

Rost's Norm Principle concerns the group $\overline{A}_0(X, \mathcal{K}_1)$, which we now define.

Definition 0.2. (Rost, [6]) For any regular scheme X, the group $A_0(X, \mathcal{K}_1)$ is defined to be the group generated by symbols $[x, \alpha]$, where x is a closed point of X and $\alpha \in k(x)^\times$, modulo the relations (i) $[x, \alpha][x, \alpha'] = [x, \alpha\alpha']$ and (ii) for every point y of dimension 1 the image of the tame symbol $K_2(k(y)) \to \oplus k(x)^\times$ is zero.

The functor $A_0(X, \mathcal{K}_1)$ is covariant in X for proper maps, because it is isomorphic to the motivic homology group $H_{-1,-1}(X) = \operatorname{Hom}_{DM}(\mathbb{Z}, M(X)(1)[1])$ (see [12, 1.1]). It is also the K-cohomology group $H^d(X, \mathcal{K}_{d+1})$, where $d = \dim(X)$.

The reduced group $\overline{A}_0(X, \mathcal{K}_1)$ is defined to be the quotient of $A_0(X, \mathcal{K}_1)$ by the difference of the two projections from $A_0(X \times X, \mathcal{K}_1)$. As observed in [12, 1.2], there is a well defined map $N : \overline{A}_0(X, \mathcal{K}_1) \to k^\times$ sending $[x, \alpha]$ to the norm of α.

A field is called *p-special* if p divides the order of every finite field extension.

Theorem 0.3 (Norm Principle). *Suppose that k is a p-special field and that X is a norm variety for some nontrivial symbol $\{\underline{a}\}$. Let $[z, \beta] \in \overline{A}_0(X, \mathcal{K}_1)$ be such that $[k(z) : k] = p^\nu$ for $\nu > 1$. Then there exists a point $x \in X$ with $[k(x) : k] = p$ and $\alpha \in k(x)^\times$ such that $[z, \beta] = [x, \alpha]$ in $\overline{A}_0(X, \mathcal{K}_1)$.*

We will prove the Norm Principle 0.3 in Sect. 9.

Our proofs of these two results are based on Rost's 1998 preprint [7], his web site [8] and Rost's lectures [9] in 1999–2000 and 2005. The idea for writing up these notes in publishable form originated during his 2005 course, and was reinvigorated by conversations with Markus Rost at the Abel Symposium 2007 in Oslo. As usual, all mistakes in this paper are the responsibility of the authors.

0.1 Rost Varieties

In the rest of this introduction, we explain how 0.1 and 0.3 imply the problematic Theorem 0.7, and hence complete the proof of the Bloch–Kato conjecture. We first recall the notions of a ν_i-variety and a Rost variety.

Let X be a smooth projective variety of dimension $d > 0$. Recall from [5, Sect. 16] that there is a characteristic number $s_d : K_0(X) \to \mathbb{Z}$ corresponding to the symmetric polynomial $\sum t_j^d$ in the Chern roots t_j of a bundle; we write $s_d(X)$ for s_d of the tangent bundle T_X. When $d = p^\nu - 1$, we know that $s_d(X) \equiv 0 \mod p$; see [5, 16.6 and 16-E], [11, pp. 128–129] or [1, II.7].

Definition 0.4. (see [12, 1.20]) A ν_{n-1}-*variety* over a field k is a smooth projective variety X of dimension $d = p^{n-1} - 1$, with $s_d(X) \not\equiv 0 \mod p^2$.

For example, $s_d(\mathbb{P}^d) = d + 1$ by [5, 16.6]. Thus the projective space $\mathbb{P}^{p-1}$ is a ν_1-variety, and so is any Brauer–Severi variety of dimension $p - 1$. In Sect. 8, we will show that the bundle $\mathbb{P}(\mathcal{A})$ over S is a ν_n-variety.

Definition 0.5. A *Rost variety* for a sequence $\underline{a} = (a_1, \dots, a_n)$ of units in k is a ν_{n-1}-variety such that: $\{a_1, \dots, a_n\}$ vanishes in $K_n^M(k(X))/p$; for each $i < n$ there is a ν_i-variety mapping to X; and the motivic homology sequence

$$H_{-1,-1}(X \times X) \xrightarrow{\pi_0^* - \pi_1^*} H_{-1,-1}(X) \to H_{-1,-1}(k) \quad (= k^\times).$$

is exact. Part of Theorem 0.7 states that Rost varieties exist for every $\underline{a}$.

Remark 0.6. Rost originally defined a *norm variety* for $\{\underline{a}\}$ to be a projective splitting variety of dimension p^{n-1} which is a ν_{n-1}-variety. (See [9, 10/20/99].) Theorem 0.7(2) says that our definition agrees with Rost's when k is p-special.

Here is the statement of Theorem 0.7, quoted from [12, 1.21]. It assumes that the Bloch–Kato conjecture holds for $n - 1$.

Theorem 0.7. *Let $n \geq 2$ and $0 \neq \{\underline{a}\} = \{a_1, \dots, a_n\} \in K_n^M(k)/p$. Then:*

(0) There exists a geometrically irreducible norm variety for $\{\underline{a}\}$.

Assume further that k is p-special. If X is a norm variety for $\{\underline{a}\}$, then:

(1) X is geometrically irreducible.

(2) X is a ν_{n-1}-variety.

(3) Each element of $\overline{A}_0(X, \mathcal{K}_1)$ is of the form $[x, \alpha]$, where $x \in X$ is a closed point of degree p and $\alpha \in k(x)^\times$. (See Definition 0.2 above).

The construction of geometrically irreducible norm varieties was carried out in [12, pp. 254–256]; this proves part (0) of Theorem 0.7. Part (1) was proven in [12, 5.4]. Part (2) was proven in [12, 5.2], assuming Rost's Chain Lemma (see 0.1), and part (3) was proven in [12, p. 271], assuming not only the Chain Lemma but also the Norm Principle (see 0.3 above).

As stated in the introduction of [12], the construction of norm varieties and the proof of Theorem 0.7 are part of an inductive proof of the Bloch–Kato conjecture. We point out that in the present paper, the inductive assumption (that the Bloch–Kato conjecture for $n-1$ holds) is never used. It only appears in [12] to prove that the candidates for norm varieties constructed there are p-generic splitting varieties. (However, the Norm Principle 0.3 is itself a statement about norm varieties.) In particular, the Chain Lemma 0.1 holds in all degrees independently of the Bloch–Kato conjecture.

1 Forms on Vector Bundles

We begin with a presentation of some well known facts about p-forms.

If V is a vector space over a field k, a *p-form* on V is a symmetric p-linear function on V, i.e., a linear map $\phi : \mathrm{Sym}^p(V) \to k$. It determines a *$p$-ary form*, i.e., a function $\varphi : V \to k$ satisfying $\varphi(\lambda v) = \lambda^p \varphi(v)$, by $\varphi(v) = \phi(v, v, \dots, v)$. If $p!$ is invertible in k, p-linear forms are in 1–1 correspondence with p-ary forms.

If $V = k$ then every p-form may be written as $\varphi(\lambda) = a\lambda^p$ or $\phi(\lambda_1, \dots) = a\prod \lambda_i$ for some $a \in k$. Up to isometry, non-zero 1-dimensional p-forms are in 1–1 correspondence with elements of $k^\times/k^{\times p}$. Therefore an n-tuple of forms φ_i determine a well-defined element of $K_n^M(k)/p$ which we write as $\{\varphi_1, \dots, \varphi_n\}$.

Of course the notion of a p-form on a projective module over a commutative ring makes sense, but it is a special case of p-forms on locally free modules (algebraic vector bundles), which we now define.

Definition 1.1. If $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module over a scheme X then a p-form on $\mathcal{E}$ is a symmetric p-linear function on $\mathcal{E}$, i.e., a linear map $\phi : \mathrm{Sym}^p(\mathcal{E}) \to \mathcal{O}_X$. If $\mathcal{E}$ is invertible, we will sometimes identify the p-form with the diagonal p-ary form $\varphi = \phi \circ \Delta : \mathcal{E} \to \mathcal{O}_X$; locally, if v is a section generating $\mathcal{E}$ then the form is determined by $a = \varphi(v)$: $\varphi(tv) = a\,t^p$.

The geometric vector bundle over a scheme X whose sheaf of sections is $\mathcal{E}$ is $\mathbb{V} = \mathbf{Spec}(S^*(\mathcal{E}^\vee))$, where $\mathcal{E}^\vee$ is the dual $\mathcal{O}_X$-module of $\mathcal{E}$. We will sometimes describe p-forms in terms of $\mathbb{V}$.

The projective space bundle associated to $\mathcal{E}$ is $\pi : \mathbb{P}(\mathcal{E}) = \mathbf{Proj}(S^*) \to X$, $S^* = S^*(\mathcal{E}^\vee)$. The tautological line bundle on $\mathbb{P}(\mathcal{E})$ is $\mathbb{L} = \mathbf{Spec}(\mathrm{Sym}\,\mathcal{O}(1))$, and its sheaf of sections is $\mathcal{O}(-1)$. The multiplication $S^* \otimes \mathcal{E}^\vee \to S^*(1)$ in the symmetric algebra induces a surjection of locally free sheaves $\pi^*(\mathcal{E}^\vee) \to \mathcal{O}(1)$ and hence an injection $\mathcal{O}(-1) \to \pi^*(\mathcal{E})$; this yields a canonical morphism $\mathbb{L} \to \pi^*(\mathbb{V})$ of the associated geometric vector bundles.

Definition 1.2. Any p-form $\psi : \mathrm{Sym}^p(\mathcal{E}) \to \mathcal{O}_X$ on $\mathcal{E}$ induces a canonical p-form ϵ on the tautological line bundle $\mathbb{L}$:

$$\epsilon : \mathcal{O}(-p) = \mathrm{Sym}^p(\mathcal{O}(-1)) \to \mathrm{Sym}^p(\pi^*\mathcal{E}) = \pi^*\mathrm{Sym}^p(\mathcal{E}) \xrightarrow{\psi} \pi^*\mathcal{O}_X = \mathcal{O}_{\mathbb{P}(\mathcal{E})}.$$

We will use the following notational shorthand. For a scheme Z, a point q on some Z-scheme and a vector bundle V on Z we write $V|_q$ for the fiber of V at q, i.e., the $k(q)$ vector space $q^*(V)$ for $q \to Z$. If φ is a p-form on a line bundle L, $0 \neq u \in L|_q$ and $a = \varphi|_q(u^p)$, then $\varphi|_q : (L|_q)^p \to k(q)$ is the p-form $\varphi|_q(tu^p) = at^p$.

Example 1.3. Given an invertible sheaf L on X, and a p-form φ on L, the bundle $V = \mathcal{O} \oplus L$ has the p-form $\psi(t,u) = t^p - \varphi(u)$. Then $\mathbb{P}(V) \to X$ is a $\mathbb{P}^1$-bundle, and its tautological line bundle $\mathbb{L}$ has the p-form ϵ described in 1.2.

Over a point in $\mathbb{P}(V)$ of the form $\infty = (0 : u)$, the p-form on $\mathbb{L}|_\infty$ is $\epsilon(0, \lambda u) = -\lambda^p \varphi(u)$. If $q = (1 : u)$ is any other point on $\mathbb{P}(V)$ then the 1-dimensional subspace $\mathbb{L}|_q$ of the vector space $V|_q$ is generated by $v = (1, u)$ and the p-form $\epsilon|_q$ on $\mathbb{L}|_q$ is determined by $\epsilon(v) = \psi(1, u) = 1 - \varphi(u)$ in the sense that $\epsilon(\lambda\, v) = \lambda^p(1 - \varphi(u))$.

One application of these ideas is the formation of the sheaf of Kummer algebras associated to a p-form. Recall that if L is a line bundle then the $(p-1)$st symmetric power of $\mathbb{P}(\mathcal{O} \oplus L)$ is $\mathrm{Sym}^{p-1}\mathbb{P}(\mathcal{O} \oplus L) = \mathbb{P}(\mathcal{A}(L))$, where $\mathcal{A}(L) = \bigoplus_{i=0}^{p-1} L^{\otimes i}$.

Definition 1.4. If L is a line bundle on X, equipped with a p-form ϕ, the *Kummer algebra* $\mathcal{A}_\phi(L)$ is the vector bundle $\mathcal{A}(L) = \bigoplus_{i=0}^{p-1} L^{\otimes i}$ regarded as a bundle of algebras as in [12, 3.11]; locally, if u is a section generating L then $\mathcal{A}(L) \cong \mathcal{O}[u]/(u^p - \phi(u))$. If $x \in X$ and $a = \phi|_x(u)$ then the $k(x)$-algebra $\mathcal{A}|_x$ is the Kummer algebra $k(x)(\sqrt[p]{a})$, which is a field if $a \notin k(x)^p$ and $\prod k(x)$ otherwise.

Since the norm on $\mathcal{A}_\phi(L)$ is given by a homogeneous polynomial of degree p, we may regard the norm as a map from $\mathrm{Sym}^p \mathcal{A}_\phi(L)$ to $\mathcal{O}$. The canonical p-form ϵ on the tautological line bundle $\mathbb{L}$ on the projective bundle $\mathbb{P} = \mathbb{P}(\mathcal{A}(L))$, given in 1.2, agrees with the natural p-form:

$$\mathbb{L}^{\otimes p} \to \mathrm{Sym}^p \pi^* \mathcal{A}(L) \xrightarrow{N} \mathcal{O}_{\mathbb{P}},$$

where $\pi : \mathbb{P} \to X$ is the structure map and the canonical inclusion of $\mathbb{L}$ into $\pi^*(\mathcal{A}(L)) = \oplus_0^{p-1} \pi^* L^{\otimes i}$ induces the first map.

Recall from 1.2 and 1.4 that ϕ is a p-form on L, $\psi = (1, -\phi)$ is a p-form on $\mathcal{O} \oplus L$ and ϵ is the canonical p-form on $\mathbb{L}$ induced from ψ.

Lemma 1.5. *Suppose that $x \in X$ has $\phi|_x \neq 0$ and that $0 \neq u \in L|_x$. Then $\epsilon|_{(0:u)} \neq 0$. Moreover, $\phi(u) \in k(x)^{\times p}$ iff there is a point $\ell \in \mathbb{P}(\mathcal{O} \oplus L)$ over x so that $\epsilon|_\ell = 0$.*

Proof. Let $w = (t, su)$ be a point of $\mathbb{L}|_x$ over $\ell = (t : su) \in \mathbb{P}(\mathcal{O} \oplus L)|_x$. If $t = 0$ then $\ell = (0 : u)$ and $\epsilon(w) = -s^p\phi(u)$, which is nonzero for $s \neq 0$. If $t \neq 0$ then $\epsilon|_\ell$ is determined by the scalar $\epsilon(w) = \psi(t, su) = t^p - s^p\phi(u)$. Thus $\epsilon|_\ell = 0$ iff $\phi(u) = (t/s)^p$. $\square$

Here is an alternative proof of 1.5, using the Kummer algebra $K = k(x)(a)$, $a = \sqrt[p]{\phi(u)}$. Since $\epsilon(w) = \psi(t, su)$ is the norm of the nonzero element $t - sa$ in K, the norm $\epsilon(w)$ is zero iff the Kummer algebra is split, i.e., $\phi(u) = a^p \in k(x)^{\times p}$.

Finally, the notation $\{\gamma,\ldots,\gamma'_{n-1}\}$ in the Chain Lemma 0.1 is a special case of the notation in the following definition.

Definition 1.6. Given line bundles $H_1,\ldots,H_n$ on X, p-forms α_i on H_i, and a point $x\in X$ at which each form $\alpha_i|_x$ is nonzero, we write $\{\alpha_1,\ldots,\alpha_n\}|_x$ for the element $\{\alpha_1|_x,\ldots,\alpha_n|_x\}$ of $K_n^M(k(x))/p$ described before Definition 1.1: if u_i is a generator of $H_i|_x$ and $\alpha_i|_x(u_i)=a_i$ then $\{\alpha_1,\ldots,\alpha_n\}|_x=\{a_1,\ldots,a_n\}$.

We record the following useful consequence of this construction. Recall that if $(R,\mathfrak{m})$ is a regular local ring with quotient field F, then any regular sequence $r_1,\ldots$ generating $\mathfrak{m}$ determines a specialization map $K_*^M(F)\to K_*^M(R/\mathfrak{m})$; if $a_1,\ldots$ are units of R, any of these specializations sends $\{a_1,\ldots\}$ to $\{\bar{a}_1,\ldots\}$.

Lemma 1.7. *Suppose that the p-forms α_i are all nonzero at the generic point η of a smooth X. On the open subset U of X of points x on which each $\alpha_i|_x\neq 0$, the symbol $\{\alpha_1|_x,\ldots,\alpha_n|_x\}$ in $K_n^M(k(x))/p$ is obtained by specialization from the symbol in $K_n^M(k(X))/p$.*

2 The Chain Lemma When $n=2$

The goal of this section is to construct certain iterated projective bundles together with line bundles and p-forms on them as needed in the case $n=2$ of the Chain Lemma 0.1. Our presentation is based upon Rost's lectures [9].

We begin with a generic construction, which starts with a pair K_0, K_{-1} of line bundles on a variety $X_0=X_{-1}$ and produces a tower of varieties X_r, equipped with distinguished lines bundles K_r. Each X_r is a product of $p-1$ projective line bundles over X_{r-1}, so X_r has relative dimension $r(p-1)$ over X_0.

Definition 2.1. Given a morphism $f_{r-1}:X_{r-1}\to X_{r-2}$ and line bundles K_{r-1} on X_{r-1}, K_{r-2} on X_{r-2}, we form the projective line bundle $\mathbb{P}(\mathcal{O}\oplus K_{r-1})$ over X_{r-1} and its tautological line bundle $\mathbb{L}$. By definition, X_r is the product $\prod_1^{p-1}\mathbb{P}(\mathcal{O}\oplus K_{r-1})$ over X_{r-1}. Writing f_r for the projection $X_r\to X_{r-1}$, and $\mathbb{L}_r$ for the external product $\mathbb{L}\boxtimes\cdots\boxtimes\mathbb{L}$ on X_r, we define the line bundle K_r on X_r to be $K_r=(f_r\circ f_{r-1})^*(K_{r-2})\otimes\mathbb{L}_r$.

$$X_r\xrightarrow{f_r}X_{r-1}\xrightarrow{f_{r-1}}X_{r-2}\cdots X_1\xrightarrow{f_1}X_0=X_{-1}.$$

Example 2.2 (k-tower). The *k-tower* is the tower obtained when we start with $X_0=\mathrm{Spec}(k)$, using the trivial line bundles K_{-1}, K_0. Note that $X_1=\prod\mathbb{P}^1$ and $K_1=\mathbb{L}_1$, while X_2 is a product of projective line bundles over $\prod\mathbb{P}^1$, and $K_2=\mathbb{L}_2$.

In the Chain Lemma (Theorem 0.1) for $n=2$ we have $S=X_p$ in the k-tower, and the line bundles are $J=J_1=K_p$, $J_1'=f_p^*(K_{p-1})$. Before defining the p-forms γ_1 and γ_1' in 2.7, we quickly establish 2.6; this verifies part (6) of Theorem 0.1, that the degree of $c_1(K_p)^{p^2-p}$ is prime to p.

If L is a line bundle over X, and $\lambda = c_1(L)$, the Chow ring of $\mathbb{P} = \mathbb{P}(\mathcal{O} \oplus L)$ is $CH(\mathbb{P}) = CH(X)[z]/(z^2 - \lambda z)$, where $z = c_1(\mathbb{L})$. If $\pi : \mathbb{P} \to X$ then $\pi_*(z) = -1$ in $CH(X)$. Applying this observation to the construction of X_r out of $X = X_{r-1}$ with $\lambda_{r-1} = c_1(K_{r-1})$, we have

$$CH(X_r) = CH(X_{r-1})[z_{r,1}, \dots, z_{r,p-1}]/(\{z_{r,j}^2 - \lambda_{r-1} z_{r,j} \mid j = 1, \dots, p-1\}),$$

where $z_{r,j}$ is the first Chern class of the jth tautological line bundle $\mathbb{L}$. (Formally, $CH(X_{r-1})$ is identified with a subring of $CH(X_r)$ via the pullback of cycles.) By induction on r, this yields the following result:

Lemma 2.3. *$CH^*(X_r)$ is a free $CH^*(X_0)$-module. A basis consists of the monomials $\prod z_{i,j}^{e_{i,j}}$ for $e_{i,j} \in \{0, 1\}$, $0 < i \le r$ and $0 < j < p$. As a graded algebra, $CH^*(X_r)/p \cong CH^*(X_0)/p \otimes_{R_0} R_r$, where $R_0 = \mathbb{F}_p[\lambda_0, \lambda_{-1}]$ and*

$$R_r = \mathbb{F}_p[\lambda_{-1}, \lambda_0, \dots, \lambda_r, z_{1,1}, \dots, z_{r,p-1}]/I_r,$$

$$I_r = (\{z_{i,j}^2 - \lambda_{i-1} z_{i,j} \mid 1 \le i \le r,\ 0 < j < p\},\ \{\lambda_i - \lambda_{i-2} - \textstyle\sum_{j=1}^{p-1} z_{i,j} \mid 1 \le i \le r\}).$$

Definition 2.4. For $r = 1, \dots, p$, set $z_r = \sum_{j=1}^{p-1} z_{r,j}$ and $\zeta_r = \prod z_{r,j}$. It follows from Lemma 2.3 that $\lambda_i = \lambda_{i-2} + z_i$ and $z_i^p = \sum z_{i,j}^p = \sum z_{i,j} \lambda_{i-1}^{p-1} = z_i \lambda_{i-1}^{p-1}$ in R_r and hence in $CH(X_r)/p$.

By Lemma 2.3, if $1 \le r \le p$ then multiplication by $\prod \zeta_i \in CH^{r(p-1)}(X_r)$ is an isomorphism $CH_0(X_0)/p \xrightarrow{\sim} CH_0(X_r)/p$. If $X_0 = \mathrm{Spec}(k)$ then $CH_0(X_r)/p \cong \mathbb{F}_p$, and is generated by $\prod \zeta_i$.

Lemma 2.5. *If $y \in CH_0(X_0)$, the degree of $y \cdot \zeta_1 \cdots \zeta_r$ is $(-1)^{r(p-1)} \deg(y)$.*

Proof. The degree on X_r is the composition of the $(f_i)_*$. The projection formula implies that $(f_r)_*(\zeta_r) = (-1)^{p-1}$, and

$$(f_r)_*(y \cdot \zeta_1 \cdots \zeta_r) = (y \cdot \zeta_1 \cdots \zeta_{r-1}) \cdot (f_r)_*(\zeta_r) = (-1)^{p-1} y \cdot \zeta_1 \cdots \zeta_{r-1}.$$

Hence the result follows by induction on r. □

Proposition 2.6. *For every 0-cycle y on X_0 and $1 \le r \le p$, $\lambda_r = c_1(K_r)$ satisfies $y\,\lambda_r^{r(p-1)} \equiv (-1)^{r-1} y\, \zeta_1 \cdots \zeta_r$ in $CH_0(X_r)/p$, and*

$$\deg(y \lambda_r^{r(p-1)}) \equiv \deg(y) \mod p.$$

For the k-tower 2.2 (with $y = 1$), we have $\deg(\lambda_p^{p^2-p}) \equiv 1 \mod p$.

Proof. If $r = 1$ this follows from $y\lambda_{-1} = y\lambda_0 = 0$ in $CH(X_0)$: $\lambda_1 = z_1 + \lambda_{-1}$ and $y \cdot \zeta_1 \equiv y\, \lambda_1^{p-1}$. For $r \ge 2$, we have $\lambda_r = z_r + \lambda_{r-2}$ and $z_r^p = z_r \lambda_{r-1}^{p-1}$ by 2.4. Because $p - r \ge 0$, we have

$$\begin{aligned}\lambda_r^{r(p-1)} &= (z_r + \lambda_{r-2})^{p(r-1)+(p-r)} \equiv (z_r^p + \lambda_{r-2}^p)^{r-1} \cdot (z_r + \lambda_{r-2})^{p-r} \mod p\\ &= (z_r \lambda_{r-1}^{p-1} + \lambda_{r-2}^p)^{r-1} (z_r + \lambda_{r-2})^{p-r} \equiv -\zeta_r \, \lambda_{r-1}^{(r-1)(p-1)} + T \mod p,\end{aligned}$$

where $T \in CH(X_{r-1})[z_r]$ is a homogeneous polynomial of total degree $< p-1$ in z_r.

By 2.3, the coefficients of yT are elements of $CH(X_{r-1})$ of degree $> \dim(X_{r-1})$, so yT must be zero. Then by the inductive hypothesis,

$$y\, \lambda_r^{r(p-1)} \equiv (-1)^{r-2} y\, \zeta_r \lambda_{r-1}^{(r-1)(p-1)} \equiv (-1)^{r-1} y\, \zeta_r \cdot (\zeta_1 \cdots \zeta_{r-1})$$

in $CH^*(X_r)/p$, as claimed. Now the degree assertion follows from Lemma 2.5. □

2.1 The p-Forms

We now turn to the p-forms in the Chain Lemma 0.1, using the k-tower 2.2. We will inductively equip the line bundles $\mathbb{L}_r$ and K_r of 2.2 with p-forms Ψ_r and φ_r; the γ_1 and γ_1' of the Chain Lemma 0.1 will be φ_p and φ_{p-1}.

When $r = 0$, we equip the trivial line bundles K_{-1}, K_0 on $X_0 = \mathrm{Spec}(k)$ with the p-forms $\varphi_{-1}(t) = a_2 t^p$ and $\varphi_0(t) = a_1 t^p$. The p-form φ_{r-1} on K_{r-1} induces a p-form $\psi(t,u) = t^p - \varphi_{r-1}(u)$ on $\mathcal{O} \oplus K_{r-1}$ and a p-form ϵ on the tautological line bundle $\mathbb{L}$, as in Example 1.3. As observed in Example 1.3, at the point $q = (1 : x)$ of $\mathbb{P}(\mathcal{O} \oplus K_{r-1})$ we have $\epsilon(y) = \psi(1,x) = 1 - \varphi_{r-1}(x)$.

Definition 2.7. The p-form Ψ_r on $\mathbb{L}_r$ is the product form $\prod \epsilon$:

$$\Psi_r(y_1 \boxtimes \cdots \boxtimes y_{p-1}) = \prod \epsilon(y_i).$$

The p-form φ_r on $K_r = (f_{r-1} \circ f_r)^*(K_{r-2}) \otimes \mathbb{L}_r$ is defined to be

$$\varphi_r = (f_{r-1} \circ f_r)^*(\varphi_{r-2}) \otimes \Psi_r.$$

Proposition 2.8. *Let $x = (x_1, \ldots, x_{p-1}) \in X_r$ be a point with residue field $E = k(x)$. For $-1 \le i \le r$, choose generators u_i and v_i for the one-dimensional E vector spaces $K_i|_x$ and $\mathbb{L}_i|_x$ respectively, in such a way that $u_i = u_{i-2} \otimes v_i$.*

1. *If $\varphi_i|_x = 0$ for some $1 \le i \le r$ then $\{a_1, a_2\}_E = 0 \in K_2(E)/p$.*
2. *If $\varphi_i|_x \neq 0$ for all i, $1 \le i \le r$, then*

$$\{a_1, a_2\}_E = (-1)^r \{\varphi_{r-1}(u_{r-1}), \varphi_r(u_r)\} \in K_2(E)/p.$$

Proof. By induction on r. Both parts are obvious if $r = 0$. To prove the first part, we may assume that $\varphi_i|_x \neq 0$ for $1 \le i \le r-1$, but $\varphi_r|_x = 0$. We have $u_r = u_{r-2} \otimes v_r$ and by the definition of φ_r, we conclude that

$$0 = \varphi_r(u_r) = \varphi_{r-2}(u_{r-2})\Psi_r(v_r),$$

whence $\Psi_r(v_r) = 0$. Now the element $v_r \neq 0$ is a tensor product of sections w_j and $\Psi_r(v_r) = \prod \epsilon(w_j)$ so $\epsilon(w_j) = 0$ for a nonzero section w_j of $\mathbb{L}|_{x_j}$. By Lemma 1.5, $\varphi_{r-1}(u_{r-1})$ is a pth power in E. Consequently, $\{\varphi_{r-2}(u_{r-2}), \varphi_{r-1}(u_{r-1})\}_E = 0$ in $K_2(E)/p$. This symbol equals $\pm\{a_1, a_2\}_E$ in $K_2(E)/p$, by (2) and induction. This finishes the proof of the first assertion.

For the second claim, we can assume by induction that

$$\{a_1, a_2\}_E = \pm\{\varphi_{r-2}(u_{r-2}), \varphi_{r-1}(u_{r-1})\}_E.$$

Now $\varphi_r(u_r) = \varphi_{r-2}(u_{r-2})\Psi_r(v_r)$. Letting $K = k(x)(\alpha)$, $\alpha = \sqrt[p]{\varphi_{r-1}}(u_{r-1})$, we have $N_{K/k(x)}(t - s\alpha) = t^p - s\varphi_{r-1}u_{r-1}) = \epsilon(t, su_{r-1})$, and hence $\Psi_r(v_r) = N_{K/k(x)}(v')$ for some $v' \in K$. But $\{\varphi_{r-1}(u_{r-1}), N_{K/k(x)}(v')\} = 0$ by Lemma 2.9 below. We conclude that

$$\{\varphi_{r-2}(u_{r-2}), \varphi_{r-1}(u_{r-1})\}_E \equiv -\{\varphi_{r-1}(u_{r-1}), \varphi_r(u_r)\}_E \mod p;$$

this concludes the proof of the second assertion. □

Lemma 2.9. *For any field k, any $a \in k^\times$ and any b in $K_a = k[\sqrt[p]{a}]$, the symbol $\{a, N_{K_a/k}(b)\}$ is trivial in $K_2(k)/p$.*

Proof. Because $\{a, b\} = p\{\sqrt[p]{a}, b\}$ vanishes in $K_2(K_a)/p$, we have $\{a, N(b)\} = N\{a, b\} = pN(\{\sqrt[p]{a}, b\}) = 0$. □

Proof of the Chain Lemma 0.1 (for $n = 2$). We verify the conditions for the variety $S = X_p$ in the k-tower 2.2; the line bundles $J = J_1 = K_p$, $J_1' = f_p^*(K_{p-1})$; the p-forms γ_1 and γ_1' in 0.1 are the forms φ_p and φ_{p-1} of Definition 2.7. Part (1) of Theorem 0.1 is immediate from the construction of $S = X_p$; parts (2) and (4) were proven in Proposition 2.8; parts (3) and (5) follow from (2) and (4); and part (6) is Proposition 2.6 with $y = 1$. □

2.2 Norm Principle for n = 2

The Norm Principle for $n = 2$ was implicit in the Merkurjev–Suslin paper [4, 4.3]. We reproduce their short proof, which uses the Severi–Brauer variety X of the cyclic division algebra $D = A_\zeta(a, b)$ attached to a nontrivial symbol $\{a, b\}$ in $K_2(k)/p$ and a pth root of unity ζ; X is a norm variety for the symbol $\{a, b\}$.

Theorem 2.10 (Norm Principle for $n = 2$). *If $x \in X$ and $[k(x) : k] = p^m$ for $m > 1$ then for all $\lambda \in k(x)$ there exists $x' \in X$ and $\lambda' \in k(x')$ so that $[k(x') : k] \leq p$ and $[x, \lambda] = [x', \lambda']$ in $\overline{A}_0(X, \mathcal{K}_1)$.*

Proof. By Merkurjev–Suslin [4, 8.7.2], $N : \overline{A}_0(X, \mathcal{K}_1) \to k^\times$ is an injection with image $\mathrm{Nrd}(D) \subseteq k^\times$. Therefore the unit $N([x, \lambda])$ of k can be written as the reduced norm of an element $\lambda' \in D$. The subfield $E = k(\lambda')$ of D has degree $\le p$, and corresponds to a point $x' \in X$. Since $N([x', \lambda']) = \mathrm{Nrd}(\lambda') = N([x, \lambda])$, we have $[x, \lambda] = [x', \lambda']$ in $\overline{A}_0(X, \mathcal{K}_1)$. $\square$

3 The Symbol Chain

Here is the pattern of the chain lemma in all weights.

We start with a sequence $a_1, a_2, \ldots$ of units of k, and the function $\Phi_0(t) = t^p$. For $r \ge 1$, we inductively define functions Φ_r in p^r variables and Ψ_r in $p^r - p^{r-1}$ variables, taking values in k, and prove (in 3.4) that $\{a_1, \ldots, a_r, \Phi_r(\mathbf{x})\} \equiv 0$ mod p. Note that Φ_r and Ψ_r depend only upon the units $a_1, \ldots, a_r$. We write $\mathbf{x}_i$ for a sequence of p^r variables x_{ij} (where $j = (j_1, \ldots, j_r)$ and $0 \le j_t < p$), and we inductively define

$$\Psi_{r+1}(\mathbf{x}_1, \ldots, \mathbf{x}_{p-1}) = \prod_{i=1}^{p-1} \big[1 - a_{r+1}\Phi_r(\mathbf{x}_i)\big], \tag{1}$$

$$\Phi_{r+1}(\mathbf{x}_0, \ldots, \mathbf{x}_{p-1}) = \Phi_r(\mathbf{x}_0)\Psi_{r+1}(\mathbf{x}_1, \ldots, \mathbf{x}_{p-1}). \tag{2}$$

We say that two rational functions are *birationally equivalent* if they can be transformed into one another by an automorphism (over the base field k) of the field of rational functions.

Example 3.1. $\Psi_1(x_1, \ldots, x_{p-1}) = \prod(1 - a_1 x_i^p)$ and $\Phi_1(x_0, \ldots, x_{p-1})$ is $x_0^p \prod(1 - a_1 x_i^p)$, the norm of the element $x_0 \prod(1 - x_i\alpha_1)$ in the Kummer extension $k(\mathbf{x})(\alpha_1)$, $\alpha_1 = \sqrt[p]{a_1}$. Thus Φ_1 is birationally equivalent to symmetrizing in the x_i, followed by the norm from $k[\sqrt[p]{a_1}]$ to k. More generally, $\Psi_r(\mathbf{x}_1, \ldots, \mathbf{x}_{p-1})$ is the product of norms of elements in Kummer extensions $k(\mathbf{x}_1, \ldots, \mathbf{x}_{p-1})(\sqrt[p]{b_i})$ of $k(\mathbf{x}_1, \ldots, \mathbf{x}_{p-1})$.

Example 3.2. It is useful to interpret the map Φ_1 geometrically. Let $R_{k(\alpha)/k}\mathbb{A}^1$ denote the variety, isomorphic to $\mathbb{A}^p$, which is the Weil restriction [17] of the affine line over $k(\alpha)$, so that there is a morphism $N : R_{k(\alpha)/k}\mathbb{A}^1 \to \mathbb{A}^1$ corresponding to the norm map. The function $k^p \to k(\alpha)$ defined by

$$(x_0, s_1, \ldots, s_{p-1}) \mapsto x_0(1 - s_1\alpha + s_2\alpha^2 - \cdots \pm s_{p-1}\alpha^{p-1})$$

induces a birational map $\mathbb{A}^p \xrightarrow{m} R_{k(\alpha)/k}\mathbb{A}^1$. Finally, let $q : \mathbb{A}^{p-1} \to \mathbb{A}^{p-1}/\Sigma_{p-1} \cong \mathbb{A}^{p-1}$ be the symmetrizing map sending $(x_1, \ldots)$ to the elementary symmetric functions $(s_1, \ldots)$. Then the following diagram commutes:

$$\begin{tikzcd}
\mathbb{A}^p = \mathbb{A}^1 \times \mathbb{A}^{p-1} \arrow[r, "1\times q"] \arrow[drr, "\Phi_1"'] & \mathbb{A}^1 \times \mathbb{A}^{p-1} \arrow[r, "m", "\text{birat.}"'] & R_{k(\alpha)/k}\mathbb{A}^1 = \mathbb{A}^p \arrow[d, "N"] \\
& & \mathbb{A}^1.
\end{tikzcd}$$

Remark 3.3. If $p = 2$, $\Phi_1(x_0, x_1) = x_0^2(1 - a_1 x_1^2)$ is birationally equivalent to the norm form $u^2 - a_1 v^2$ for $k(\sqrt{a_1})/k$, and $\Phi_2 = \Phi_1(\mathbf{x}_0)[1 - a_2\Phi_1(\mathbf{x}_1)]$ is birationally equivalent to the norm form $\langle\langle a_1, a_2\rangle\rangle = (u^2 - a_1 v^2)[1 - a_2(w^2 - a_1 t^2)]$ for the quaternionic algebra $A_{-1}(a_1, a_2)$.

More generally, Φ_n is birationally equivalent to the Pfister form

$$\langle\langle a_1, \dots, a_r\rangle\rangle = \langle\langle a_1, \dots, a_{r-1}\rangle\rangle \perp a_n \langle\langle a_1, \dots, a_{r-1}\rangle\rangle$$

and Ψ_r is equivalent to the restriction of the Pfister form to the subspace defined by the equations $\mathbf{x}_0 = (1, \dots, 1)$.

Suppose that $p = 3$. Then Φ_2 is birationally equivalent to (symmetrizing, followed by) the reduced norm of the algebra $A_\zeta(a_1, a_2)$ and Rost showed in [10] that Φ_3 is equivalent to the norm form of the exceptional Jordan algebra $J(a_1, a_2, a_3)$. When $r = 4$, Rost showed that the set of nonzero values of Φ_4 is a subgroup of $k^\times$.

For the next lemma, it is useful to introduce the function field F_r over k in the p^r variables $x_{j_1,\dots,j_r}$, $0 \le j_t < p$. Note that F_r is isomorphic to the tensor product of p copies of F_{r-1}.

Lemma 3.4. $\{a_1, \dots, a_r, \Phi_r(\mathbf{x})\} = \{a_1, \dots, a_r, \Psi_r(\mathbf{x})\} = 0 \in K_{r+1}^M(F_r)/p$.
If $b \in k$ is a nonzero value of Φ_r, then $\{a_1, \dots, a_r, b\} = 0 \in K_{r+1}^M(k)/p$.

Proof. By Lemma 2.9, $\{a_r, \Psi_r(\mathbf{x})\} = 0$ because $\Psi_r(\mathbf{x})$ is a product of norms of elements of $k(\mathbf{x})(\sqrt[p]{b_i})$ by 3.1. If $r = 1$ then $\{a_1, \Phi_1(\mathbf{x})\} = \{a_1, x_0^p\} \equiv 0$ as well. The result for F_r follows by induction:

$$\{a_1, \dots, a_{r+1}, \Phi_{r+1}(\mathbf{x})\} = \{a_1, \dots, a_{r+1}, \Phi_r(\mathbf{x}_0)\}\{a_1, \dots, a_{r+1}, \Psi_{r+1}(\mathbf{x})\} = 0.$$

The result for b follows from the first assertion, and specialization from F_r to k, using the regular local ring at the point $\mathbf{c}$ where $\Phi_r(\mathbf{c}) = b$. $\square$

Remark 3.5. For any value $b \in k^\times$ of Φ_n, any desingularization X of the projective closure of the affine hypersurface $X_b = \{\mathbf{x} : \Phi_n(\mathbf{x}) = b\}$ is a norm variety for the symbol $\{a_1, \dots, a_n, b\}$ in $K_{n+1}^M(k)/p$. Note that $\dim(X_b) = p^n - 1$.

Indeed, we see from Lemma 3.4 that every affine point of X_b splits the symbol. In particular, the generic point of X_b is a splitting field for this symbol. By specialization, every point of X_b and X splits the symbol.

The symmetric group Σ_{p-1} acts on $\{\mathbf{x}_1, \dots, \mathbf{x}_{p-1}\}$ and fixes Φ_n, so it acts on X_b. It is easy to see that X_b/Σ_{p-1} is birationally isomorphic to the norm variety constructed in [12, Sect. 2] using the hypersurface W defined by $N = b$ in the vector bundle of *loc. cit.* By [12, 1.19], X is also a norm variety.

The Chain Lemma is based upon the observation that certain manipulations (or "moves") of Milnor symbols do not change the class in $K_n^M(k)/p$. Here is the class of moves we will model geometrically in Sect. 4; strings of these moves will be used in Sect. 5 to prove the Chain Lemma.

Definition 3.6. A move of type C_n on a sequence $a_1, \dots, a_n$ in $k^\times$ is a transformation of the kind:

$$\text{Type } C_n: \qquad (a_1, \dots, a_n) \mapsto (a_1, \dots, a_{n-2}, a_n \Psi_{n-1}(\mathbf{x}), a_{n-1}).$$

Here Ψ_{n-1} is a function of $p^{n-1} - p^{n-2}$ new variables $\mathbf{x}_i = \{\mathbf{x}_{1,1}, \dots, \mathbf{x}_{1,p-1}\}$.

By Lemma 3.4, $\{a_1, \dots, a_n\} = -\{a_1, \dots, a_{n-2}, a_n \Psi_{n-1}(\mathbf{x}), a_{n-1}\}$, so the move does not change the symbol in $K_n^M(k)/p$. If we do this move p times, always with a new set of variables $\mathbf{x}_i$, we obtain a move $(a_1, \dots, a_n) \mapsto (a_1, \dots, a_{n-2}, \gamma_{n-1}, \gamma'_{n-1})$ in which $\gamma_{n-1}, \gamma'_{n-1}$ are functions of $p^n - p^{n-1}$ variables $\mathbf{x}_{i,j}$, $1 \le i \le p$, $1 \le j < p$.

Since these moves do not change the symbol, we have

$$\{a_1, \dots, a_n\} = \{a_1, \dots, a_{n-2}, \gamma_{n-1}, \gamma'_{n-1}\} \tag{3}$$

in $K_n^M(k)/p$. The functions γ_{n-1} and γ'_{n-1} in (3) are the ones appearing in the Chain Lemma 0.1.

Formally, if $k(\mathbf{x}_1)$ is the function field of the move of type C_n, then the function field F'_n of the move (3) is $k(\mathbf{x}_1, \dots, \mathbf{x}_p)$. We will define a variety S_{n-1} with function field F'_n.

Using $p^{n-1} - p^{n-2}$ more variables $\mathbf{x}'_{i,j}$ $(1 \le i < p, 1 \le j \le p)$ we do p moves of type C_{n-1} on $(a_1, \dots, a_{n-2}, \gamma_{n-1})$ to get the sequence $(a_1, \dots, a_{n-3}, \gamma_{n-2}, \gamma'_{n-2}, \gamma'_{n-1})$. The function field of this move is $F'_{n-1} \otimes F'_n$, and we will define a variety S_{n-2} with this function field, together with a morphism $S_{n-2} \to S_{n-1}$.

Next, apply p moves of type C_{n-2}, then p moves of type C_{n-3}, and so on, ending with p moves of type C_2. We have the sequence $(\gamma_1, \gamma'_1, \gamma'_2, \dots, \gamma'_{n-1})$ in $p^n - p$ variables $\mathbf{x}_1, \dots, \mathbf{x}_{p-1}$. Moreover, we see from Lemma 3.4 that

$$\{a_1, \dots, a_n\} = \{\gamma_1, \gamma'_1, \gamma'_2, \dots, \gamma'_{n-1}\} \quad \text{in } K_n^M(k)/p. \tag{4}$$

The net effect will be to construct a tower

$$S = S_1 \xrightarrow{f_r} S_2 \longrightarrow \cdots \to S_{n-2} \longrightarrow S_{n-1} \longrightarrow S_n = \operatorname{Spec}(k). \tag{5}$$

Let S be any variety containing $U = \mathbb{A}^{p^n - p}$ as an affine open, so that $k(S) = k(\mathbf{x}_1, \dots, \mathbf{x}_{p-1})$, each $\mathbf{x}_i$ is p^{n-1} variables $x_{i,j}$ and all line bundles on U are trivial. Then parts (1) and (2) of the Chain Lemma 0.1 are immediate from (3) and (4).

Now the only thing left to do is to construct $S = S_1$, extend the line bundles (and forms) from U to S, and prove parts (4) and (6) of 0.1.

4 Model P_{n-1} for Moves of Type C_n

In this section, we construct a tower of varieties P_r and Q_r over a fixed base scheme S', with p-forms on lines bundles over them, which will produce a model of the forms Ψ_r and Φ_r in (1) and (2). This tower, depicted in (1), is defined in 4.4.

$$P_{n-1} \to \cdots \to P_r \longrightarrow Q_{r-1} \to P_{r-1} \longrightarrow \cdots \to Q_1 \to P_1 \longrightarrow Q_0 = S' \quad (1)$$

The passage from S' to the variety P_{n-1} is a model for the moves of type C_n defined in 3.6.

Definition 4.1. Let X be a variety over some fixed base S'. Given line bundles K, L on X, we can form the vector bundle $V = \mathcal{O} \oplus L$, the $\mathbb{P}^1$-bundle $\mathbb{P}(V)$ over X, and $\mathbb{L}$. Taking products over S', set

$$P = \prod\nolimits_1^{p-1} \mathbb{P}(\mathcal{O} \oplus L); \quad Q = X \times_{S'} P.$$

On P and Q, we have the external products of the tautological line bundles:

$$\mathbb{L}(1,\dots,1) = \mathbb{L} \boxtimes \mathbb{L} \boxtimes \cdots \boxtimes \mathbb{L} \text{ on } P, \quad K \boxtimes \mathbb{L}(1,\dots,1) \text{ on } Q.$$

Given p-forms φ and ϕ on K and L, respectively, the line bundle $\mathbb{L}$ has the p-form ϵ, as in Example 1.3, and the line bundles $\mathbb{L}(1,\dots,1)$ and $K \boxtimes \mathbb{L}(1,\dots,1)$ are equipped with the product p-forms $\Psi = \prod \epsilon$ and $\Phi = \varphi \otimes \Psi$.

Let $x = (x_1,\dots,x_{p-1})$ denote the generic point of X^{p-1}. The function fields of P and Q are $k(P) = k(x)(y_1,\dots,y_{p-1})$ and $k(Q) = k(P)(x_0)$. We may represent the generic point of P in coordinate form as a $(p-1)$-tuple $\{(1 : y_i)\}$, where the y_i generate L over x_i. Then $y = \{(1, y_i)\}$ is a generator of $\mathbb{L}(1,\dots,1)$ at the generic point, and $\Psi(y) = \prod(1 - \phi(y_i))$ by 1.3. If v_0 is a generator of K at the generic point x_0 of X, then $\Phi(y) = \varphi(v_0)\Psi(y)$.

Example 4.2. An important special case arises when we begin with two line bundles H on S', K on X, with p-forms α and φ. In this case, we set $L = H \otimes K$ and equip it with the product form $\phi(u \otimes v) = \alpha(u)\varphi(v)$. At the generic point q of Q we can pick a generator $u \in H|_q$ and set $y_i = u \otimes v_i$; the forms resemble the forms of (1) and (2):

$$\Psi(y) = \prod\bigl(1 - \alpha(u)\varphi(v_i)\bigr), \quad \Phi(y) = \varphi(v_0)\,\Psi(y).$$

Remark 4.3. Suppose a group G acts on S', X, K and L, and K_0, L_0 are nontrivial 1-dimensional representations so that at every fixed point x of X (a) $k(x) = k$, (b) $L_x \cong L_0$. Then G acts on P (resp., Q) with 2^{p-1} fixed points y over each fixed point of X^{p-1} (resp., of X^p), each with $k(y) = k$, and each fiber of $\mathbb{L} = \mathbb{L}(1,\dots,1)$ (resp., $K \boxtimes \mathbb{L}$) is the representation L_0^j (resp., $K_0 \otimes L_0^j$) for some

j $(0 \leq j < p)$. Indeed, G acts nontrivially on each term $\mathbb{P}^1$ of the fiber $\prod \mathbb{P}^1$, so that the fixed points in the fiber are the points $(y_1, \dots, y_{p-1})$ with each y_i either $(0:1)$ or $(1:0)$.

We now define the tower (1) of P_r and Q_r over a fixed base S', by induction on r. We start with line bundles $H_1, \dots, H_r$, and $K_0 = \mathcal{O}_{S'}$ on S', and set $Q_0 = S'$.

Definition 4.4. Given a variety Q_{r-1} and a line bundle K_{r-1} on Q_{r-1}, we form the varieties $P_r = P$ and $Q_r = Q$ using the construction in Definition 4.1, with $X = Q_{r-1}$, $K = K_{r-1}$ and $L = H_r \otimes K_{r-1}$ as in 4.2. To emphasize that P_r only depends upon S' and $H_1, \dots, H_r$, we will sometimes write $P_r(S'; H_1, \dots, H_r)$. As in 4.1, P_r has the line bundle $\mathbb{L}(1, \dots, 1)$, and Q_r has the line bundle $K_r = K_{r-1} \boxtimes \mathbb{L}(1, \dots, 1)$.

Suppose that we are given p-forms $\alpha_i \neq 0$ on H_i, and we set $\Phi_0(t) = t^p$ on K_0. Inductively, the line bundle K_{r-1} on Q_{r-1} is equipped with a p-form Φ_{r-1}. As described in 4.1 and 4.2, the line bundle $\mathbb{L}(1, \dots, 1)$ on P_r obtains a p-form Ψ_r from the p-form $\alpha_r \otimes \Phi_{r-1}$ on $L = H_r \otimes K_{r-1}$, and K_r obtains a p-form $\Phi_r = \Phi_{r-1} \otimes \Psi_r$.

Example 4.5. $Q_1 = P_1$ is $\prod_1^{p-1} \mathbb{P}^1(\mathcal{O} \oplus H_1)$ over S', equipped with the line bundle $K_1 = \mathbb{L}(1, \dots, 1)$. If H_1 is a trivial bundle with p-form $\alpha_1(t) = a_1 t^p$ then Φ_1 is the p-form Φ_1 of Example 3.1.

P_2 is $\prod_1^{p-1} \mathbb{P}^1(\mathcal{O} \oplus H_2 \otimes K_1)$ over Q_1^{p-1}, and $K_2 = K_1 \boxtimes \mathbb{L}(1, \dots, 1)$.

Lemma 4.6. *If* $r > 0$ *then* $\dim(P_r/S') = (p^r - p^{r-1})$ *and* $\dim(Q_r/S') = p^r - 1$.

Proof. Set $d_r = \dim(Q_r/S')$. This follows easily by induction from the formulas $\dim(P_{r+1}/S') = (p-1)(d_r+1)$, $\dim(Q_{r+1}/S') = p(d_r+1) - 1$. □

Choosing generators u_i for H_i at the generic point of S', we get units $a_i = \alpha_i(u_i)$.

Lemma 4.7. *At the generic points of* P_r *and* Q_r, *the* p*-forms* Ψ_n *and* Φ_n *of 4.4 agree with the forms defined in (1) and (2).*

Proof. This follows by induction on r, using the analysis of 4.2. Given a point $q = (q_1, \dots, q_p)$ of Q_{r-1}^{p-1} and a point $\{(1 : y_i)\}$ on P_r over it, $y = \{(1, y_i)\}$ is a nonzero point on $\mathbb{L}(1, \dots, 1)$ and $y_i = 1 \otimes v_i$ for a section v_i of K_{r-1}. Since $\epsilon(1, y_i) = 1 - a_r \Phi_{r-1}(v_i)$ and $\Psi_r(y) = \prod \epsilon(1, y_i)$, the forms Ψ_r agree. Similarly, if v_0 is the generator of K_{r-1} over the generic point q_0 then $y' = v_0 \otimes y$ is a generator of K_r and

$$\Phi_r(y') = \Phi_{r-1}(v_0)\Psi_r(y),$$

which is also in agreement with the formula in (2). □

Recall that K_0 is the trivial line bundle, and that Φ_0 is the standard p-form $\Phi_0(v) = v^p$ on K_0. Every point of $P_r = \prod \mathbb{P}(\mathcal{O} \oplus L)$ has the form $w = (w_1, \dots, w_{p-1})$, and the projection $P_r \to \prod Q_{r-1}$ sends $w \in P_r$ to a point $x = (x_1, \dots, x_{p-1})$.

Proposition 4.8. *Let $s \in S'$ be a point such that $a_1|_s, \dots, a_r|_s \neq 0$.*
1. If $\Psi_r|_w = 0$ for some $w \in P_r$, then $\{a_1, \dots, a_r\}$ vanishes in $K_r^M(k(w))/p$.
2. If $\Phi_r|_q = 0$ for some $q = (x_0, w) \in Q_r$, $\{a_1, \dots, a_r\}$ vanishes in $K_r^M(k(q))/p$.

Proof. Since $\Phi_r = \Phi_{r-1} \otimes \Psi_r$, the assumption that $\Psi_r|_w = 0$ implies that $\Phi_r|_q = 0$ for any $x_0 \in Q_{r-1}$ over s. Conversely, if $\Phi_r|_q = 0$ then either $\Psi_r|_w = 0$ or $\Phi_{r-1}|_{x_0} = 0$. Since $\Phi_0 \neq 0$, we may proceed by induction on r and assume that $\Phi_{r-1}|_{x_j} \neq 0$ for each j, so that $\Phi_r|_q = 0$ is equivalent to $\Psi_r|_w = 0$.

By construction, the p-form on $L = H_r \otimes K_{r-1}$ is $\phi(u_r \otimes v) = a_r \Phi_{r-1}(v)$, where u_r generates the vector space $H_r|_s$ and v is a section of K_{r-1}. Since $\Psi_r|_w$ is the product of the forms $\epsilon|_{w_j}$, some $\epsilon|_{w_j} = 0$. Lemma 1.5 implies that $a_r \Phi_{r-1}(v)$ is a pth power in $k(x_j)$, and hence in $k(w)$, for any generator v of $K_{r-1}|_{x_j}$. By Lemma 3.4, $\{a_1, \dots, a_{r-1}, \Phi_{r-1}\} = 0$ and hence

$$\{a_1, \dots, a_r\} = \{a_1, \dots, a_{r-1}, a_r \Phi_{r-1}\} = 0$$

in $K_r^M(k(w))/p$, as claimed. □

We conclude this section with some identities in $CH(P_n)/p\,CH(P_n)$, given in 4.11. To simplify the statements and proofs, we write $\mathrm{ch}(X)$ for $CH(X)/p\,CH(X)$, and adopt the following notation.

Definition 4.9. Set $\eta = c_1(H_n) \in \mathrm{ch}^1(S')$, and $\gamma = c_1(\mathbb{L}(1, \dots, 1)) \in \mathrm{ch}^1(P_n)$. Writing $\mathbb{P}$ for the bundle $\mathbb{P}(\mathcal{O} \oplus H_n \otimes K_{n-1})$ over Q_{n-1}, let $c \in \mathrm{ch}(\mathbb{P})$ denote $c_1(\mathbb{L})$ and let $\kappa \in \mathrm{ch}(Q_{n-1})$ denote $c_1(K_{n-1})$. We write $c_j, \kappa_j \in \mathrm{ch}(P_n)$ for the images of c and κ under the jth coordinate pullbacks $\mathrm{ch}(Q_{n-1}) \to \mathrm{ch}(\mathbb{P}) \to \mathrm{ch}(P_n)$.

Lemma 4.10. *Suppose that $H_1, \dots, H_{n-1}$ are trivial. Then*
(a) $\gamma^{p^n} = \gamma^{p^{n-1}} \eta^d$ in $\mathrm{ch}(P_n)$, where $d = p^n - p^{n-1}$
(b) If in addition H_n is trivial, then $\gamma^d = -\prod c_j \kappa_j^e$, where $e = p^{n-1} - 1$
(c) If $S' = \operatorname{Spec} k$ then the zero-cycles $\kappa^e \in \mathrm{ch}_0(Q_{n-1})$ and $\gamma^d \in \mathrm{ch}_0(P_n)$ have

$$\deg(\kappa^e) \equiv (-1)^{n-1} \quad \textit{and} \quad \deg(\gamma^d) \equiv -1 \mod p$$

Proof. First note that because K_{n-1} is defined over the e-dimensional variety $Q_{n-1}(\operatorname{Spec} k; H_1, \dots, H_{n-1})$, the element $\kappa = c_1(K_{n-1})$ satisfies $\kappa^{p^{n-1}} = 0$. Thus $(\eta+\kappa)^{p^{n-1}} = \eta^{p^{n-1}}$ and hence $(\eta+\kappa)^d = \eta^d$. Now the element $c = c_1(\mathbb{L})$ satisfies the relation $c^2 = c(\eta + \kappa)$ in $\mathrm{ch}(\mathbb{P})$ and hence

$$c^{p^n} = c^{p^{n-1}} (\eta + \kappa)^d = c^{p^{n-1}} \eta^d$$

in $\mathrm{ch}^{p^n}(\mathbb{P})$. Now recall that $P_n = \prod \mathbb{P}$. Then $\gamma = \sum c_j$ and

$$\gamma^{p^n} = \sum c_j^{p^n} = \sum c_j^{p^{n-1}} \eta^d = \gamma^{p^{n-1}} \eta^d.$$

When H_n is trivial we have $\eta = 0$ and hence $c^2 = c\,\kappa$. Setting $b_j = c_j^{p^{n-1}} = c_j \kappa_j^e$, we have $\gamma^d = \gamma^{p^{n-1}(p-1)} = (\sum b_j)^{p-1}$. To evaluate this, we use the algebra trick that since $b_j^2 = 0$ for all j and $p = 0$ we have $(\sum b_j)^{p-1} = (p-1)! \prod b_j = -\prod b_j$.

For (c), note that if $S' = \operatorname{Spec} k$ then $\eta = 0$ and γ^d is a zero-cycle on P_n. By the projection formula for $\pi : P_n \to \prod Q_{n-1}$, part (b) yields $\pi_* \gamma^d = (-1)^p \prod \kappa_j^e$. Since each Q_{n-1} is an iterated projective space bundle, $CH(\prod Q_{n-1}) = \otimes_1^{p-1} CH(Q_{n-1})$, and the degree of $\prod \kappa_j^e$ is the product of the degrees of the κ_j^e. By induction on n, these degrees are all the same, and nonzero, so $\deg(\prod \kappa_j^e) \equiv 1 \mod p$.

It remains to establish the inductive formula for $\deg(\kappa^e)$. Since it is clear for $n = 0$, and the Q_i are projective space bundles, it suffices to compute that $c_1(K_n)^{p^n-1} = \kappa^e \gamma^d$ in $\mathrm{ch}(Q_n) = \mathrm{ch}(Q_{n-1}) \otimes \mathrm{ch}(P_n)$. Since $\kappa^{e+1} = 0$ and $c_1(K_n) = \kappa + \gamma$ we have

$$c_1(K_n)^{p^{n-1}} = \kappa^{e+1} + \gamma^{p^{n-1}} = \gamma^{p^{n-1}},$$

and hence $c_1(K_n)^d = \gamma^d$. Since $\gamma^{d+1} = 0$, this yields the desired calculation:

$$c_1(K_n)^{p^n-1} = c_1(K_n)^e c_1(K_n)^d = (\kappa + \gamma)^e \gamma^d = \kappa^e \gamma^d. \qquad \square$$

Corollary 4.11. *There is a ring homomorphism* $\mathbb{F}_p[\lambda, z]/(z^p - \lambda^{p-1} z) \to \mathrm{ch}(P_n)$, *sending* λ *to* $\eta^{p^{n-1}}$ *and* z *to* $\gamma^{p^{n-1}}$.

5 Model for p Moves

In this section we construct maps $S_{n-1} \to S_n$ which model the p moves of type C_n defined in 3.6. Each such move introduces $p^{n-1} - p^{n-2}$ new variables, and will be modelled by a map $Y_r \to Y_{r-1}$ of relative dimension $p^{n-1} - p^{n-2}$, using the P_{n-1} construction in 4.4. The result (Definition 5.1) will be a tower of the form:

$$\begin{array}{ccccccccc}
J_{n-1} = L_p & & L_{p-1} & & & & L_2 & & L_1 & & L_0 = J_n \\
S_{n-1} = Y_p & \xrightarrow{f_p} & Y_{p-1} & \longrightarrow & \cdots \to & & Y_2 & \xrightarrow{f_2} & Y_1 & \xrightarrow{f_1} & Y_0 = S_n.
\end{array}$$

Fix $n \geq 2$, a variety S_n, and line bundles $H_1, \ldots, H_{n-2}$, H_n and $J_n = H_{n-1}$ on S_n. The first step in the tower is to form $Y_0 = S_n$ and $Y_1 = P_{n-1}(S_n; H_1, \ldots, H_{n-2}, J_n)$, with line bundles $L_0 = J_n$ and $L_1 = H_n \otimes \mathbb{L}(1, \ldots, 1)$ as in 4.4. In forming the other Y_r, the base in the P_{n-1} construction 4.4 will become Y_{r-1} and only the final line bundle will change (from J_n to L_{r-1}). Here is the formal definition.

Definition 5.1. For $r > 1$, we define morphisms $f_r : Y_r \to Y_{r-1}$ and line bundles $\mathbb{L}_r^{\boxtimes}$ and L_r on Y_r as follows. Inductively, we are given a morphism $f_{r-1} : Y_{r-1} \to Y_{r-2}$ and line bundles L_{r-1} on Y_{r-1}, L_{r-2} on Y_{r-2}. Set $\mathbb{L}_r^{\boxtimes} = \mathbb{L}(1, \ldots, 1)$,

$$Y_r = P_{n-1}(Y_{r-1}; H_1, \ldots, H_{n-2}, L_{r-1}) \xrightarrow{f_r} Y_{r-1}, \quad L_r = f_r^* f_{r-1}^* (L_{r-2}) \otimes \mathbb{L}_r^{\boxtimes}.$$

Finally, we write S_{n-1} for Y_p and set $J_{n-1} = L_p$, $J'_{n-1} = f_p^*(L_{p-1})$. By Lemma 4.6, $\dim(Y_r/Y_{r-1}) = p^{n-1} - p^{n-2}$ and hence $\dim(S_{n-1}/S_n) = p^n - p^{n-1}$.

For example, when $n = 2$ and H_1 is trivial, this tower is exactly the tower of 2.1: we have $Y_r = P_1(Y_{r-1}; L_{r-1}) = \prod \mathbb{P}^1(\mathcal{O} \oplus L_{r-1})$.

We remark that the line bundles J_{n-1} and J'_{n-1} will be the line bundles of the Chain Lemma 0.1. The rest of tower (5) will be obtained in Definition 5.10 by repeating this construction and setting $S = S_1$.

The rest of this section, culminating in Theorem 5.11, is devoted to proving part (6) of the Chain Lemma, that the degree of the zero-cycle $c_1(J_1)^{\dim S}$ is relatively prime to p. In preparation, we need to compare the degrees of the zero-cycles $c_1(J_{n-1})^{\dim S_{n-1}}$ on S_{n-1} and $c_1(J_n)^{\dim S_n}$ on S_n. In order to do so, we introduce the following algebra.

Definition 5.2. We define the graded $\mathbb{F}_p$-algebra A_r and $\bar{A}_r$ by $\bar{A}_r = A_r/\lambda_{-1}A$ and:

$$A_r = \mathbb{F}_p[\lambda_{-1}, \lambda_0, \ldots, \lambda_r, z_1, \ldots, z_r]/(\{z_i^p - \lambda_{i-1}^{p-1} z_i, \lambda_i - \lambda_{i-2} - z_i \mid i = 1, \ldots r\}).$$

Remark 5.3. By Corollary 4.11, there is a homomorphism $A_p \xrightarrow{\rho} \mathrm{ch}(Y_p)$, sending λ_r to $c_1(L_r)^{p^{n-2}}$ and z_r to $c_1(\mathbb{L}_r^{\boxtimes})^{p^{n-2}}$. When H_{n-1} is trivial, ρ factors through $\bar{A}_p$.

Lemma 5.4. *In $\bar{A}_r$, every element u of degree 1 satisfies $u^{p^2} = u^p \lambda_0^{p^2-p}$.*

Proof. We will show that $\bar{A}_r$ embeds into a product of graded rings of the form $\Lambda_k = \mathbb{F}_p[\lambda_0][v_1, \ldots, v_k]/(v_1^p, \ldots, v_k^p)$. In each entry, $u = a\lambda_0 + v$ with $v^p = 0$ and $a \in \mathbb{F}_p$, so $u^p = a\lambda_0^p$ and $u^{p^2} = a\lambda_0^{p^2}$, whence the result.

Since $\bar{A}_{r+1} = \bar{A}_r[z]/(z^p - \lambda_r^{p-1} z)$ is flat over $\bar{A}_r$, it embeds by induction into a product of graded rings of the form $\Lambda' = \Lambda_k[z]/(z^p - u^{p-1}z)$, $u \in \Lambda_k$. If $u \neq 0$, there is an embedding of Λ' into $\prod_{i=0}^{p-1} \Lambda_k$ whose ith component sends z to iu. If $u = 0$, then $\Lambda' \cong \Lambda_{k+1}$. $\square$

Remark 5.5. It follows that if $m > 0$ and $(p^2 - p) \mid m$ then $u^{kp+m} = \lambda_0^m u^{kp}$.

Proposition 5.6. *In $\bar{A}_r$, $\lambda_p^{p^N-p} = \lambda_0^{p^N-p^2}(\prod z_i^{p-1} + T\lambda_0)$, where* $\deg(T) = p^2 - p - 1$.

Proof. By Definition 5.2, $\bar{A}_p$ is free over $\mathbb{F}_p[\lambda_0]$, with the elements $\prod z_i^{m_i}$ $(0 \le m_i < p)$ forming a basis. Thus any term of degree $p^N - p$ is a linear combination of $F = \lambda_0^{p^N-p^2} \prod z_i^{p-1}$ and terms of the form $\lambda_0^{m_0} \prod z_i^{m_i}$ where $\sum m_i = p^N - p^2$ and $m_0 > p^N - p^2$. It suffices to determine the coefficient of F in $\lambda_p^{p^N-p}$. Since $\lambda_p^{p^N-p} = \lambda_0^{p^N-p^2} \lambda_p^{p^2-p}$ by Remark 5.5, it suffices to consider $N = 2$, when $F = \prod z_i^{p-1}$.

As in the proof of Proposition 2.6, if $p \ge r \ge 2$ we compute in the ring $\bar{A}_r$ that

$$\begin{aligned}\lambda_r^{r(p-1)} &= (z_r + \lambda_{r-2})^{p(r-1)+(p-r)} = (z_r^p + \lambda_{r-2}^p)^{r-1} \cdot (z_r + \lambda_{r-2})^{p-r} \\ &= (z_r \lambda_{r-1}^{p-1} + \lambda_{r-2}^p)^{r-1} (z_r + \lambda_{r-2})^{p-r} = z_r^{p-1} \lambda_{r-1}^{(r-1)(p-1)} + T,\end{aligned}$$

where $T \in \bar{A}_{r-1}[z_r]$ is a homogeneous polynomial of total degree $< p-1$ in z_r. By induction on r, the coefficient of $(z_1 \cdots z_r)^{p-1}$ in $\lambda_r^{r(p-1)}$ is 1 for all r. □

Lemma 5.7. *If* $S_n = \mathrm{Spec}(k)$ *and* $c = c_1(J_{n-1}) \in CH^1(S_{n-1})$*, then*

$$\deg(c^{\dim S_{n-1}}) \equiv -1 \mod p.$$

Proof. Set $d = \dim(S_{n-1}) = p^n - p^{n-1}$; under the map $A_p \xrightarrow{\rho} \mathrm{ch}(S_{n-1})$ of 5.3, the degree $p^2 - p$ part of A_p maps to $CH^d(S_{n-1})$. In particular, the zero-cycle $c^d = \rho(\lambda_p)^{p^2-p}$ equals the product of the $\rho(z_i)^{p-1} = c_1(\mathbb{L}_i^{\boxtimes})^{d/p}$ by Proposition 5.6 (the $T\lambda_0^*$ term maps to zero for dimensional reasons). Because $S_{n-1} = Y_p$ is a product of iterated projective space bundles, $CH_0(Y_p)$ is the tensor product of their CH_0 groups, and the degree of c^d is the product of the degrees of the $c_1(\mathbb{L}_i^{\boxtimes})^{d/p}$, each of which is -1 by Lemma 4.10. It follows that $\deg(c^d) \equiv -1 \mod p$. □

Theorem 5.8. *If* S_n *has dimension* $p^M - p^n$ *and* $H_1, \dots, H_{n-1}$ *are trivial then the zero-cycles* $c_1(J_{n-1})^{\dim S_{n-1}} \in CH_0(S_{n-1})$ *and* $c_1(J_n)^{\dim S_n} \in CH_0(S_n)$ *have the same degree modulo* p*:*

$$\deg(c_1(J_{n-1})^{\dim S_{n-1}}) \equiv \deg(c_1(J_n)^{\dim S_n}) \mod p.$$

Proof. By Remark 5.3, there is a homomorphism $A_p \xrightarrow{\rho} \mathrm{ch}(S_{n-1})$, sending λ_r to $c_1(L_r)^{p^{n-2}}$ and z_r to $c_1(\mathbb{L}_r^{\boxtimes})^{p^{n-2}}$. Because H_{n-1} is trivial, ρ factors through $\bar{A}_p$.

Set $N = M - n + 2$ and $y = \lambda_0^{p^N - p^2}$, so $\rho(y) = c_1(J_n)^{\dim S_n} \in \mathrm{ch}_0(S_n)$. From Proposition 5.6 we have $\lambda_p^{p^N - p} \equiv y \prod z_i^{p-1}$ modulo $\ker(\rho)$. From Lemma 2.5, the degree of this element equals the degree of y modulo p. □

The *p*-forms. We now define the p-forms on the line bundles J_{n-1} and J'_{n-1} using the tower 5.1. Suppose that the line bundles $L_{-1} = H_n$ and $L_0 = J_n$ on S_n are equipped with the p-forms β_{-1} and β_0. We endow the line bundle L_1 in Definition 5.1 with the p-form $\beta_1 = f^*(\beta_{-1}) \otimes \Psi_{n-1}(\beta_0)$; inductively, we endow the line bundle L_r with the p-form

$$\beta_r = f^*(\beta_{r-2}) \otimes \Psi_{n-1}(\beta_{r-1}).$$

For example, when $n = 2$ and H_1 is trivial, we saw that the tower 5.1 is exactly the tower of 2.1. In addition, the p-form $\beta_r = \Psi_1(\beta_{r-1})$ agrees with the p-form $\varphi_r = f^*(\varphi_{r-2}) \otimes \Psi_r$ of 2.7.

Lemma 5.9. *If* $\beta_0 = \alpha_{n-1}$ *and* $\beta_{-1} = \alpha_n$*, then (at the generic point of* Y_1*) the* p*-form* β_p *agrees with the form* $\alpha_n \Psi_{n-1}$ *in 3.6.*

Proof. By Lemma 4.7, the form agrees with the form of (1). □

Definition 5.10. The tower (5) of varieties S_i is obtained by downward induction, starting with $S_n = \mathrm{Spec}(k)$ and $J_n = H_{n-1}$. Construction 5.1 yields S_{n-1}, J_{n-1} and

J'_{n-1}. Inductively, we repeat construction 5.1 for i, starting with the output S_{i+1} and J_{i+1} of the previous step, to produce S_i, J_i and J'_i.

By downward induction in the tower (5), each J_i and J'_i carries a p-form, which we call γ_i and γ'_i, respectively. By 5.9, these forms agree with the forms γ_i and γ'_i of (3) and (4).

Since $\dim(S_i/S_{i-1}) = p^{i+1} - p^i$ we have $\dim(S_i/S_n) = p^n - p^i$. Thus if we combine Lemma 5.7 and Theorem 5.8, we obtain the following result.

Theorem 5.11. *For each $i < n$,* $\deg(c_1(J_i)^{\dim S_i}) \equiv -1 \mod p$.

Theorem 5.11 establishes part (6) of the Chain Lemma 0.1, that the degree of the zero-cycle $c_1(J_1)^{\dim S_1}$ is relatively prime to p.

Proof of the Chain Lemma 0.1. We verify the conditions for the variety $S = S_1$ in the tower (5); the line bundles J_i and J'_i and their p-forms are obtained by pulling back from the bundles and forms defined in 5.10. Part (1) of Theorem 0.1 is immediate from the construction of S; part (6) is Theorem 5.11, combined with Lemma 5.7. Part (2) was just established, and part (4) was proven in Proposition 4.8; parts (3) and (5) follow from (2) and (4). This completes the proof of the Chain Lemma. □

6 Nice G-Actions

We will extend the Chain Lemma to include an action by $G = \mu_p^n$ on S, J_i, and J'_i leaving γ_i and γ'_i invariant, such that the action is admissible in the following sense. Here μ_p is the cyclic group of p^{th} roots of unity.

Definition 6.1. (Rost, cf. [7, p.2]) Let G be a group acting on a k-variety X. We say that the action is *nice* if $\mathrm{Fix}_G(X)$ is 0-dimensional, and consists of k-points.

When G also acts on a line bundle L over X, the action on the geometric bundle L is *nice* exactly when G acts nontrivially on $L|_x$ for every fixed point $x \in X$, and in this case $\mathrm{Fix}_G(L)$ is the zero-section over $\mathrm{Fix}_G(X)$.

Suppose that G acts nicely on each of several line bundles L_i over X. We say that G *acts nicely* on $\{L_1, \dots, L_r\}$ if for each fixed point $x \in X$ the image of the canonical representation $G \to \prod \mathrm{Aut}(L_i|_x) = \prod k(x)^\times$ is $\prod G_i$, with each G_i nontrivial.

Remark 6.2. If $X_i \to S$ are equivariant maps and the X_i are nice, then G also acts nicely on $X_1 \times_S X_2$. However, even if G acts nicely on line bundles L_i it may not act nicely on $L_1 \boxtimes L_2$, because the representation over (x_1, x_2) is the product representation $L_1|_{x_1} \otimes L_2|_{x_2}$.

Example 6.3. Suppose that G acts nicely on a line bundle L over X. Then the induced G-action on $\mathbb{P} = \mathbb{P}(\mathcal{O} \oplus L)$ and its canonical line bundle $\mathbb{L}$ is nice. Indeed, if $x \in X$ is a fixed point then the fixed points of $\mathbb{P}|_x$ consist of the two k-points

$\{[\mathcal{O}], [L]\}$, and if $L|_x$ is the representation ρ then G acts on $\mathbb{L}$ at these fixed points as ρ and ρ^{-1}, respectively.

By 6.2, G also acts nicely on the products $P = \prod \mathbb{P}(\mathcal{O} \oplus L)$ and $Q = X \times_{S'} P$ of Definition 4.1, but it may not act nicely on $\mathbb{L}(1, \dots, 1)$.

Example 6.4. If a p-group G acts nicely on L, it also acts nicely on the projective space $\mathbb{P}(\mathcal{A})$ of the Kummer algebra bundle $\mathcal{A} = \mathcal{A}(L)$ of 1.4. Indeed, an elementary calculation shows that $\mathrm{Fix}_G \mathbb{P}(\mathcal{A})$ consists of the p sections $[L^i]$, $0 \le i < p$ over $\mathrm{Fix}_G(X)$. In each fiber, the (vertical) tangent space at each fixed point is the representation $\rho \oplus \cdots \oplus \rho^{p-1}$. If $G = \mu_p$, this is the reduced regular representation.

Over any fixed point $x \in X$, $L|_x$ is trivial, and the cyclic group of order p acts on the bundle $\mathcal{A}|_x$ by $L^i \mapsto L^{i+1}$, rotating the fixed points. This induces G-isomorphisms between the tangent spaces at these points.

Example 6.5. The action of G on $Y = \mathbb{P}(\mathcal{O} \oplus \mathcal{A})$ is not nice. In this case, an elementary calculation shows that $\mathrm{Fix}_G(Y)$ consists of the points $[L^i]$ of $\mathbb{P}(\mathcal{A})$, $0 < i < p$, together with the projective line $\mathbb{P}(\mathcal{O} \oplus \mathcal{O})$ over every fixed point x of X. For each x, the (vertical) tangent space at $[L^i]$ is $1 \oplus \rho \oplus \cdots \oplus \rho^{p-1}$; if $G = \mu_p$, this is the regular representation.

When $G = \mu_p^n$, the following lemma allows us to assume that the action on $L|_x$ is induced by the standard representation $\mu_p \subset k^\times$, via a projection $G \to \mu_p$.

Lemma 6.6. *Any nontrivial 1-dimensional representation ρ of $G = \mu_p^n$ factors as the composition of a projection $G \to \mu_p$ with the standard representation of μ_p.*

Proof. The representation ρ is a nonzero element of $(\mathbb{Z}/p)^n = G^* = \mathrm{Hom}(\mu_p^n, \mathbb{G}_m)$, and π is the Pontryagin dual of the induced map $\mathbb{Z}/p \to G^*$ sending 1 to ρ. $\square$

The construction of the P_r and Q_r in 4.4 is natural in the given line bundles $H_1, \dots, H_n$ over S', and so is the construction of the Y_r, S_r and S in 5.1 and 5.10. Since $\prod_{i=1}^n \mathrm{Aut}(H_i)$ acts on the H_i, this group (and any subgroup) will act on the variety S of the Chain Lemma. We will show that it acts nicely on S.

Recall from Definition 4.4 that P_r and Q_r are defined by the construction 4.1 using the line bundle $L_r = H_r \otimes K_{r-1}$ over Q_{r-1}.

Lemma 6.7. *If $S' = \mathrm{Spec}(k)$, then $G = \mu_p^r$ acts nicely on L_r, P_r and Q_r.*

This implies that any subgroup of $\prod_{i=1}^r \mathrm{Aut}(H_i)$ containing μ_p^r also acts nicely.

Proof. We proceed by induction on r, the case $r = 1$ being 6.3, so we may assume that μ_p^{r-1} acts nicely on Q_{r-1}. By 6.2, it suffices to show that $G = \mu_p^r$ acts nicely on $\mathbb{P}(\mathcal{O} \oplus L_r)$, where $L_r = H_r \otimes K_{r-1}$. Since the final component μ_p of G acts trivially on K_{r-1} and Q_{r-1} and nontrivially on H_r, $G = \mu_p^{r-1} \times \mu_p$ acts nicely on L_r. By Example 6.3, G acts nicely on $\mathbb{P}(\mathcal{O} \oplus L_r)$. $\square$

The proof of Lemma 6.7 goes through in slightly greater generality.

Corollary 6.8. *Suppose that $G = \mu_p^n$ acts nicely on S' and on the line bundles $\{H_1, \dots, H_r\}$ over it. Then G acts nicely on L_r, P_r and Q_r.*

Proof. Without loss of generality, we may replace S' by a fixed point $s \in S'$, in which case G acts nicely on $\{H_1, \dots, H_r\}$ through the surjection $\mu_p^n \to \mu_p^r$. Now we are in the situation of Lemma 6.7. □

Example 6.9. Since μ_p^{n-1} acts nicely on $Y = P_{n-1}(S'; H_1, \dots, H_{n-1})$ and on the bundle K_{n-1}, while μ_p of $G = \mu_p^n$ acts solely on H_n, it follows that the group $\mu_p^n = \mu_p^{n-1} \times \mu_p$ acts nicely on $\{H_1, \dots, H_{n-1}, H_n \otimes \mathbb{L}(1, \dots, 1)\}$ over Y.

We can now process the tower of varieties Y_r defined in 5.1. For notational convenience, we write H_{n-1} for J_n. The case $r = 0$ of the following assertion uses the convention that $L_0 = H_{n-1}$ and $L_{-1} = H_n$.

Proposition 6.10. *Suppose that $G = G_0 \times \mu_p^n$ acts nicely on S_n and (via $G \to \mu_p^n$) on $\{H_1, \dots, H_n\}$. Then G acts nicely on each Y_r, and on its line bundles $\{H_1, \dots, H_{n-2}, L_r, L_{r-1}\}$.*

Proof. The question being local, we may replace S' by a fixed point $s \in S'$, and G by μ_p^n. We proceed by induction on r, the case $r = 1$ being Example 6.9, since $L_1 = H_n \otimes \mathbb{L}(1, \dots, 1)$. Inductively, suppose that G acts nicely on Y_r and on $\{H_1, \dots, H_{n-2}, L_r, L_{r-1}\}$. Thus there is a factor of G isomorphic to μ_p which acts nontrivially on L_r but acts trivially on $\{H_1, \dots, H_{n-2}, L_r\}$. Hence this factor acts trivially on $Y_{r+1} = P_{n-1}(Y_r; H_1, \dots, H_{n-2}, L_r)$ and its line bundle $\mathbb{L}^{\boxtimes}$, and nontrivially on $L_{r+1} = L_{r-1} \otimes \mathbb{L}^{\boxtimes}$. The assertion follows. □

Corollary 6.11. *$G = \mu_p^n$ acts nicely on (S, J).*

Proof. By Definition 5.1, $S_{n-1} = Y_p$, $J_{n-1} = L_p$ and $J'_{n-1} = L_{p-1}$. By 6.10 with $r = p$, G acts nicely on S_{n-1} and on $\{H_1, \dots, H_{n-2}, J_{n-1}, J'_{n-1}\}$. By downward induction, $G = \mu_p^{n-i} \times \mu_p^i$ acts nicely on S_i and $\{H_1, \dots, H_{i-1}, J_i, J'_i\}$ for all $i \le n$. The case $i = 1$ is the conclusion, since $(S, J) = (S_1, J_1)$. □

Remark 6.12. If $G = \mu_p^n$ acts nicely on S', Rost [7, p.2] would say that a fixed point $s \in S'$ is *twisting* for $\{H_1, \dots, H_r\}$ if the map $G \to \mu_p^r \subset \prod k(s)^\times = \prod \mathrm{Aut}(H_i|_s)$ is a surjection.

7 G-Fixed Point Equivalences

Let $\mathcal{A} = \mathcal{A}(J)$ be the Kummer algebra over the variety S of the Chain Lemma 0.1, as in 1.4. The group $G = \mu_p^n$ acts nicely on S and J by 6.11, and on $\mathcal{A}$ and $\mathbb{P}(\mathcal{A})$ by 6.4. In this section, we introduce two G-varieties $\overline{Y}$ and Q, parametrized by norm conditions, and show that they are G-fixed point equivalent to $\mathbb{P}(\mathcal{A})$ and $\mathbb{P}(\mathcal{A})^p$, respectively. This will be used in the next section to show that $\overline{Y}$ is G-fixed point equivalent to the Weil restriction of Q_E for any Kummer extension E of k.

We begin by defining fixed point equivalence and the variety Q.

Definition 7.1. Let G be an algebraic group. We say that two G-varieties X and Y are *G-fixed point equivalent* if $\mathrm{Fix}_G X$ and $\mathrm{Fix}_G Y$ are 0-dimensional, lie in the smooth locus of X and Y, and there is a separable extension K of k and a bijection $\mathrm{Fix}_G(X_K) \to \mathrm{Fix}_G(Y_K)$ under which the families of tangent spaces at the fixed points are isomorphic as G-representations over K.

Definition 7.2. Recall from 1.4 that the norm $\mathcal{A} \xrightarrow{N} \mathcal{O}_S$ is equivariant, and homogeneous of degree p. We define the G-variety Q over $S \times \mathbb{A}^1$, and its fiber Q_w over $w \in k$, by the equation $N(\beta) = t^p w$:

$$\begin{aligned} Q &= \{([\beta, t], w) \in \mathbb{P}(\mathcal{A} \oplus \mathcal{O}) \times \mathbb{A}^1 : N(\beta) = t^p w\}, \\ Q_w &= \{[\beta, t] \in \mathbb{P}(\mathcal{A} \oplus \mathcal{O}) : N(\beta) = t^p w\}, \qquad \text{for } w \in k. \end{aligned}$$

Since $\dim(S) = p^n - p$ we have $\dim(Q_w) = p^n - 1$. If $w \neq 0$, then it is proved in [12, Sect. 2] that Q_w is geometrically irreducible and that the open subscheme where $t \neq 0$ is smooth.

If $w \neq 0$, Q_w is disjoint from the section $\sigma : S \cong \mathbb{P}(\mathcal{O}) \to \mathbb{P}(\mathcal{A} \oplus \mathcal{O})$; over each point of S, the point $(0 : 1)$ is disjoint from Q_w. Hence the projection $\mathbb{P}(\mathcal{A} \oplus \mathcal{O}) - \sigma(S) \to \mathbb{P}(\mathcal{A})$ from these points induces an equivariant morphism $\pi : Q_w \to Y = \mathbb{P}(\mathcal{A})$, $\pi(\beta, t) = \beta$. This is a cover of degree p over its image, since $\pi(\beta, t) = \pi(\beta, \zeta t)$ for all $\zeta \in \mu_p$.

Theorem 7.3. *If $w \neq 0$, G acts nicely on Q_w and $\mathrm{Fix}_G Q_w \cap (Q_w)_{\mathrm{sing}} = \emptyset$. Moreover, Q_w and $Y = \mathbb{P}(\mathcal{A})$ are G-fixed point equivalent over the field $\ell = k(\sqrt[p]{b})$.*

Proof. Since the maps $Q_w \xrightarrow{\pi} Y \to S$ are equivariant, π maps $\mathrm{Fix}_G Q_w$ to $\mathrm{Fix}_G Y$, and both lie over the finite set $\mathrm{Fix}_G S$ of k-rational points. Since the tangent space T_y is the product of $T_s S$ and the tangent space of the fiber Y_s, and similarly for Q_w, it suffices to consider a G-fixed point $s \in S$.

By 6.10 and Lemma 6.6, G acts nontrivially on $L = J|_s$ via a projection $G \to \mu_p$. By Example 6.4, G acts nicely on $\mathbb{P}(\mathcal{A})$. Thus there is no harm in assuming that $G = \mu_p$ and that L is the standard one-dimensional representation.

Let $y \in Y$ be a G-fixed point lying over s. By 6.3, the tangent space of $Y|_s$ at y is the reduced regular representation, and y is one of $[1], [L], \dots [L^{p-1}]$.

We saw in Example 6.5 that a fixed point $[a_0 : a_1 : \cdots : a_{p-1} : t]$ of G in $\mathbb{P}(\mathcal{A} \oplus \mathcal{O})|_s$ is either one of the points $e_i = [\cdots 0 : a_i : 0 \cdots : 0]$, which do not lie on Q_w, or a point on the projective line $\{[a_0 : 0 : t]\}$. By inspection, $Q_w \otimes_k \ell$ meets the projective line in the ℓ-points $[\zeta \sqrt[p]{b} : 0 : \cdots : 0 : 1]$, $\zeta \in \mu_p$. Each of these p points is smooth on Q_w, and the tangent space (over s) is the reduced regular representation of G. □

Warning. Since $\pi([\zeta \sqrt[p]{b} : 0 : \cdots : 0 : 1]) = [1 : 0 : \cdots : 0]$ for all $\zeta \in \mu_p$, $\mathrm{Fix}_G(Q_w) \xrightarrow{\pi} \mathrm{Fix}_G(Y)$ is *not* a scheme isomorphism over ℓ.

Remark 7.4. For any $w \in k^\times$ of N, any desingularization Q' of Q_w is a smooth, geometrically irreducible splitting variety for the symbol $\{a_1, \dots, a_n, w\}$ in $K^M_{n+1}(k)/p$. Assuming the Bloch–Kato conjecture for n, Suslin and Joukhovitski show Q' is a norm variety in [12, Sect. 2]. Note that the variety X_w of 3.5 is birationally a cover of Q_w.

To construct $\overline{Y}$, we fix a Kummer extension $E = k(\epsilon)$ of k. Let $\mathcal{B}$ be the $\mathcal{O}_S$-subbundle $(\mathcal{A}\otimes 1) \oplus (\mathcal{O}_S \otimes \epsilon)$ of $\mathcal{A}_E = \mathcal{A} \otimes_k E$ and let $N_{\mathcal{B}} : \mathcal{B} \to \mathcal{O}_S \otimes_k E$ be the map induced by the norm on $\mathcal{A}_E$.

Definition 7.5. Let U be the variety $\mathbb{P}(\mathcal{A}) \times_k \mathbb{P}(\mathcal{B})^{\times(p-1)}$ over $S^{\times p}$, and let L be the line bundle $\mathbb{L}(\mathcal{A}) \boxtimes \mathbb{L}(\mathcal{B})^{\boxtimes(p-1)}$ over U, given as the external product of the tautological bundles. The product of the various norms defines an algebraic morphism $N : L \to \mathcal{O}_S \otimes E$.

Lemma 7.6. *Let $u \in U$ be a point over $(s_0, s_1, \dots, s_{p-1})$, and write A_i for the $k(s_i)$-algebra $\mathcal{A}|_{s_i}$. Then the following hold.*

1. *If $\{\underline{a}\}$ does not split at any of the points $s_0, \dots, s, s_{p-1}$, then the norm map $N : L_u \to k(u) \otimes E$ is non-zero.*
2. *If $\{\underline{a}\}|_{s_0} \neq 0$ in $K^M_n(k(s_0))/p$, then A_0 is a field.*
3. *For $i \geq 1$, if $\{\underline{a}\}|_{E(s_i)} \neq 0$ in $K^M_n(E(s_i))/p$ then $A_i \otimes E$ is a field.*

Proof. The first assertion follows from part (4) of the Chain Lemma 0.1, since by 1.4 the norm on L is induced from the p-form γ_1 on J. Assertions (2–3) follow from part (2) of the Chain Lemma, since $\{\underline{a}\} \neq 0$ implies that γ is nontrivial. $\square$

Definition 7.7. Let $\mathbb{A}^E$ denote the Weil restriction $\mathrm{Res}_{E/k}\mathbb{A}^1$, characterized by $\mathbb{A}^E(F) = F \otimes_k E$ as in [17]. Let $\overline{Y}$ denote the subvariety of $\mathbb{P}(L \oplus \mathcal{O}) \times \mathbb{A}^E$ consisting of all points $([\alpha : t], w)$ such that $N(\alpha) = t^p w$ in E. We write $\overline{Y}_w$ for the fiber over a point $w \in \mathbb{A}^E$. Note that $\dim(\overline{Y}_w) = p^{n+1} - p = p \dim(Q_w)$.

Notation 7.8. Let $([\alpha : t], w)$ be a k-rational point on $\overline{Y}$, so that $w \in \mathbb{A}^E(k) = E$. We may regard $[\alpha : t] \in \mathbb{P}(L \oplus \mathcal{O})(k)$ as being given by a point $u \in U(k)$, lying over a point $(s_0, \dots, s_{p-1}) \in S(k)^{\times p}$, and a nonzero pair $(\alpha, t) \in L_u \times k$ (up to scalars). From the definition of L, we see that (up to scalars) α determines a p-tuple $(b_0, b_1 + t_1\epsilon, \dots, b_{p-1} + t_{p-1}\epsilon)$, where $b_i \in \mathcal{A}|_{s_i}$ and $t_i \in k$. When $\alpha \neq 0$, $b_0 \neq 0$ and for all $i > 0$, $b_i \neq 0$ or $t_i \neq 0$. Finally, writing A_i for $\mathcal{A}|_{s_i}$, the norm condition says that in E:

$$N_{A_0/k}(b_0) \prod_{i=1}^{p-1} N_{A_i \otimes E/E}(b_i + t_i\epsilon) = t^p w.$$

If $k \subseteq F$ is a field extension, then an F-point of $\overline{Y}$ is described as above, replacing k by F and E by $E \otimes_k F$ everywhere.

Note that if $w \neq 0$ then $\alpha \neq 0$, because $N(\alpha) = t^p w$ and $(\alpha, t) \neq (0, 0)$.

Lemma 7.9. *If $\overline{Y}$ has a k-point with $t = 0$ then $\{\underline{a}\}|_E = 0$ in $K_n^M(E)/p$.*

Proof. We use the description of a k-point of $\overline{Y}$ from 7.8. If $t = 0$, then $\alpha \neq 0$, therefore $b_0 \neq 0 \in A_0$ and $b_i + t_i\epsilon \neq 0 \in A_i \otimes E$. By Lemma 7.6, if $\{\underline{a}\}|_E \neq 0$ in $K_n^M(E)/p$ then A_0 and all the algebras $A_i \otimes E$ are fields, so that $N(\alpha) = N_{A_0/k}(b_0)\prod_{i=1}^{p-1} N_{A_i\otimes E/E}(b_i + t_i\epsilon) \neq 0$, a contradiction to $t^p w = 0$. □

Consider the projection $\overline{Y} \to \mathbb{A}^E$ onto the second factor, and write $\overline{Y}_w$ for the (scheme-theoretic) fiber over $w \in \mathbb{A}^E$. Combining 7.6 with 7.9 we obtain the following consequence (in the notation of 7.8):

Corollary 7.10. *If $\{\underline{a}\} \neq 0$ in $K_n^M(E)/p$ and $w \neq 0$ is such that $\overline{Y}_w$ has a k-point, then A_0 and the $A_i \otimes E$ are fields and w is a product of norms of an element of A_0 and elements in the subsets $A_i + \epsilon$ of $A_i \otimes_k E$.*

Later on, in Theorem 7.14, we will see that if w is a generic element of E then such a k-point exists.

The group $G = \mu_p^n$ acts nicely on S and J by 6.11, and on $\mathcal{A}$ and $\mathbb{P}(\mathcal{A})$ by 6.4. It acts trivially on $\mathbb{A}^E$, so G acts on B, U and $\overline{Y}$ (but not nicely; see 6.2).

In the notation of 7.8, if $([\alpha : t], w)$ is a fixed point of the G-action on $\overline{Y}$ then the points $u_0 \in \mathbb{P}(\mathcal{A})$ and $s_i \in S$ are fixed, and therefore are k-rational (see 6.1). If u is defined over F, each point $(b_i : t_i)$ is fixed in $\mathcal{B}|_{s_i}$. Since S acts nicely on J, Example 6.5 shows that if $t = 0$ then either $t_i \neq 0$ (and $b_i \in F \subset A_i \otimes F$) or else $t_i = 0$ and $0 \neq b_i \in J|_{s_i}^{\otimes r_i} \otimes F \subseteq A_i \otimes F$ is for some r_i, $0 \le r_i < p$.

Lemma 7.11. *For all w, $\mathrm{Fix}_G\overline{Y}_w$ is disjoint from the locus where $t = 0$.*

Proof. Suppose $([\alpha : 0], w)$ is a fixed point defined over a field F containing k. As explained above, $b_0 \neq 0$ and (for each $i > 0$) $b_i + t_i\epsilon \neq 0$ and either $t_i \neq 0$ or there is an r_i so that $b_i \in J^{r_i}|_{s_i} \otimes F$. Let I be the set of indices such that $t_i \neq 0$.

By Example 6.4, $b_0 \in J|_{s_0}^{\otimes r_0}$ for some r_0, and hence $N_{A_0}(b_0)$ is a unit in k, because the p-form γ is nontrivial on $J|_{s_0}$. Likewise, if $i \notin I$, then $N_{A_i\otimes F/F}(b_i)$ is a unit in F.

Now suppose $i \in I$, i.e., $t_i \neq 0$, and recall that in this case $b_i \in F \subset A_i \otimes F$. If we write EF for the algebra $E \otimes F \cong F[\epsilon]/(\epsilon^p - e)$, then the norm from $A_i \otimes EF$ to EF is simply the p-th power on elements in EF, so $N_{A_i\otimes EF/EF}(b_i + t_i\epsilon) = (b_i + t_i\epsilon)^p$ as an element in the algebra EF. Taking the product, and keeping in mind $t = 0$, we get the equation

$$\prod_{i\in I} N_{A_i\otimes EF/EF}(b_i + t_i\epsilon) = \prod_{i\in I}(b_i + t_i\epsilon)^p = 0.$$

Because EF is a separable F-algebra, it has no nilpotent elements. We conclude that

$$\prod_{i\in I}(b_i + t_i\epsilon) = 0.$$

The left hand side of this equation is a polynomial of degree at most $p - 1$ in ϵ; since $\{1, \epsilon, \ldots, \epsilon^{p-1}\}$ is a basis of $F \otimes E$ over F, that polynomial must be zero. This implies that $b_i = t_i = 0$ for some i, a contradiction. □

Proposition 7.12. *If $w \in \mathbb{A}^E$ is generic then* $\mathrm{Fix}_G\overline{Y}_w$ *lies in the open subvariety where* $t\prod_{i=1}^p t_i \neq 0$.

Remark 7.13. The open subvariety in 7.12 is G-isomorphic (by setting t and all t_i to 1) to a closed subvariety of $\mathbb{A}(\mathcal{A})^p$, namely the fiber over w of the map $N_{\mathcal{A}\otimes E/E}$: $\mathbb{A}(\mathcal{A})^p \to \mathbb{A}^E$ defined by

$$N(b_0,\ldots,b_{p-1}) = N_{A_0/k}(b_0)\prod\nolimits_1^{p-1} N_{A_i\otimes E/E}(b_i+\epsilon).$$

Indeed, $\mathbb{A}(\mathcal{A})^p$ is G-isomorphic to an open subvariety of $\overline{Y}$ and $N_{A_i\otimes E/E}$ is the restriction of $\alpha \mapsto N(\alpha)$.

Proof. By Lemma 7.11, $\mathrm{Fix}_G\overline{Y}_w$ is disjoint from the locus where $t=0$, so we may assume that $t=1$. Since w is generic, we may also take $w\neq 0$. So let $([\alpha:1],w)$ be a fixed point defined over $F\supseteq k$ for which $t_j=0$. As in the proof of the previous lemma, we collect those indices i such that $t_i\neq 0$ into a set I, and write EF for $E\otimes_k F$. Recall that for $i\in I$, we have $b_i\in F$. Since $j\notin I$, we have that $|I|\leq p-2$. For $i\notin I$,

$$N_{A_i\otimes EF/EF}(b_i+t_i\epsilon) = N_{A_i\otimes F/F}(b_i)\in F^\times$$

(the norm cannot be 0 as $t^p w = w\neq 0$ by assumption). So we get that

$$\prod\nolimits_{i\in I}(b_i+t_i\epsilon)^p = \xi w$$

for some $\xi\in F^\times$. If we view ξw as a point in $\mathbb{P}(E)(F) = (EF-\{0\})/F^\times$, then we get an equation of the form

$$\left[\prod\nolimits_{i\in I}(b_i+t_i\epsilon)^p\right] = [w].$$

But the left-hand side lies in the image of the morphism $\prod_{i\in I}\mathbb{P}^1\to\mathbb{P}(E)$ which sends $[b_i:t_i]\in\mathbb{P}^1(F)$ to $[\prod(b_i+t_i\epsilon)^p]\in\mathbb{P}(E)(F)$. Since $|I|\leq p-2$, this image is a proper closed subvariety, proving the assertion for generic w. □

Theorem 7.14. *For a generic closed point $w\in\mathbb{A}^E$, $\overline{Y}_w$ is G-fixed point equivalent to the disjoint union of $(p-1)!$ copies of $\mathbb{P}(\mathcal{A})^p$.*

Proof. Since both lie over S, it suffices to consider a G-fixed point $s=(s_0,\ldots,s_{p-1})$ in $S(k)^p$ and prove the assertion for the fixed points over s. Because G acts nicely on S and J, $k(s)=k$ and (by Lemma 6.6) G acts on J_s via a projection $G\to\mu_p$ as the standard representation of μ_p. Note that $J_s=J_{s_i}$ for all i.

By Example 6.4, there are precisely p fixed points on $\mathbb{P}(\mathcal{A})$ lying over a given fixed point $s_i\in S(k)$, and at each of these points the (vertical) tangent space is the reduced regular representation of μ_p. Thus each fixed point in $\mathbb{P}(\mathcal{A})^p$ is k-rational, the number of fixed points over s is p^p, and each of their tangent spaces is the sum of p copies of the reduced regular representation.

Since w is generic, we saw in 7.12 that all the fixed points of $\overline{Y}_w$ satisfy $t \neq 0$ and $t_i \neq 0$ for $1 \leq i \leq p-1$. By Remark 7.13, they lie in the affine open $\mathbb{A}(\mathcal{A})^p$ of $\mathbb{P}(L \oplus \mathcal{O})$. Because μ_p acts nicely on J_s, an F-point $b = (b_0, \dots, b_{p-1})$ of $\mathbb{A}(\mathcal{A})^p$ is fixed if and only if each $b_i \in F$. That is, $\mathrm{Fix}_G(\mathbb{A}(\mathcal{A})^p) = \mathbb{A}^p$. Now the norm map restricted to the fixed-point set is just the map $\mathbb{A}^p \to \mathbb{A}^E$ sending b to $b_0^p \prod_{i=1}^{p-1}(b_i + \epsilon)^p$. This map is finite of degree $p^p(p-1)!$, and étale for generic w, so $\mathrm{Fix}_G(\overline{Y}_w)$ has $p^p(p-1)!$ geometric points for generic w. This is the same number as the fixed points in $(p-1)!$ copies of $\mathbb{P}(\mathcal{A})$ over s, so it suffices to check their tangent space representations.

At each fixed point b, the tangent space of $\mathbb{A}(\mathcal{A})^p$ (or $\overline{Y}$) is the sum of p copies of the regular representation of μ_p. Since this tangent space is also the sum of the tangent space of $\mathbb{A}^p$ (a trivial representation of G) and the normal bundle of $\mathbb{A}^p$ in $\overline{Y}$, the normal bundle must then be p copies of the reduced regular representation of μ_p. Since the tangent space of $\mathbb{A}^p$ maps isomorphically onto the tangent space of $\mathbb{A}^E$ at w, the tangent space of $\overline{Y}_w$ is the same as the normal bundle of $\mathbb{A}^p$ in $\overline{Y}$, as required. □

Remark 7.15. The fixed points in $\overline{Y}_w$ are not necessarily rational points, and we only know that the isomorphism of the tangent spaces at the fixed points holds on a separable extension of k. This is parallel to the situation with the fixed points in Q_w described in Theorem 7.3.

8 A v_n-Variety

The following result will be needed in the proof of the norm principle.

Theorem 8.1. *Let S be the variety of the chain lemma for some symbol $\{\underline{a}\} \in K_n^M(k)/p$ and $\mathcal{A} = \bigoplus_{i=0}^{p-1} J^{\otimes i}$ the sheaf of Kummer algebras over S. Then the projective bundle $\mathbb{P}(\mathcal{A})$ has dimension $d = p^n - 1$ and $p^2 \nmid s_d(\mathbb{P}(\mathcal{A}))$.*

Proof. Let $\pi : \mathbb{P}(\mathcal{A}) \to S$ be the projection. The statement about the dimension is trivial. In the Grothendieck group $K_0(\mathbb{P}(\mathcal{A}))$, we have that

$$[T_{\mathbb{P}(\mathcal{A})}] = \pi^*([T_S]) + [T_{\mathbb{P}(\mathcal{A})/S}]$$

where $T_{\mathbb{P}(\mathcal{A})/S}$ is the relative tangent bundle. The class s_d is additive, and the dimension of S is less than d, so we conclude that $s_d(\mathbb{P}(\mathcal{A})) = s_d(T_{\mathbb{P}(\mathcal{A})/S})$. Now $[T_{\mathbb{P}(\mathcal{A})/S}] = [\pi^*(\mathcal{A}) \otimes \mathcal{O}(1)_{\mathbb{P}(\mathcal{A})/S}] - 1$; applying additivity again, together with the definition of s_d and the decomposition of $\mathcal{A}$ and hence $\pi^*(\mathcal{A})$ into line bundles, we obtain

$$s_d(\mathbb{P}(\mathcal{A})) = \deg \sum_{i=0}^{p-1} c_1(\pi^* J^{\otimes i} \otimes \mathcal{O}(1))^d.$$

The projective bundle formula presents the Chow ring $CH^*(\mathbb{P}(\mathcal{A}))$ as:

$$CH^*(\mathbb{P}(\mathcal{A})) = CH^*(S)[y]/(\prod_{i=0}^{p-1}(y - ix))$$

where $x = -c_1(J) \in CH^1(S)$ and $y = c_1(\mathcal{O}(1)) \in CH^1(\mathbb{P}(\mathcal{A}))$. Then $s_d(\mathbb{P}(\mathcal{A}))$ is the degree of the following element of the ring $CH^*(\mathbb{P}(\mathcal{A}))$:

$$s'_d(\mathbb{P}(\mathcal{A})) = \sum_{i=0}^{p-1}(y - ix)^d = \sum_{i=0}^{p-1} a_i y^i x^{d-i}$$

for some integer coefficients a_i. Since $x \in CH^1(S)$, we have $x^r = 0$ for any $r > \dim(S) = p^n - p$. It follows that $s'_d(\mathbb{P}(\mathcal{A})) = a_{p-1} y^{p-1} x^{\dim(S)}$. By part (6) of the Chain Lemma 0.1, the degree of $x^{\dim(S)} = (-1)^{\dim(S)} c_1(J)^{\dim(S)}$ is prime to p. In addition, $\pi_*(y^{p-1}) = \pi_*(c_1(\mathcal{O}(1))^{p-1}) = [S] \in CH^0(S)$. By the projection formula $s_d(\mathbb{P}(\mathcal{A})) = a_{p-1} \deg x^{\dim(S)}$. Thus to prove the theorem, it suffices to show that $a_{p-1} \equiv p \mod p^2$; this algebraic calculation is achieved in Lemma 8.2. □

Lemma 8.2. *In the ring $R = \mathbb{Z}/p^2[x, y]/(\prod_{i=0}^{p-1}(y - ix))$, the coefficient of y^{p-1} in $u_m = \sum_{i=0}^{p-1}(y - ix)^{p^m-1}$ is px^b, with $b = p^m - p$.*

Proof. Since u_m is homogeneous of degree $p^m - 1$, it suffices to determine the coefficient of y^{p-1} in u_m in the ring

$$R/(x-1) = \mathbb{Z}/p^2[y]/(\prod_{i=0}^{p-1}(y - i)) \cong \prod_{i=0}^{p-1} \mathbb{Z}/p^2.$$

If $m = 1$, then $u_1 = \sum_{i=0}^{p-1}(y - i)^{p-1}$ is a polynomial of degree $p - 1$ with leading term py^{p-1}. Inductively, we use the fact that for all $a \in \mathbb{Z}/p^2$, we have

$$a^{p^2-p} = \begin{cases} 0, & \text{if } p \mid a \\ 1, & \text{otherwise.} \end{cases}$$

Thus for $m \geq 2$, if we set $k = (p^{m-1} - 1)/(p - 1)$, then $a^{p^m-1} = a^{(p-1)+k(p^2-p)} = a^{p-1} \in \mathbb{Z}/p^2$, and therefore

$$u_m = \sum_{i=0}^{p-1}(y - i)^{p^m-1} = \sum_{i=0}^{p-1}(y - i)^{p-1} = u_1$$

holds in $R/(x - 1)$; the result follows. □

9 The Norm Principle

We now turn to the Norm Principle, which concerns the group $A_0(X, \mathcal{K}_1)$ associated to a variety X. In the literature, this group is also known as $H_{-1,-1}(X)$ and $H^d(X, \mathcal{K}_{d+1})$, where $d = \dim(X)$. We recall the definition from 0.2.

Definition 9.1. If X is a regular scheme then $A_0(X,\mathcal{K}_1)$ is the cokernel of the map $\oplus_y K_2(k(y)) \xrightarrow{(\partial_{xy})} \oplus_x k(x)^\times$. In this expression, the first sum is taken over all points $y \in X$ of dimension 1, and the second sum is over all closed points $x \in X$. The map $\partial_{xy} : K_2(k(y)) \to k(x)^\times$ is the tame symbol associated to the discrete valuation on $k(y)$ associated to x; if x is not a specialization of y then $\partial_{xy} = 0$. If $x \in X$ is closed and $\alpha \in k(x)^\times$ we write $[x,\alpha]$ for the image of α in $A_0(X,\mathcal{K}_1)$.

The group $A_0(X,\mathcal{K}_1)$ is covariant for proper morphisms $X \to Y$, and clearly $A_0(\mathrm{Spec}(k),\mathcal{K}_1) = k^\times$ for every field k. Thus if $X \to \mathrm{Spec}(k)$ is proper then there is a morphism $N : A_0(X,\mathcal{K}_1) \to k^\times$, whose restriction to the group of units of a closed point x is the norm map $k(x)^\times \to k^\times$. That is, $N[x,\alpha] = N_{k(x)/k}(\alpha)$.

Definition 9.2. When X is smooth and proper over k, we write $\overline{A}_0(X,\mathcal{K}_1)$ for the quotient of $A_0(X,\mathcal{K}_1)$ by the relation that $[x_1, N_{x/x_1}(\alpha)] = [x_2, N_{x/x_2}(\alpha)]$ for every closed point x of $X \times_k X$ lying over (x_1,x_2) and every $\alpha \in k(x)^\times$.

It is proven in [12, 1.5–1.7] that if X has a k-rational point then $\overline{A}_0(X,\mathcal{K}_1) = k^\times$; if $X(k) = \emptyset$, then both the kernel and cokernel of $N : \overline{A}_0(X,\mathcal{K}_1) \to k^\times$ have exponent n, where n is the gcd of the degrees $[k(x) : k]$ for closed $x \in X$. In addition, if x, x' are two points of X then for any field map $k(x') \to k(x)$ over k and any $\alpha \in k(x)^\times$ we have $[x,\alpha] = [x', N_{x/x'}\alpha]$ in $\overline{A}_0(X,\mathcal{K}_1)$.

To illustrate the advantage of passing to $\overline{A}_0$, consider a cyclic field extension E/k. Then $A_0(\mathrm{Spec} E,\mathcal{K}_1) = E^\times$ and by Hilbert 90, there is an exact sequence

$$0 \to \overline{A}_0(\mathrm{Spec} E,\mathcal{K}_1) \to k^\times \to \mathrm{Br}(K/k) \to 0.$$

We now suppose that k is a p-special field, so that the kernel and cokernel of $N : \overline{A}_0(X,\mathcal{K}_1) \to k^\times$ are p-groups, and that X is a norm variety (a p-generic splitting variety of dimension $p^n - 1$). The Norm Principle is concerned with reducing the degrees of the field extensions $k(x)$ used to represent elements of $\overline{A}_0(X,\mathcal{K}_1)$. For this, the following definition is useful.

Definition 9.3. Let $\widetilde{A}_0(k)$ denote the subset of elements θ of $\overline{A}_0(X,\mathcal{K}_1)$ represented by $[x,\alpha]$ where $k(x) = k$ or $[k(x) : k] = p$. If E/k is a field extension, $\widetilde{A}_0(E)$ denotes the corresponding subset of $\overline{A}_0(X_E,\mathcal{K}_1)$.

Lemma 9.4. *If k is p-special and X is a norm variety, then $\widetilde{A}_0(k)$ is a subgroup of $\overline{A}_0(X,\mathcal{K}_1)$.*

Proof. By the Multiplication Principle [12, 5.7], which depends upon the Chain Lemma 0.1, we know that for each $[x,\alpha]$, $[x',\alpha']$ in $\widetilde{A}_0(k)$, there is a $[x'',\alpha''] \in \widetilde{A}_0(k)$ so that $[x,\alpha] + [x',\alpha'] = [x'',\alpha'']$ in $\overline{A}_0(X,\mathcal{K}_1)$. Hence $\widetilde{A}_0(k)$ is closed under addition. It is nonempty because $E = k[\sqrt[p]{a_1}]$ splits the symbol and therefore $X(E) \neq \emptyset$. It is a subgroup because $[x,\alpha] + [x,\alpha^{-1}] = [x,1] = 0$. □

Lemma 9.5 ([12, 1.24]). *If k is p-special and X is a norm variety, the restriction of $\overline{A}_0(X,\mathcal{K}_1) \xrightarrow{N} k^\times$ to $\widetilde{A}_0(k)$ is an injection.*

Proof. Let $[x,\alpha]$ represent $\theta \in \widetilde{A}_0(k)$. If $N(\theta) = N_{k(x)/k}(\alpha) = 1$ then $\alpha = \sigma(\beta)/\beta$. for some β by Hilbert's Theorem 90. But $[x,\sigma(\beta)] = [x,\beta]$ in $\widetilde{A}_0(k)$; see [12, 1.5]. $\square$

Example 9.6. If X has a k-point z, then the norm map N of 0.2 is an isomorphism $\widetilde{A}_0(k) \cong \overline{A}_0(X,\mathcal{K}_1) \xrightarrow{\simeq} k^\times$, split by $\alpha \mapsto [z,\alpha]$. Indeed, for every closed point x of X we have $[x,\alpha] = [z, N_{k(x)/k}\alpha]$ in $\overline{A}_0(X,\mathcal{K}_1)$, by [12, 1.5].

Our goal in the next section is to prove the following theorem. Let E/k be a field extension with $[E:k] = p$. Since k has pth roots of unity, we can write $E = k(\epsilon)$ with $\epsilon^p \in k$.

Theorem 9.7. *Suppose that k is p-special, $\{\underline{a}\}_E \neq 0$ and that X is a norm variety for $\{\underline{a}\}$. For $[z,\alpha] \in \widetilde{A}_0(E)$, there exist finitely many points $x_i \in X$ of degree p over k, $t_i \in k$ and $b_i \in k(x_i)$ such that*

$$N_{E(z)/E}(\alpha) = \prod N_{E(x_i)/E}(b_i + t_i\epsilon).$$

The proof of Theorem 9.7 uses Theorem 10.4, which in turn depends upon the cobordism results in the appendix. Theorem 9.7 is the key ingredient in the proof of Theorem 9.8, and we will see that the Norm Principle follows easily from Theorem 9.8.

Theorem 9.8. *If k is p-special and $[E:k] = p$ then $\overline{A}_0(X_E,\mathcal{K}_1) \xrightarrow{N_{E/k}} \overline{A}_0(X,\mathcal{K}_1)$ sends $\widetilde{A}_0(E)$ to $\widetilde{A}_0(k)$.*

Proof. If $\{\underline{a}\}_E = 0$ then the generic splitting variety X has an E-point x, and Theorem 9.8 is immediate from Example 9.6. Indeed, in this case X_E has an E-point x' over x, every element of $\widetilde{A}_0(E) \cong E^\times$ has the form $[x',\alpha]$, and $N_{E/k}[x',\alpha] = [x,\alpha]$. Hence we may assume that $\{\underline{a}\}_E \neq 0$. This has the advantage that $E(x_i) = E \otimes_k k(x_i)$ is a field for every $x_i \in X$.

Choose $\theta = [z,\alpha] \in \widetilde{A}_0(E)$ and let $x_i \in X$, t_i and b_i be the data given by Theorem 9.7. Each x_i lifts to an $E(x_i)$-point $x_i \otimes E$ of X_E so we may consider the element

$$\theta' = \theta - \sum [x_i \otimes E, b_i + t_i\epsilon] \in \overline{A}_0(X_E,\mathcal{K}_1).$$

By 9.4 over E, θ' belongs to the subgroup $\widetilde{A}_0(E)$. By Theorem 9.7, its norm is

$$N(\theta') = N_{E(z)/E}(\alpha)/\prod N_{E(x_i)/E}(b_i + t_i\epsilon) = 1.$$

By Lemma 9.5, $\theta' = 0$. Hence $N_{E/k}(\theta) = \sum \left[x_i, N_{E(x_i)/k(x_i)}(b_i + t_i\epsilon)\right]$ in $\overline{A}_0(X,\mathcal{K}_1)$. Since $\widetilde{A}_0(k)$ is a group by 9.4, this is an element of $\widetilde{A}_0(k)$. $\square$

Corollary 9.9 (Theorem 0.7(3)). *If k is p-special then $\widetilde{A}_0(k) = \overline{A}_0(X,\mathcal{K}_1)$, and $N : \overline{A}_0(X,\mathcal{K}_1) \to k^\times$ is an injection.*

Proof. We may suppose that $X(k) = \emptyset$. For every closed $z \in X$ there is an intermediate subfield E with $[k(z) : E] = p$ and a $k(z)$-point z' in X_E over z. Since $[z', \alpha] \in \widetilde{A}_0(E)$, Theorem 9.8 implies that $[z, \alpha] = N[z', \alpha]$ is in $\widetilde{A}_0(k)$. This proves the first assertion. The second follows from this and Lemma 9.5. $\square$

The Norm Principle of the Introduction follows from Theorem 9.8.

Proof of the Norm Principle (Theorem 0.3). We consider a generator $[z, \alpha]$ of $\overline{A}_0(X, \mathcal{K}_1)$. Since $[k(z) : k] = p^\nu$ for $\nu > 0$, there is a subfield E of $k(z)$ with $[k(z) : E] = p$, and z lifts to a $k(z)$-point z' of X_E. By construction, $[z', \alpha] \in \widetilde{A}_0(E)$ and $\overline{A}_0(X_E, \mathcal{K}_1) \to \overline{A}_0(X, \mathcal{K}_1)$ sends $[z', \alpha]$ to $[z, \alpha]$. By Theorem 9.8, $[z, \alpha]$ is in $\widetilde{A}_0(k)$, *i.e.*, is represented by an element $[x, \alpha]$ with $[k(x) : k] = p$. $\square$

10 Expressing Norms

Recall that $E = k(\epsilon)$ is a fixed Kummer extension of a p-special field k, and X is a norm variety over k for the symbol $\{\underline{a}\}$. The purpose of this section is to prove Theorem 9.7, that if an element $w \in E$ is a norm for a Kummer point of X_E then w is a product of norms of the form specified in Theorem 9.7.

Recall from 7.2 that $Q \subseteq \mathbb{P}(\mathcal{A} \oplus \mathcal{O}) \times \mathbb{A}^1_k$ is the variety of all points $([\beta, t], w)$ such that $N(\beta) = t^p w$, and let $q : Q \to \mathbb{A}^1_k$ be the projection. Extending the base field to E and applying the Weil restriction functor, we obtain a morphism

$$Rq = \mathrm{Res}_{E/k}(q_E) : RQ = \mathrm{Res}_{E/k}(Q_E) \to \mathbb{A}^E.$$

Moreover, choose once and for all a resolution of singularities $\tilde{Q} \to Q$, which is an isomorphism where $t \neq 0$. This is possible since Q is smooth where $t \neq 0$, see 7.2.

Remark 10.1. Since k is p-special, so is E. As stated in Lemma 9.5, the norm map $\widetilde{A}_0(E) \to E^\times$ is injective; we identify $\widetilde{A}_0(E)$ with its image. Thus $[z, \alpha] \in \widetilde{A}_0(E)$ is identified with $N_{E(z)/E}(\alpha) \in E^\times$. By [12, Theorem 5.5], there is a point $s \in S$ such that $E(z) = \mathcal{A}_s \otimes E$. Under the correspondence $E(z) \cong \mathbb{A}(\mathcal{A})_s(E)$, we identify α with a point of $\mathbb{A}(\mathcal{A})(E)$, lying over $s \in S$. Then $N_{E(z)/E}(\alpha) = Rq([\alpha, 1], N(\alpha))$. In other words, $\widetilde{A}_0(E) \subseteq E^\times$ is equal to $q(Q(E)) - \{0\}$.

To prove Theorem 9.7 it therefore suffices to show that $\overline{Y}_w(k)$ is non-empty when $w = Rq([\beta, 1], w)$. To do this, we will produce a correspondence $Z \to \overline{Y} \times_{\mathbb{A}^E} RQ$ that is dominant and of degree prime to p over RQ. We construct the correspondence Z using the Multiplication Principle of [12, 5.7] in the following form.

Lemma 10.2 (Multiplication Principle). *Let k be a p-special field. Then the set of values of the map $N : \mathbb{A}(\mathcal{A})(k) \to k$ is a multiplicative subset of $k^\times$.*

Proof. Given Remark 10.1, this is a consequence of Lemma 9.4. □

Lemma 10.3. *Let $F = k(\overline{Y})$ be the function field. Then there exists a finite extension L/F, of degree prime to p, and a point $\xi \in RQ(L)$ lying over the generic point of $\mathbb{A}^E$.*

Proof. Let F' be the maximal prime-to-p extension of F; then the field $EF' = E \otimes_k F'$ is p-special. We may regard the generic point of $\overline{Y}$ as an element in $\overline{Y}(F)$. Applying the inclusion $F \subset F'$ to this element, followed by the projection $\overline{Y} \to \mathbb{A}^E$, we obtain an element ω of $\mathbb{A}^E(F') = EF'$. By 7.8, ω is a product of norms from $\mathbb{A}(\mathcal{A})(EF')$. By the Multiplication Principle 10.2, there exists $\beta \in \mathbb{A}(\mathcal{A})(EF')$ such that $N(\beta) = \omega$. Now let ξ be the point $([\beta, 1], \omega) \in RQ(F')$. Then $Rq(\xi) = \omega$ and ξ is defined over some finite intermediate extension $F \subseteq L \subseteq F'$, with $[L : F]$ prime to p. □

Write η_L for the point of $\overline{Y}(L)$ defined by the inclusion $F \subseteq L$. We can now define $\overline{Y} \xleftarrow{f} Z \xrightarrow{g} RQ$ to be a (smooth, projective) model of the point $(\eta_L, \xi) \in (\overline{Y} \times_{\mathbb{A}^E} RQ)(L)$.

Theorem 10.4. *The morphism $g : Z \to RQ$ is proper and dominant (hence onto) and of degree prime to p.*

Proof. Let $\omega \in \mathbb{A}^E$ be the generic point, $k(\omega)$ the function field and $E(\omega) = E \otimes k(\omega)$. As degree is a generic notion and invariant under extension of the base field, we may replace $\overline{Y} \leftarrow Z \to RQ$ by its basechange along the morphism

$$\mathrm{Spec}(E(\omega)) \to \mathrm{Spec}(k(\omega)) \xrightarrow{\omega} \mathbb{A}^E,$$

to obtain morphisms $f : Z_{E(\omega)} \to \overline{Y}_{E(\omega)}$ and $g : Z_{E(\omega)} \to RQ_{E(\omega)}$. Using the normal basis theorem, we can write $E(\omega) = E(\omega_1, \dots, \omega_p)$ for transcendentals ω_i that are permuted under the action of the cyclic group $\mathrm{Gal}(E/k)$.

We will apply the DN Theorem A.1 with base field $k' = E(\omega)$. In the notation of Theorem A.1, we let $r = p$; we write Y for some desingularization of $\overline{Y}_{E(\omega)}$; we let X be $R\tilde{Q}_{E(\omega)}$, and we let W be a model for $Z_{E(\omega)}$ mapping to Y and X. Finally, we let $u_i = \{a_1, \dots, a_n, \omega_i\} \in K^M_{n+1}(k')/p$.

Observe that our base field contains E, so $R\tilde{Q}_{E(\omega)} = \mathrm{Res}_{E/k}(\tilde{Q}_E) \times_{\mathbb{A}^E} E(\omega)$ splits as a product $R\tilde{Q}_{E(\omega)} = \prod_{i=1}^p \tilde{Q}_{\omega_i}$, where Q_{ω_i} is the fiber of $Q \to \mathbb{A}^1$ over the point $\omega_i \in \mathbb{A}^1(E(\omega)) = E(\omega)$. Therefore we have $X = \prod_{i=1}^p X_i$ where X_i is $\tilde{Q}_{\omega_i}$, the resolution of singularities of Q_{ω_i}. By Remark 7.4, X_i is a smooth, geometrically irreducible splitting variety for the symbol u_i of dimension $p^n - 1$. Thus, hypothesis (1) of the DN Theorem A.1 is satisfied.

By Theorem A.10, $t_{d,1}(X_i) = t_{d,1}(\mathbb{P}(\mathcal{A}))$; by Lemma A.6, we conclude that $s_d(X_i) \equiv v s_d(\mathbb{P}(\mathcal{A})) \mod p^2$ for some unit $v \in \mathbb{Z}/p$. Since $s_d(\mathbb{P}(\mathcal{A})) \not\equiv 0$ by Theorem 8.1, we conclude that hypothesis (3) of the DN Theorem A.1 is satisfied.

Furthermore, $K = k'(X_1 \times \cdots \times X_{i-1})$ is contained in a rational function field over E; in fact, the field $E(\omega_j)(Q_{\omega_j})$ becomes a rational function field once we adjoin $\sqrt[p]{\gamma}$. Since E does not split $\{\underline{a}\}$, K does not split $\{\underline{a}\}$ either. It follows that K does not split $u_i = \{\underline{a}\} \cup \{\omega_i\}$, verifying hypothesis (2) of Theorem A.1.

We have now checked the hypotheses (1–3) of Theorem A.1. It remains to check that X and Y are G-fixed point equivalent up to a prime-to-p factor. In fact, we proved in Theorem 7.14 that $\overline{Y}_{E(\omega)}$ is G-fixed point equivalent to $(p-1)!$ copies of $\mathbb{P}(\mathcal{A})^p$, hence so is Y (since the fixed points lie in the smooth locus), and in Theorem 7.3 that X_i is G-fixed point equivalent to $\mathbb{P}(\mathcal{A})$. That is, Y is G-fixed point equivalent to $(p-1)!$ copies of X. Therefore the DN Theorem applies to show that g is dominant and of degree prime to p, as asserted. $\square$

Proof (of Theorem 9.7). We have proved that there is a diagram $\overline{Y} \xleftarrow{f} Z \xrightarrow{g} RQ$ such that the degree of g is prime to p. By blowing up if necessary we may assume that $g : Z \to RQ$ factors through $\tilde{g} : Z \to R\tilde{Q}$, with $\deg(\tilde{g})$ prime to p.

Let $[z, \alpha] \in \widetilde{A}_0(E)$, and set $w = N_{E(z)/E}(\alpha)$. By Remark 10.1, there exists a point $([\beta, 1], w) \in RQ(k)$. Lift this to a point in $R\tilde{Q}(k)$ (recall that $R\tilde{Q} \to RQ$ is an isomorphism where $t \neq 0$). Since $Z \to R\tilde{Q}$ is a morphism of smooth projective varieties of degree prime to p and k is p-special, we can lift $([\beta, 1], w)$ to a k-point of Z, and then apply $f : Z \to \overline{Y}$ to get a k-point in $\overline{Y}_w$. By the definition of $\overline{Y}$ and Corollary 7.10, this means that we can find Kummer extensions $k(x_i)/k$ (corresponding to points $s_i \in S$, and determining points $x_i \in X$ because X is a p-generic splitting variety), elements $b_i \in k(x_i)$ and $t_i \in k$ such that $w = \prod_i N_{E(x_i)/E}(b_i + t_i\epsilon)$, as asserted. $\square$

A Appendix: The DN Theorem

In this appendix, we give a proof of the following Degree Theorem, which is used in the proof of Theorem 9.8, which is the key step in establishing the Norm Principle. Throughout, k will be a fixed field of characteristic 0, p will be a prime, $n \geq 1$ will be an integer and we fix $d = p^n - 1$.

Recall from Definition 7.1 that if X and Y are G-fixed point equivalent then $\dim(X) = \dim(Y)$, the fixed points are 0-dimensional and their tangent space representations are isomorphic (over $\bar{k}$).

Theorem A.1 (DN Theorem). *For $r \geq 1$, let $u_1, \dots, u_r$ be symbols in $K^M_{n+1}(k)/p$ and let $X = \prod_1^r X_i$, where the X_i are irreducible smooth projective G-varieties of dimension $d = p^n - 1$ such that:*

1. *$k(X_i)$ splits u_i*
2. *u_i is non-zero over $k(X_1 \times \cdots \times X_{i-1})$ and*
3. *$p^2 \nmid s_d(X_i)$*

Let Y be a smooth irreducible projective G-variety which is G-fixed point equivalent to the disjoint union of m copies of X, where $p \nmid m$. Let F be a finite extension of $k(Y)$ of degree prime to p, and Spec$(F) \to X$ *a point, with model $f : W \to X$. Then f is dominant and of degree prime to p.*

$$\begin{tikzcd} \mathrm{Spec}(F) \arrow[r] \arrow[d, "\text{finite}"'] & W \arrow[d, "g"'] \arrow[dr, "f\ \text{(dominant)}"] & \\ \mathrm{Spec}(k(Y)) \arrow[r] & Y & X. \end{tikzcd}$$

The proof will use two ingredients: the degree formulas A.2 and A.5, due to Levine and Morel; and a standard localization result A.10 in (complex) cobordism theory. The former concern the algebraic cobordism ring $\Omega_*(k)$, and the latter concern the complex bordism ring MU_*. These are related via the Lazard ring $\mathbb{L}_*$; combining Quillen's theorem [1, II.8] and the Morel–Levine theorem [3, 4.3.7], we have graded ring isomorphisms:

$$\Omega_*(k) \cong \mathbb{L}_* \cong MU_{2*}.$$

Here is the Levine–Morel generalized degree formula for an irreducible projective variety X, taken from [3, Theorem 4.4.15]. It concerns the ideal $M(X)$ of $\Omega_*(k)$ generated by the classes $[Z]$ of smooth projective varieties Z such that there is a k-morphism $Z \to X$, and $\dim(Z) < \dim(X)$.

Theorem A.2 (Generalized Degree Formula). *Let $f:Y \to X$ be a morphism of smooth projective k-varieties. If $\dim(X) = \dim(Y)$ then $[Y] - \deg(f)[X] \in M(X)$.*

Trivially, if $[Z] \in M(X)$ then $M(Z) \subseteq M(X)$. We also have:

Lemma A.3. *Let X be a smooth projective k-variety. If Z and Z' are birationally equivalent, then $[Z] \in M(X)$ holds if and only if $[Z'] \in M(X)$.*

Proof. By [3, 4.4.17], the class of Z modulo $M(Z)$ is a birational invariant. Thus $[Z'] - [Z] \in M(Z)$. Because $M(Z) \subseteq M(X)$, the result follows. □

We shall also need the Levine–Morel "higher degree formula" A.5, which is taken from [3, Theorem 4.4.24], and concerns the mod p characteristic numbers $t_{d,r}(X)$ of Definition A.4, where p is prime, $n \geq 1$ and $d = p^n - 1$.

Choose a graded ring homomorphism $\psi : \mathbb{L}_* \to \mathbb{F}_p[v_n]$ corresponding to some height n formal group law, where v_n has degree d; many such group laws exist, and the class $t_{d,r}$ will depend on this choice, but only up to a unit. (See [3, 4.4].)

Definition A.4. For $r > 0$, the homomorphism $t_{d,r} : \Omega_{rd}(k) \cong \mathbb{L}_{rd} \to \mathbb{F}_p$ sends x to the coefficient of v_n^r in $\psi(x)$. If X is a smooth projective variety over k, of dimension rd, then X determines a class $[X]$ in $\Omega_{rd}(k)$, and $t_{d,r}(X)$ is $t_{d,r}([X])$.

Theorem A.5 (Higher Degree Formula). *Let $f: W \to X$ be a morphism of smooth projective varieties of dimension rd and suppose that X admits a sequence of surjective morphisms*

$$X = X^{(r)} \to X^{(r-1)} \to \cdots \to X^{(0)} = \mathrm{Spec}(k)$$

such that

1. $\dim(X^{(i)}) = i\, d$.
2. *p divides the degree of every zero-cycle on $X^{(i)} \times_{X^{(i-1)}} k(X^{(i-1)})$.*

Then $t_{d,r}(W) = \deg(f)\, t_{d,r}(X)$.

Here are some properties of this characteristic number that we shall need. Recall that if $\dim(X) = d$ then p divides $s_d(X)$, so that $s_d(X)/p$ is an integer.

Lemma A.6. *Let X/k be a smooth projective variety, and $k \subseteq \mathbb{C}$ an embedding.*

1. *For $r = 1$, there is a unit $u \in \mathbb{F}_p$ such that $t_{d,1}(X) \equiv u\, s_d(X)/p$.*
2. *If $X = \prod_{i=1}^r X_i$ and $\dim(X_i) = d$ for all i, then $t_{d,r}(X) = \prod_{i=1}^r t_{d,1}(X_i)$.*
3. *$t_{d,r}(X)$ depends only on the class of $(X \times_k \mathbb{C})^{an}$ in the complex cobordism ring.*

Proof. Part (1) is [3, Proposition 4.4.22]. Part (2) is immediate from the definition of $t_{d,r}$ and the graded multiplicative structure on $\Omega_*(k)$. Finally, part (3) is a consequence of the fact that the natural homomorphism $\Omega_*(k) \to MU_{2*}$ is an isomorphism (since both rings are isomorphic to the Lazard ring). □

We remark that the class called s_d in this article is the S_d in [3]; the class called $s_d(X)$ in [3] is our class $s_d(X)/p$.

The next lemma is a variant of Theorem A.5. It uses the same hypotheses.

Lemma A.7. *Let X be as in Theorem A.5. Then $\psi(M(X)) = 0$.*

Proof. Consider Z with $[Z] \in M(X)$. If d does not divide $\dim(Z)$, then $\psi([Z]) = 0$ for degree reasons. If $\dim(Z) = 0$, then the image of Z is a closed point of X; since the degree of such a closed point is divisible by p, we have $\psi([Z]) = 0$. Hence we may assume that $\dim(Z) = sd$ for some $0 < s < r$. The cases $r = 1$ and $s = 0$ are immediate, so we proceed by induction on r and s.

Let $f : Z \to X$ be a k-morphism with $\dim(Z) = sd$, and let $f_s : Z \to X^{(s)}$ be the obvious composition. As $\dim(Z) = \dim(X^{(s)})$, the generalized degree formula A.2 applies to show that $[Z] - \deg(f_s)([X^{(s)}]) \in M(X^{(s)})$. By induction on r, $\psi(M(X^{(s)})) = 0$, so $\psi([Z]) = \deg(f_s)\psi([X^{(s)}])$. We claim that $\deg(f_s) \equiv 0 \mod p$, which yields $\psi([Z]) = 0$, as desired.

If f_s is not dominant, then $\deg(f_s) = 0$ by definition. On the other hand, if f_s is dominant, then the generic point of Z maps to a closed point η of $X^{(s+1)} \times_{X^{(s)}} k(X^{(s)})$. By condition (2) of Theorem A.5, p divides $\deg(\eta) = \deg(f_s)$. □

We will need to show that $\psi(M(Y)) = 0$ for the Y appearing in Theorem A.1. This is accomplished in the next lemma.

Lemma A.8. *Suppose X, Y and W are smooth projective varieties of dimension rd over k, and $f : W \to X$ and $g : W \to Y$ are morphisms. Suppose further that $\psi(M(X)) = 0$ and that p does not divide $\deg(g)$. Then $\psi(M(Y)) = 0$.*

Proof. Suppose $[Z] \in M(Y)$. As $g : W \to Y$ is a proper morphism of smooth varieties, of degree prime to p, we can lift the generic point $\mathrm{Spec}(k(Z)) \to Y$ to a point $q : \mathrm{Spec}(F) \to W$ for some field extension $F/k(Z)$ of degree e prime to p. Let $\tilde{Z}$ be a smooth projective model of F possessing a morphism to Z and a morphism to X extending the k-morphism $f \circ q : \mathrm{Spec}(F) \to X$. Hence $[\tilde{Z}] \in M(X)$. By the degree formula for the map $\tilde{Z} \to Z$, $e\,[Z] - [\tilde{Z}] \in M(Z)$. If $\dim(Z) = 0$, then $M(Z) = (0)$. In general, $M(Z)$ is generated by the classes of varieties of dimension less than $\dim(Z)$ that map to Z (hence a fortiori also map to Y) over k. By induction on the dimension of Z, we may assume that $\psi(M(Z)) = 0$. Moreover, $\psi([\tilde{Z}]) = 0$ by assumption; since p does not divide e, we conclude that $\psi([Z]) = 0$ as asserted. $\square$

Finally, we will use the following standard bordism localization result.

Lemma A.9. *Suppose that the abelian p-group $G = \mu_p^n$ acts without fixed points on an compact complex analytic manifold M, by holomorphic maps. Then $\psi([M]) = 0$ in $\mathbb{F}_p$.*

Proof. By [13], $[M]$ is in the ideal of MU_* generated by $\{p, [M_1] \dots, [M_{n-1}]\}$, where $\dim_{\mathbb{C}}(M_i) = p^i - 1$. Since p is the only generator of this ideal whose dimension is a multiple of $d = p^n - 1$, ψ is zero on every generator and hence on the ideal. $\square$

Theorem A.10. *Let G be μ_p^n and let X and Y be compact complex G-manifolds which are G-fixed point equivalent. Then $\psi([X]) = \psi([Y])$.*

Proof. Remove equivariantly isomorphic small balls about the fixed points of X and Y, and let $M = X \cup -Y$ denote the result of joining the rest of X and Y, with the opposite orientation on Y. Then X and Y determine canonical weakly complex structures on punctured neighborhoods of the fixed points. As these structures are completely determined by the normal bundles at the fixed points (see [2, 3.1]), they glue to give M a canonical weakly complex structure. Moreover, G acts on M with no fixed points, and $[X]-[Y] = [M]$ in MU_*. By Lemma A.9, $\psi([X]) - \psi([Y]) = \psi([M]) = 0$. $\square$

We can now prove Theorem A.1. Note that the inclusion $k(Y) \subset F$ induces a dominant rational map $W \to Y$; we may replace W by a blowup to eliminate the points of indeterminacy and obtain a morphism $g : W \to Y$, whose degree is prime to p, without affecting the statement of Theorem A.1.

Proof of the DN Theorem A.1. We will apply Theorem A.5 to X and the $X^{(t)} = \prod_{i=1}^{t} X_i$. We must first check that the hypotheses are satisfied. The first condition is obvious. For the second condition, it is convenient to fix t and set $F = k(X_1 \times \cdots \times X_{t-1})$, $X' = X^{(t)} \times_{X^{(t-1)}} F$. By hypotheses (1–2) of Theorem A.1, the symbol u_t is nonzero over F but splits over the generic point of X'; by specialization, it splits over all closed points. A transfer argument implies that the degree of any closed point η of X' is divisible by p; this is the second condition. Hence Theorem A.5 applies and we have $t_{d,r}(W) = \deg(f)\, t_{d,r}(X)$.

By Lemmas A.8 and A.7, we have that $\psi(M(Y)) = 0$; by the generalized degree formula A.2, we conclude that $\psi([W]) = \deg(g)\,\psi([Y])$, so that $t_{d,r}(W) = \deg(g)\,t_{d,r}(Y) \neq 0$. Hence

$$\deg(f)\,t_{d,r}(X) = \deg(g)\,t_{d,r}(Y).$$

By Theorem A.10 and Lemma A.6(3), $m t_{d,r}(X) = t_{d,r}(Y)$. Condition (3) of Theorem A.1 and Lemma A.6 imply that $t_{d,1}(X_i) \neq 0$ for all i and hence that $t_{d,r}(X) \neq 0$. It follows that $m\deg(g) \equiv \deg(f) \neq 0$ modulo p, as required. □

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On the Whitehead Spectrum of the Circle

Lars Hesselholt

Abstract The seminal work of Waldhausen, Farrell and Jones, Igusa, and Weiss and Williams shows that the homotopy groups in low degrees of the space of homeomorphisms of a closed Riemannian manifold of negative sectional curvature can be expressed as a functor of the fundamental group of the manifold. To determine this functor, however, it remains to determine the homotopy groups of the topological Whitehead spectrum of the circle. The cyclotomic trace of Bökstedt, Hsiang, and Madsen and a theorem of Dundas, in turn, lead to an expression for these homotopy groups in terms of the equivariant homotopy groups of the homotopy fiber of the map from the topological Hochschild $\mathbb{T}$-spectrum of the sphere spectrum to that of the ring of integers induced by the Hurewicz map. We evaluate the latter homotopy groups, and hence, the homotopy groups of the topological Whitehead spectrum of the circle in low degrees. The result extends earlier work by Anderson and Hsiang and by Igusa and complements recent work by Grunewald, Klein, and Macko.

Introduction

Let M be a closed smooth manifold of dimension $m \geqslant 5$. Then, the stability theorem of Igusa [22] and a theorem of Weiss and Williams [35, Theorem A] show that, for all integers q less both $(m-4)/3$ and $(m-7)/2$, there is a long-exact sequence

$$\cdots \to \mathbb{H}_{q+2}(C_2, \tau_{\geqslant 2}\,\mathrm{Wh}^{\mathrm{Top}}(M)) \to \pi_q(\mathrm{Homeo}(M)) \to \pi_q(\widetilde{\mathrm{Homeo}}(M)) \to \dots$$

where the middle group is the qth homotopy group of the space of homeomorphisms of M. In particular, the group $\pi_0(\mathrm{Homeo}(M))$ is the mapping class group of M. The right-hand term is the qth homotopy group of the space of block homeomorphisms

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of M and is the subject of surgery theory. The left-hand term is the $(q+2)$th homotopy group of the Borel quotient of the 2-connective cover of the topological Whitehead spectrum of M by the canonical involution. It is one of the great past achievements that the left-hand term can be expressed by Waldhausen's algebraic K-theory of spaces [32–34].

Suppose, in addition, that M carries a Riemannian metric of negative, but not necessarily constant, sectional curvature. Another great achievement is the topological rigidity theorems [11, Remark 1.10, Theorem 2.6] of Farrell and Jones which, in this case, give considerable simplifications of the left and right-hand terms in the above sequence. For the right-hand term, there are canonical isomorphisms

$$\pi_q(\widetilde{\mathrm{Homeo}}(M)) \xrightarrow{\sim} \pi_q(\widetilde{\mathrm{HoAut}}(M)) \xleftarrow{\sim} \pi_q(\mathrm{HoAut}(M)),$$

where $\mathrm{HoAut}(M)$ and $\widetilde{\mathrm{HoAut}}(M)$ are the spaces of self-homotopy equivalences and block self-homotopy equivalences of M, respectively. We note that, as M is aspherical with $\pi_1(M)$ centerless [27, Theorems 22, 24], it follows from [13, Theorem III.2] that the canonical map from $\mathrm{HoAut}(M)$ to the discrete group $\mathrm{Out}(\pi_1(M))$ is a weak equivalence. For the left-hand term, there is a canonical isomorphism

$$\bigoplus_{(C)} \mathrm{Wh}_q^{\mathrm{Top}}(S^1) \xrightarrow{\sim} \mathrm{Wh}_q^{\mathrm{Top}}(M),$$

where the sum ranges over the set of conjugacy classes of maximal cyclic subgroups of the torsion-free group $\pi_1(M)$; see also [24, Theorem 139]. Hence, in order to evaluate the groups $\pi_q(\mathrm{Homeo}(M))$, it remains to evaluate

$$\mathrm{Wh}_q^{\mathrm{Top}}(S^1) = \pi_q(\mathrm{Wh}^{\mathrm{Top}}(S^1))$$

and the canonical involution on these groups. We prove the following result.

Theorem 1. *The groups* $\mathrm{Wh}_0^{\mathrm{Top}}(S^1)$ *and* $\mathrm{Wh}_1^{\mathrm{Top}}(S^1)$ *are zero. Moreover, there are canonical isomorphisms*

$$\mathrm{Wh}_2^{\mathrm{Top}}(S^1) \xrightarrow{\sim} \bigoplus_{r\geqslant 1} \bigoplus_{j\in\mathbb{Z}\smallsetminus 2\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$$

$$\mathrm{Wh}_3^{\mathrm{Top}}(S^1) \xrightarrow{\sim} \bigoplus_{r\geqslant 0} \bigoplus_{j\in\mathbb{Z}\smallsetminus 2\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{r\geqslant 1} \bigoplus_{j\in\mathbb{Z}\smallsetminus 2\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}.$$

The statement for $q=0$ and $q=1$ was proved earlier by Anderson and Hsiang [1] by different methods. It was also known by work of Igusa [21] that the two sides of the statement for $q=2$ are abstractly isomorphic. The statement for $q=3$ is new. We also note that in recent work, Grunewald et al. [15] have proved that for p an odd prime and $q \leqslant 4p-7$, the p-primary torsion subgroup of $\mathrm{Wh}_q^{\mathrm{Top}}(S^1)$ is a countably dimensional $\mathbb{F}_p$-vector space, if $q=2p-2$ or $2p-1$, and zero, otherwise. Hence, we will here focus the attention on the 2-primary torsion subgroup.

We briefly outline the proof of Theorem 1. The seminal work of Waldhausen establishes a cofibration sequence of spectra

$$S^1_+ \wedge K(\mathbb{S}) \xrightarrow{\alpha} K(\mathbb{S}[x^{\pm 1}]) \to \mathrm{Wh}^{\mathrm{Top}}(S^1) \xrightarrow{\partial} \Sigma S^1_+ \wedge K(\mathbb{S}),$$

which identifies the topological Whitehead spectrum of the circle as the mapping cone of the assembly map in algebraic K-theory [33, Theorem 3.3.3], [34, Theorem 0.1]. Here $\mathbb{S}$ is the sphere spectrum and $\mathbb{S}[x^{\pm 1}]$ is the Laurent polynomial extension. If we replace the sphere spectrum by the ring of integers, the assembly map

$$\alpha\colon S^1_+ \wedge K(\mathbb{Z}) \to K(\mathbb{Z}[x^{\pm 1}])$$

becomes a weak equivalence by the fundamental theorem of algebraic K-theory [28, Theorem 8, Corollary]. Hence, we obtain a cofibration sequence of spectra

$$S^1_+ \wedge K(\mathbb{S}, I) \xrightarrow{\alpha} K(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}]) \to \mathrm{Wh}^{\mathrm{Top}}(S^1) \xrightarrow{\partial} \Sigma S^1_+ \wedge K(\mathbb{S}, I),$$

where the spectra $K(\mathbb{S}, I)$ and $K(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}])$ are defined to be the homotopy fibers of the maps of K-theory spectra induced by the Hurewicz maps $\ell\colon \mathbb{S} \to \mathbb{Z}$ and $\ell\colon \mathbb{S}[x^{\pm 1}] \to \mathbb{Z}[x^{\pm 1}]$, respectively. The Hurewicz maps are rational equivalences, as was proved by Serre. This implies that $K(\mathbb{S}, I)$ and $K(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}])$ are rationally trivial spectra. It follows that, for all integers q,

$$\mathrm{Wh}^{\mathrm{Top}}_q(S^1) \otimes \mathbb{Q} = 0.$$

Therefore, it suffices to evaluate, for every prime number p, the homotopy groups with p-adic coefficients,

$$\mathrm{Wh}^{\mathrm{Top}}_q(S^1; \mathbb{Z}_p) = \pi_q(\mathrm{Wh}^{\mathrm{Top}}(S^1)_p),$$

that are defined to be the homotopy groups of the p-completion [6].

The cyclotomic trace map of Bökstedt et al. [4] induces a map

$$\mathrm{tr}\colon K(\mathbb{S}, I) \to \mathrm{TC}(\mathbb{S}, I; p)$$

from the relative K-theory spectrum to the relative topological cyclic homology spectrum. It was proved by Dundas [8] that this map becomes a weak equivalence after p-completion. The same is true for the Laurent polynomial extension. Hence, we have a cofibration sequence of implicitly p-completed spectra

$$S^1_+ \wedge \mathrm{TC}(\mathbb{S}, I; p) \xrightarrow{\alpha} \mathrm{TC}(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}]; p) \to \mathrm{Wh}^{\mathrm{Top}}(S^1) \xrightarrow{\partial} \Sigma S^1_+ \wedge \mathrm{TC}(\mathbb{S}, I; p).$$

There is also a "fundamental theorem" for topological cyclic homology which was proved by Madsen and the author in [19, Theorem C]. If A is a symmetric ring spectrum whose homotopy groups are $\mathbb{Z}_{(p)}$-modules, this theorem expresses, up to an extension, the topological cyclic homology groups $\mathrm{TC}_q(A[x^{\pm 1}]; p)$ of the Laurent polynomial extension in terms of the equivariant homotopy groups

$$\mathrm{TR}^n_q(A; p) = [S^q \wedge (\mathbb{T}/C_{p^{n-1}})_+, T(A)]_{\mathbb{T}}$$

of the topological Hochschild $\mathbb{T}$-spectrum $T(A)$ and the maps

$$\begin{aligned}
&R\colon \mathrm{TR}^n_q(A; p) \to \mathrm{TR}^{n-1}_q(A; p) && \text{(restriction)}\\
&F\colon \mathrm{TR}^n_q(A; p) \to \mathrm{TR}^{n-1}_q(A; p) && \text{(Frobenius)}\\
&V\colon \mathrm{TR}^{n-1}_q(A; p) \to \mathrm{TR}^n_q(A; p) && \text{(Verschiebung)}\\
&d\colon \mathrm{TR}^n_q(A; p) \to \mathrm{TR}^n_{q+1}(A; p) && \text{(Connes' operator)}
\end{aligned}$$

which relate these groups. Here $\mathbb{T}$ is the multiplicative group of complex numbers of modulus 1, and $C_{p^{n-1}} \subset \mathbb{T}$ is the subgroup of the indicated order. We recall the groups $\mathrm{TR}^n_q(A; p)$ in Section 1 and give a detailed discussion of the fundamental theorem in Section 2. In the following Sections 3 and 4, we briefly recall the cyclotomic trace map and the skeleton spectral sequence which we use extensively in later sections. A minor novelty here is Proposition 4 which generalizes of the fundamental long-exact sequence [17, Theorem 2.2] to a long-exact sequence

$$\cdots \to \mathbb{H}_q(C_{p^m}, TR^n(A; p)) \to \mathrm{TR}^{m+n}_q(A; p) \xrightarrow{R^n} \mathrm{TR}^m_q(A; p) \to \cdots$$

valid for all positive integers m and n.

The problem to evaluate $\mathrm{Wh}^{\mathrm{Top}}_q(S^1)$ is thus reduced to the homotopy theoretical problem of evaluating the equivariant homotopy groups $\mathrm{TR}^n_q(\mathbb{S}, I; p)$ along with the maps listed above. In the paper [15] mentioned earlier, the authors approximate the Hurewicz map $\ell\colon \mathbb{S} \to \mathbb{Z}$ by a map of suspension spectra $\theta\colon \mathbb{S}[SG] \to \mathbb{S}$ and use the Segal-tom Dieck splitting to essentially evaluate the groups $\mathrm{TR}^n_q(\mathbb{S}, I; p)$, for p odd and $q \leqslant 4p - 7$. However, this approach is not available, for $q > 4p - 7$, where a genuine understanding of the domain and target of the map

$$\mathrm{TR}^n_q(\mathbb{S}; p) \to \mathrm{TR}^n_q(\mathbb{Z}; p)$$

appears necessary. We evaluate $\mathrm{TR}^n_q(\mathbb{S}, I; 2)$, for $q \leqslant 3$, and we partly evaluate the four maps listed above. The result, which is Theorem 25, is the main result of the paper, and the proof occupies Sects. 5–7. The homotopy theoretical methods we employ here are perhaps somewhat simple-minded and more sophisticated methods will certainly make it possible to evaluate the groups $\mathrm{TR}^n_q(\mathbb{S}, I; p)$ in a wider range of degrees. In particular, it would be very interesting to understand the corresponding homology groups. However, to evaluate the groups $\mathrm{TR}^n_q(\mathbb{S}, I; p)$ is at least as difficult as to evaluate the stable homotopy groups of spheres. In the Section 8, we apply the fundamental theorem to the result of Theorem 25 and prove Theorem 1.

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1 The Groups $\mathrm{TR}^n_q(A;p)$

Let A be a symmetric ring spectrum [20, Section 5.3]. The topological Hochschild $\mathbb{T}$-spectrum $T(A)$ is a cyclotomic spectrum in the sense of [17, Definition 2.2]. In particular, it is an object of the $\mathbb{T}$-stable homotopy category. Let $C_r \subset \mathbb{T}$ be the subgroup of order r, and let $(\mathbb{T}/C_r)_+$ be the suspension $\mathbb{T}$-spectrum of the union of $\mathbb{T}/C_r$ and a disjoint basepoint. One defines the equivariant homotopy group

$$\mathrm{TR}^n_q(A;p) = [S^q \wedge (\mathbb{T}/C_{p^{n-1}})_+, T(A)]_{\mathbb{T}}.$$

to be the abelian group of maps in the $\mathbb{T}$-stable homotopy category between the indicated $\mathbb{T}$-spectra. The Frobenius map, Verschiebung map, and Connes' operator, which we mentioned in the Introduction, are induced by maps

$$\begin{aligned} f&:(\mathbb{T}/C_{p^{n-2}})_+ \to (\mathbb{T}/C_{p^{n-1}})_+\\ v&:(\mathbb{T}/C_{p^{n-1}})_+ \to (\mathbb{T}/C_{p^{n-2}})_+\\ \delta&:\Sigma(\mathbb{T}/C_{p^{n-1}})_+ \to (\mathbb{T}/C_{p^{n-1}})_+ \end{aligned}$$

in the $\mathbb{T}$-stable homotopy category defined as follows. The map f is the map of suspension $\mathbb{T}$-spectra induced by the canonical projection $\mathrm{pr}\colon \mathbb{T}/C_{p^{n-2}} \to \mathbb{T}/C_{p^{n-1}}$, and the map v is the corresponding transfer map. To define the latter, we choose an embedding $\iota\colon \mathbb{T}/C_{p^{n-2}} \hookrightarrow \lambda$ into a finite dimensional orthogonal $\mathbb{T}$-presentation. The product embedding $(\iota,\mathrm{pr})\colon \mathbb{T}/C_{p^{n-2}} \to \lambda \times \mathbb{T}/C_{p^{n-1}}$ has trivial normal bundle, and the linear structure of λ determines a preferred trivialization. Hence, the Pontryagin–Thom construction gives a map of pointed $\mathbb{T}$-spaces

$$S^\lambda \wedge (\mathbb{T}/C_{p^{n-1}})_+ \to S^\lambda \wedge (\mathbb{T}/C_{p^{n-2}})_+$$

and v is the induced map of suspension $\mathbb{T}$-spectra. Finally, there is a unique homotopy class of maps of pointed spaces $\delta''\colon S^1 \to (\mathbb{T}/C_{p^{n-1}})_+$ such that image by the Hurewicz map is the fundamental class $[\mathbb{T}/C_{p^{n-1}}]$ corresponding to the counterclockwise orientation of the circle $\mathbb{T} \subset \mathbb{C}$ and such that the composite of δ'' and the map $(\mathbb{T}/C_{p^{n-1}})_+ \to S^0$ that collapses $\mathbb{T}/C_{p^{n-1}}$ to the non-base point of S^0 is the null-map. The map δ'' induces the map of suspension $\mathbb{T}/C_{p^{n-1}}$-spectra

$$\delta'\colon \Sigma(\mathbb{T}/C_{p^{n-1}})_+ \to (\mathbb{T}/C_{p^{n-1}})_+$$

which, in turn, induces the map δ.

The definition of the restriction map is more delicate. We let E be the unit sphere in $\mathbb{C}^\infty$ and consider the cofibration sequence of pointed $\mathbb{T}$-spaces

$$E_+ \to S^0 \to \tilde{E} \to \Sigma E_+$$

where the left-hand map collapses E onto the non-base point of S^0; the $\mathbb{T}$-space $\tilde{E}$ is canonically homeomorphic to the one-point compactification of $\mathbb{C}^\infty$. It induces a cofibration sequence of $\mathbb{T}$-spectra

$$E_+ \wedge T(A) \to T(A) \to \tilde{E} \wedge T(A) \to \Sigma E_+ \wedge T(A),$$

and hence, a long-exact sequence of equivariant homotopy groups. By [17, Theorem 2.2], the latter sequence is canonically isomorphic to the sequence

$$\cdots \to \mathbb{H}_q(C_{p^{n-1}}, T(A)) \xrightarrow{N} \mathrm{TR}^n_q(A;p) \xrightarrow{R} \mathrm{TR}^{n-1}_q(A;p) \to \cdots$$

which is called the fundamental long-exact sequence. The left-hand term is the group homology of $C_{p^{n-1}}$ with coefficients in the underlying $C_{p^{n-1}}$-spectrum of $T(A)$ and is defined to be the equivariant homotopy group

$$\mathbb{H}_q(C_{p^{n-1}}, T(A)) = [S^q, E_+ \wedge T(A)]_{C_{p^{n-1}}}.$$

The isomorphism of the left-hand terms in the two sequences is given by the canonical change-of-groups isomorphism

$$[S^q, E_+ \wedge T(A)]_{C_{p^{n-1}}} \xrightarrow{\sim} [S^q \wedge (\mathbb{T}/C_{p^{n-1}})_+, E_+ \wedge T(A)]_{\mathbb{T}}$$

and the resulting map N in the fundamental long-exact sequence is called the norm map. The isomorphism of the right-hand terms in the two sequences involves the cyclotomic structure of the spectrum $T(A)$ as we now explain. The C_p-fixed points of the $\mathbb{T}$-spectrum $T(A)$ is a $\mathbb{T}/C_p$-spectrum $T(A)^{C_p}$. Moreover, the isomorphism

$$\rho_p\colon \mathbb{T} \to \mathbb{T}/C_p$$

given by the pth root induces an equivalence of categories that to the $\mathbb{T}/C_p$-spectrum D associates the $\mathbb{T}$-spectrum $\rho_p^* D$. Then the additional cyclotomic structure of the topological Hochschild $\mathbb{T}$-spectrum $T(A)$ consists of a map of $\mathbb{T}$-spectra

$$r\colon \rho_p^*((\tilde{E} \wedge T(A))^{C_p}) \to T(A)$$

with the property that the induced map of equivariant homotopy groups

$$[S^q \wedge (\mathbb{T}/C_{p^{n-1}})_+, \rho_p^*((\tilde{E} \wedge T(A))^{C_p})]_{\mathbb{T}} \to [S^q \wedge (\mathbb{T}/C_{p^{n-1}})_+, T(A)]_{\mathbb{T}}$$

is an isomorphism for all positive integers n. The right-hand sides of the two sequences above are now identified by the composition

$$\begin{aligned}
&[S^q \wedge (\mathbb{T}/C_{p^{n-1}})_+, \tilde{E} \wedge T(A)]_{\mathbb{T}} \xleftarrow{\sim} [S^q \wedge (\mathbb{T}/C_{p^{n-1}})_+, (\tilde{E} \wedge T(A))^{C_p}]_{\mathbb{T}/C_p} \\
&\xrightarrow{\sim} [S^q \wedge (\mathbb{T}/C_{p^{n-2}})_+, \rho_p^*((\tilde{E} \wedge T(A))^{C_p})]_{\mathbb{T}} \xrightarrow{\sim} [S^q \wedge (\mathbb{T}/C_{p^{n-2}})_+, T(A)]_{\mathbb{T}}
\end{aligned}$$

of the canonical isomorphism, the isomorphism ρ_p^*, and the isomorphism induced by the map r. By definition, the restriction map is the resulting map R in the fundamental long-exact sequence. Since r is a map of $\mathbb{T}$-spectra, the restriction map commutes with the Frobenius map, the Verschiebung map, and Connes' operator.

We mention that, if the symmetric ring spectrum A is commutative, then $T(A)$ has the structure of a commutative ring $\mathbb{T}$-spectrum which, in turn, gives the graded abelian group $\mathrm{TR}_*^n(A; p)$ the structure of an anti-symmetric graded ring, for all $n \geqslant 1$. The restriction and Frobenius maps are both ring homomorphisms, the Frobenius and Verschiebung maps satisfy the projection formula

$$xV(y) = V(F(x)y),$$

and Connes' operator is a derivation with respect to the product.

In general, the restriction map does not admit a section. However, if $A = \mathbb{S}$ is the sphere spectrum, there exists a map

$$s\colon T(\mathbb{S}) \to \rho_p^*(T(\mathbb{S})^{C_p})$$

in the $\mathbb{T}$-stable homotopy category such that the composition

$$T(\mathbb{S}) \xrightarrow{s} \rho_p^*(T(\mathbb{S})^{C_p}) \to \rho_p^*((\tilde{E} \wedge T(\mathbb{S}))^{C_p}) \xrightarrow{r} T(\mathbb{S})$$

is the identity map [25, Corollary 4.4.8]. The map s induces a section

$$S = S_n\colon \mathrm{TR}_q^{n-1}(\mathbb{S}; p) \to \mathrm{TR}_q^n(\mathbb{S}; p) \qquad \text{(Segal-tom Dieck splitting)}$$

of the restriction map. The section S is a ring homomorphism and commutes with the Verschiebung map and Connes' operator. The composition FS_n is equal to $S_{n-1}F$, for $n \geqslant 3$, and to the identity map, for $n = 2$. It follows that, for every symmetric ring spectrum A, the graded abelian group $\mathrm{TR}_*^n(A; p)$ is a graded module over the graded ring $\mathrm{TR}_*^1(\mathbb{S}; p)$ which is canonically isomorphic to the graded ring given by the stable homotopy groups of spheres. It is proved in [16, Section 1] that Connes' operator satisfies the following additional relations

$$\begin{aligned}
FdV &= d + (p-1)\eta, \\
dd &= d\eta = \eta d,
\end{aligned}$$

where η indicates multiplication by the Hopf class $\eta \in \mathrm{TR}_1^1(\mathbb{S}; p)$. It follows from the above that $FV = p$, $dF = pFd$, and $Vd = pdV$.

The zeroth space A_0 of the symmetric spectrum A is a pointed monoid which is commutative if A is commutative. There is a canonical map

$$[-]_n \colon \pi_0(A_0) \to \mathrm{TR}_0^n(A; p) \qquad \text{(Teichmüller map)},$$

which satisfies $R([a]_n) = [a]_{n-1}$ and $F([a]_n) = [a^p]_{n-1}$; see [19, Section 2.5]. If A is commutative, the Teichmüller map is multiplicative and satisfies

$$Fd([a]_n) = [a]_{n-1}^{p-1} d([a]_{n-1}).$$

2 The Fundamental Theorem

Let A be a symmetric ring spectrum, and let Γ be the free group on a generator x. We define the symmetric ring spectrum $A[x^{\pm 1}]$ to be the symmetric spectrum

$$A[x^{\pm 1}] = A \wedge \Gamma_+$$

with the multiplication map given by the composition of the canonical isomorphism from $A \wedge \Gamma_+ \wedge A \wedge \Gamma_+$ to $A \wedge A \wedge \Gamma_+ \wedge \Gamma_+$ that permutes the second and third smash factors and the smash product $\mu_A \wedge \mu_\Gamma$ of the multiplication maps of A and Γ and with the unit map given by the composition of the canonical isomorphism from $\mathbb{S}$ to $\mathbb{S} \wedge S^0$ and the smash product $e_A \wedge e_\Gamma$ of the unit maps of A and Γ. There is a natural map of symmetric ring spectra $f \colon A \to A[x^{\pm 1}]$ defined to be the composition of the canonical isomorphism from A to $A \wedge S^0$ and the smash product $\mathrm{id}_A \wedge e_\Gamma$ of the identity map of A and the unit map of Γ. It induces a natural map

$$f_* \colon \mathrm{TR}_q^n(A; p) \to \mathrm{TR}_q^n(A[x^{\pm 1}]; p).$$

Moreover, there is a map of symmetric ring spectra $g \colon \mathbb{S}[x^{\pm 1}] \to A[x^{\pm 1}]$ defined to be the smash product $e_A \wedge \mathrm{id}_\Gamma$ of the unit map of A and the identity map of Γ. The map g makes $A[x^{\pm 1}]$ into an algebra spectrum over the commutative symmetric ring spectrum $\mathbb{S}[x^{\pm 1}]$. It follows that there is a natural pairing

$$\nu \colon \mathrm{TR}_q^n(A[x^{\pm 1}]; p) \otimes \mathrm{TR}_{q'}^n(\mathbb{S}[x^{\pm 1}]; p) \to \mathrm{TR}_{q+q'}^n(A[x^{\pm 1}]; p)$$

which makes the graded abelian group $\mathrm{TR}_*^n(A[x^{\pm 1}]; p)$ a graded module over the anti-symmetric graded ring $\mathrm{TR}_*^n(\mathbb{S}[x^{\pm 1}]; p)$. The element $[x]_n \in \mathrm{TR}_0^n(\mathbb{S}[x^{\pm 1}]; p)$ is a unit with inverse $[x]_n^{-1} = [x^{-1}]_n$ and we define

$$d\log[x]_n = [x]_n^{-1} d[x]_n \in \mathrm{TR}_1^n(\mathbb{S}[x^{\pm 1}]; p).$$

It follows from the general relations that

$$F(d\log[x]_n) = R(d\log[x]_n) = d\log[x]_{n-1}.$$

Now, given an integer j and element $a \in \mathrm{TR}_q^n(A;p)$, we define

$$\begin{aligned} a[x]_n^j &= \nu(f_*(a) \otimes [x]_n^j) \in \mathrm{TR}_q^n(A[x^{\pm 1}];p) \\ a[x]_n^j d\log[x]_n &= \nu(f_*(a) \otimes [x]_n^j d\log[x]_n) \in \mathrm{TR}_{q+1}^n(A[x^{\pm 1}];p). \end{aligned}$$

The following theorem, which is similar to the fundamental theorem of algebraic K-theory, was proved by Ib Madsen and the author in [19, Theorem C]. The assumption in loc. cit. that the prime p be odd is unnecessary; the same proof works for $p = 2$. However, the formulas for F, V, and d given in loc. cit. are valid for odd primes only. Below, we give a formula for the Frobenius which holds for all primes p.

Theorem 2. *Let p be a prime number, and let A be a symmetric ring spectrum whose homotopy groups are $\mathbb{Z}_{(p)}$-modules. Then every element $\omega \in \mathrm{TR}_q^n(A[x^{\pm 1}];p)$ can be written uniquely as a (finite) sum*

$$\sum_{j\in\mathbb{Z}} \big(a_{0,j}[x]_n^j + b_{0,j}[x]_n^j d\log[x]_n\big) + \sum_{\substack{1\leqslant s<n \\ j\in\mathbb{Z}\smallsetminus p\mathbb{Z}}} \big(V^s(a_{s,j}[x]_{n-s}^j) + dV^s(b_{s,j}[x]_{n-s}^j)\big)$$

with $a_{s,j} = a_{s,j}(\omega) \in \mathrm{TR}_q^{n-s}(A;p)$ and $b_{s,j} = b_{s,j}(\omega) \in \mathrm{TR}_{q-1}^{n-s}(A;p)$. The corresponding statement for the equivariant homotopy groups with $\mathbb{Z}_p$-coefficients is valid for every symmetric ring spectrum A.

It is perhaps helpful to point out that the formula in the statement of Theorem 2 defines a canonical map from the direct sum

$$\bigoplus_{j\in\mathbb{Z}} \big(\mathrm{TR}_q^n(A;p) \oplus \mathrm{TR}_{q-1}^n(A;p)\big) \oplus \bigoplus_{\substack{1\leqslant s<n \\ j\in\mathbb{Z}\smallsetminus p\mathbb{Z}}} \big(\mathrm{TR}_q^{n-s}(A;p) \oplus \mathrm{TR}_{q-1}^{n-s}(A;p)\big)$$

to the group $\mathrm{TR}_q^n(A[x^{\pm 1}];p)$ and that the theorem states that this map is an isomorphism. We also remark that the assembly map

$$\alpha\colon \mathrm{TR}_q^n(A;p) \oplus \mathrm{TR}_{q-1}^n(A;p) \to \mathrm{TR}_q^n(A[x^{\pm 1}];p)$$

is given by the formula

$$\alpha(a,b) = a[x]_n^0 + b[x]_n^0 d\log[x]_n,$$

where $[x]_n^0 = [1]_n \in \mathrm{TR}_0^n(\mathbb{S}[x^{\pm 1}];p)$ is the multiplicative unit element.

The value of the restriction and Frobenius maps on $\mathrm{TR}_q^n(A[x^{\pm 1}];p)$ are readily derived from the general relations. Indeed, if $\omega \in \mathrm{TR}_q^n(A[x^{\pm 1}];p)$ is equal to the sum in the statement of Theorem 2, then

$$
\begin{aligned}
R(\omega) = \sum_{j \in \mathbb{Z}} & \big(R(a_{0,j})[x]_{n-1}^{j} + R(b_{0,j})[x]_{n-1}^{j} d\log[x]_{n-1}\big) \\
& + \sum_{\substack{1 \leqslant s < n-1 \\ j \in \mathbb{Z} \smallsetminus p\mathbb{Z}}} \big(V^s(R(a_{s,j})[x]_{n-1-s}^{j}) + dV^s(R(b_{s,j})[x]_{n-1-s}^{j})\big)
\end{aligned}
$$

$$
\begin{aligned}
F(\omega) = \sum_{j \in p\mathbb{Z}} & \big(F(a_{0,j/p})[x]_{n-1}^{j} + F(b_{0,j/p})[x]_{n-1}^{j} d\log[x]_{n-1}\big) \\
& + \sum_{j \in \mathbb{Z} \smallsetminus p\mathbb{Z}} \big((pa_{1,j} + db_{1,j} + (p-1)\eta b_{1,j})[x]_{n-1}^{j} \\
& \qquad\qquad + (-1)^{q-1} j b_{1,j}[x]_{n-1}^{j} d\log[x]_{n-1}\big) \\
& + \sum_{\substack{1 \leqslant s < n-1 \\ j \in \mathbb{Z} \smallsetminus p\mathbb{Z}}} \big(V^s((pa_{s+1,j} + (p-1)\eta b_{s+1,j})[x]_{n-1-s}^{j}) \\
& \qquad\qquad + dV^s(b_{s+1,j}[x]_{n-1-s}^{j})\big).
\end{aligned}
$$

We leave it to the reader to derive the corresponding formulas for the Verschiebung map and Connes' operator. The following result is an immediate consequence.

We recall that the limit system $\{M_n\}$ satisfies the Mittag-Leffler condition if, for every n, there exists $m \geqslant n$ such that, for all $k \geqslant m$, the image of $M_k \to M_n$ is equal to the image of $M_m \to M_n$. This implies that the derived limit $R^1 \lim_n M_n$ vanishes.

Corollary 3. *Let p be a prime number, let A be a symmetric ring spectrum whose homotopy groups are $\mathbb{Z}_{(p)}$-modules, and let q be an integer. If both of the limit systems $\{\mathrm{TR}_q^n(A;p)\}$ and $\{\mathrm{TR}_{q-1}^n(A;p)\}$ satisfy the Mittag-Leffler condition, then so does the limit system $\{\mathrm{TR}_q^n(A[x^{\pm 1}];p)\}$. Moreover, the element*

$$\omega = (\omega^{(n)}) \in \lim_R \mathrm{TR}_q^n(A[x^{\pm 1}];p)$$

lies in the kernel of the map $1 - F$ if and only if the coefficients

$$
\begin{aligned}
a_{s,j}^{(n)} &= a_{s,j}(\omega^{(n)}) \in \mathrm{TR}_q^{n-s}(A;p) \\
b_{s,j}^{(n)} &= b_{s,j}(\omega^{(n)}) \in \mathrm{TR}_{q-1}^{n-s}(A;p)
\end{aligned}
$$

satisfy the equations

$$
a_{s,j}^{(n-1)} = \begin{cases}
F(a_{0,j/p}^{(n)}) & (s = 0 \text{ and } j \in p\mathbb{Z}) \\
pa_{1,j}^{(n)} + db_{1,j}^{(n)} + (p-1)\eta b_{1,j}^{(n)} & (s = 0 \text{ and } j \in \mathbb{Z} \smallsetminus p\mathbb{Z}) \\
pa_{s+1,j}^{(n)} + (p-1)\eta b_{s+1,j}^{(n)} & (1 \leqslant s < n-1 \text{ and } j \in \mathbb{Z} \smallsetminus p\mathbb{Z})
\end{cases}
$$

$$b^{(n-1)}_{s,j} = \begin{cases} F(b^{(n)}_{0,j/p}) & (s = 0 \textit{ and } j \in p\mathbb{Z}) \\ (-1)^{q-1} j b^{(n)}_{1,j} & (s = 0 \textit{ and } j \in \mathbb{Z} \smallsetminus p\mathbb{Z}) \\ b^{(n)}_{s+1,j} & (1 \leqslant s < n-1 \textit{ and } j \in \mathbb{Z} \smallsetminus p\mathbb{Z}) \end{cases}$$

for all $n \geqslant 1$. *The corresponding statements for the equivariant homotopy groups with* $\mathbb{Z}_p$*-coefficients is valid for every symmetric ring spectrum* A.

We do not have a good description of the cokernel of $1 - F$. In particular, it is generally not easy to decide whether or not this map is surjective.

3 Topological Cyclic Homology

Let A be a symmetric ring spectrum. We recall the definition of the topological cyclic homology groups $\mathrm{TC}_q(A; p)$ and refer to [17, 18] for a full discussion.

We consider the $\mathbb{T}$-fixed point spectrum

$$\mathrm{TR}^n(A; p) = F((\mathbb{T}/C_{p^{n-1}})_+, T(A))^{\mathbb{T}}$$

of the function $\mathbb{T}$-spectrum $F((\mathbb{T}/C_{p^{n-1}})_+, T(A))$. There is a canonical isomorphism

$$\iota\colon \pi_q(\mathrm{TR}^n(A; p)) \xrightarrow{\sim} \mathrm{TR}^n_q(A; p)$$

and maps of spectra

$$R^\iota, F^\iota\colon \mathrm{TR}^n(A; p) \to \mathrm{TR}^{n-1}(A; p)$$

such that the following diagrams commute

$$\begin{tikzcd} \pi_q(\mathrm{TR}^n(A;p)) \arrow[r, "\iota"] \arrow[d, "R^\iota_*"] & \mathrm{TR}^n_q(A;p) \arrow[d, "R"] \\ \pi_q(\mathrm{TR}^{n-1}(A;p)) \arrow[r, "\iota"] & \mathrm{TR}^{n-1}_q(A;p) \end{tikzcd} \qquad \begin{tikzcd} \pi_q(\mathrm{TR}^n(A;p)) \arrow[r, "\iota"] \arrow[d, "F^\iota_*"] & \mathrm{TR}^n_q(A;p) \arrow[d, "F"] \\ \pi_q(\mathrm{TR}^{n-1}(A;p)) \arrow[r, "\iota"] & \mathrm{TR}^{n-1}_q(A;p). \end{tikzcd}$$

The map F^ι is induced by the map of $\mathbb{T}$-spectra $f\colon (\mathbb{T}/C_{p^{n-2}})_+ \to (\mathbb{T}/C_{p^{n-1}})_+$ and the map R^ι is defined to be the composition of the map

$$F((\mathbb{T}/C_{p^{n-1}})_+, T(A))^{\mathbb{T}} \to F((\mathbb{T}/C_{p^{n-1}})_+, \tilde{E} \wedge T(A))^{\mathbb{T}}$$

induced by the canonical inclusion of S^0 in $\tilde{E}$ and the weak equivalence

$$\begin{aligned} F((\mathbb{T}/C_{p^{n-1}})_+, \tilde{E} \wedge T(A))^{\mathbb{T}} &\xleftarrow{\sim} F((\mathbb{T}/C_{p^{n-1}})_+, (\tilde{E} \wedge T(A))^{C_p})^{\mathbb{T}/C_p} \\ &\xrightarrow{\sim} F((\mathbb{T}/C_{p^{n-2}})_+, \rho_p^*((\tilde{E} \wedge T(A))^{C_p}))^{\mathbb{T}} \xrightarrow{\sim} F((\mathbb{T}/C_{p^{n-2}})_+, T(A))^{\mathbb{T}} \end{aligned}$$

defined by the composition of the canonical isomorphism, the isomorphism ρ_p^*, and the map induced by the map r which we recalled in Section 1. We then define $\mathrm{TC}^n(A;p)$ to be the homotopy equalizer of the maps R^ι and F^ι and

$$\mathrm{TC}(A;p) = \operatorname*{holim}_n \mathrm{TC}^n(A;p)$$

to be the homotopy limit with respect to the maps R^ι. We also define

$$\mathrm{TR}(A;p) = \operatorname*{holim}_n \mathrm{TR}^n(A;p)$$

to be the homotopy limit with respect to the maps R^ι such that we have a long-exact sequence of homotopy groups

$$\cdots \to \mathrm{TC}_q(A;p) \to \mathrm{TR}_q(A;p) \xrightarrow{1-F} \mathrm{TR}_q(A;p) \xrightarrow{\partial} \mathrm{TC}_{q-1}(A;p) \to \cdots .$$

We recall Milnor's short-exact sequence

$$0 \to R^1 \lim_n \mathrm{TR}^n_{q+1}(A;p) \to \mathrm{TR}_q(A;p) \to \lim_n \mathrm{TR}^n_q(A;p) \to 0.$$

In the cases we consider below, the derived limit on the left-hand side vanishes.

The cyclotomic trace map of Bökstedt–Hsiang–Madsen [4] is a map of spectra

$$\mathrm{tr}\colon K(A) \to \mathrm{TC}(A;p).$$

A technically better definition of the cyclotomic trace map was given by Dundas–McCarthy [10, Section 2.0] and [9]. From the latter definition it is clear that every class x in the image of the composite map

$$K_q(A) \xrightarrow{\mathrm{tr}} \mathrm{TC}_q(A;p) \to \mathrm{TC}^n_q(A;p) \to \mathrm{TR}^n_q(A;p)$$

is annihilated by Connes' operator. It is also not difficult to show that, for A commutative, the cyclotomic trace is multiplicative; see [12, Appendix].

The spectrum $\mathrm{TR}^n(A;p)$ considered here is canonically isomorphic to the underlying non-equivariant spectrum associated with the $\mathbb{T}$-spectrum

$$TR^n(A;p) = \rho^*_{p^{n-1}}(T(A)^{C_{p^{n-1}}}).$$

Moreover, the fundamental long-exact sequence of [17, Theorem 2.2] has the following generalization which is used in the proof of Lemma 26.

Proposition 4. *Let A be a symmetric ring spectrum, and let m and n be positive integers. Then there is a natural long-exact sequence*

$$\cdots \to \mathbb{H}_q(C_{p^m}, TR^n(A;p)) \xrightarrow{N_n} \mathrm{TR}^{m+n}_q(A;p) \xrightarrow{R^n} \mathrm{TR}^m_q(A;p) \to \cdots$$

where the left-hand term is the group homology of C_{p^m} with coefficients in the underlying C_{p^m}-spectrum of $TR^n(A;p)$.

Proof. A map of $\mathbb{T}$-spectra $f\colon T \to T'$ is defined to be an $\mathcal{F}_p$-equivalence if it induces an isomorphism of equivariant homotopy groups

$$f_*\colon [S^q \wedge (\mathbb{T}/C_{p^v})_+, T]_{\mathbb{T}} \to [S^q \wedge (\mathbb{T}/C_{p^v})_+, T']_{\mathbb{T}}$$

for all integers q and $v \geqslant 0$. The cofibration sequence of pointed $\mathbb{T}$-spaces

$$E_+ \xrightarrow{\pi} S^0 \xrightarrow{\iota} \tilde{E} \xrightarrow{\partial} \Sigma E_+,$$

which we considered in Section 1, induces a cofibration sequence of $\mathbb{T}$-spectra

$$E_+ \wedge \rho^*_{p^s}(T(A)^{C_{p^s}}) \to \rho^*_{p^s}(T(A)^{C_{p^s}}) \to \tilde{E} \wedge \rho^*_{p^s}(T(A)^{C_{p^s}}) \to \Sigma E_+ \wedge \rho^*_{p^s}(T(A)^{C_{p^s}}).$$

We show that with $s = n-1$, the induced long-exact sequence of equivariant homotopy groups is isomorphic to the sequence of the statement. The isomorphism of the left-hand terms in the two sequences is defined as in Section 1. To define the isomorphism of the right-hand terms in the two sequences, we first show that the cyclotomic structure map r gives rise to an $\mathcal{F}_p$-equivalence

$$r'\colon \tilde{E} \wedge \rho^*_{p^{n-1}}(T(A)^{C_{p^{n-1}}}) \xrightarrow{\sim} \tilde{E} \wedge T(A).$$

Since the map $\pi\colon E_+ \to S^0$ induces a weak equivalence

$$E_+ \wedge \rho^*_{p^s}((E_+ \wedge T(A))^{C_{p^s}}) \xrightarrow{\sim} \rho^*_{p^s}((E_+ \wedge T(A))^{C_{p^s}}),$$

a diagram chase shows that the map $\iota\colon S^0 \to \tilde{E}$ induces a weak equivalence

$$\tilde{E} \wedge \rho^*_{p^s}(T(A)^{C_{p^s}}) \xrightarrow{\sim} \tilde{E} \wedge \rho^*_{p^s}(\tilde{E} \wedge T(A)^{C_{p^s}}).$$

The cyclotomic structure map r induces an $\mathcal{F}_p$-equivalence

$$\tilde{E} \wedge \rho^*_{p^s}(\tilde{E} \wedge T(A)^{C_{p^s}}) \xrightarrow{\sim} \tilde{E} \wedge \rho^*_{p^{s-1}}(T(A)^{C_{p^{s-1}}}),$$

which, composed with the former equivalence, defines an $\mathcal{F}_p$-equivalence

$$\tilde{E} \wedge \rho^*_{p^s}(T(A)^{C_{p^s}}) \xrightarrow{\sim} \tilde{E} \wedge \rho^*_{p^{s-1}}(T(A)^{C_{p^{s-1}}}).$$

The composition of these $\mathcal{F}_p$-equivalence as s varies from $n-1$ to 1 gives the desired $\mathcal{F}_p$-equivalence r'. The isomorphism of the right-hand terms in the two sequences is now given by the composition of the isomorphism

$$[S^q \wedge (\mathbb{T}/C_{p^m})_+, \tilde{E} \wedge \rho^*_{p^{n-1}}(T(A)^{C_{p^{n-1}}})]_{\mathbb{T}} \xrightarrow{\sim} [S^q \wedge (\mathbb{T}/C_{p^m})_+, \tilde{E} \wedge T(A)]_{\mathbb{T}}$$

induced by the map r' and the isomorphism

$$[S^q \wedge (\mathbb{T}/C_{p^m})_+, \tilde{E} \wedge T(A)]_{\mathbb{T}} \xrightarrow{\sim} [S^q \wedge (T/C_{p^{m-1}})_+, T(A)]_{\mathbb{T}}$$

defined in Section 1. □

4 The Skeleton Spectral Sequence

The left-hand groups in Proposition 4 are the abutment of the strongly convergent skeleton spectral sequence which we now discuss in some detail. Let G be a finite group, and let T be a G-spectrum. Then we define

$$\mathbb{H}_q(G, T) = [S^q, E_+ \wedge T]_G,$$

where E is a free contractible G–CW-complex. The group $\mathbb{H}_q(G, T)$ is well-defined up to canonical isomorphism. Indeed, if also E' is a free contractible G–CW-complex, then there is a unique homotopy class of G-maps $u\colon E \to E'$, and the induced map $u_*\colon [S^q, E_+ \wedge T]_G \to [S^q, E'_+ \wedge T]_G$ is the canonical isomorphism. The skeleton filtration of the G–CW-complex E gives rise to a spectral sequence

$$E^2_{s,t} = H_s(G; \pi_t(T)) \Rightarrow \mathbb{H}_{s+t}(G, T)$$

from the homology of the group G with coefficients in the G-module $\pi_t(T)$. We will need the precise identification of the E^2-term below. The augmented cellular complex of E is the augmented chain complex $\epsilon\colon P \to \mathbb{Z}$ defined by

$$P_s = \tilde{H}_s(E_s/E_{s-1}; \mathbb{Z})$$

with the differential d induced by the map ∂ in the cofibration sequence

$$E_{s-1}/E_{s-2} \to E_s/E_{s-2} \to E_s/E_{s-1} \xrightarrow{\partial} \Sigma E_{s-1}/E_{s-2}.$$

and with the augmentation given by $\epsilon(x) = 1$, for all $x \in E_0$. It is a resolution of the trivial G-module $\mathbb{Z}$ by free $\mathbb{Z}[G]$-modules. We define

$$H_s(G, \pi_t(T)) = H_s((P \otimes \pi_t(T))^G, d \otimes \mathrm{id}).$$

The E^1-term of the spectral sequence is defined by

$$E^1_{s,t} = [S^{s+t}, (E_s/E_{s-1}) \wedge T]_G$$

with the d^1-differential induced by the boundary map ∂ in the cofibration sequence above. The quotient E_s/E_{s-1} is homeomorphic to a wedge of s-spheres indexed by a set on which the groups G acts freely. Therefore, the Hurewicz homomorphism

$$\pi_s(E_s/E_{s-1}) \to \tilde{H}(E_s/E_{s-1}; \mathbb{Z}),$$

the exterior product map

$$\pi_s(E_s/E_{s-1}) \otimes \pi_t(T) \to \pi_{s+t}((E_s/E_{s-1}) \wedge T),$$

and the canonical map

$$[S^{s+t}, (E_s/E_{s-1}) \wedge T]_G \to (\pi_{s+t}((E_s/E_{s-1}) \wedge T))^G$$

are all isomorphisms. These isomorphisms gives rise to a canonical isomorphism

$$h\colon (P_s \otimes \pi_t(T))^G \xrightarrow{\sim} E^1_{s,t},$$

which satisfies $h \circ (d \otimes \mathrm{id}) = d^1 \circ h$. The induced isomorphism of homology groups is then the desired identification of the E^2-term.

We consider the skeleton spectral sequence with $G = C_{p^{n-1}}$ and $T = TR^v(A; p)$. Since the action by $C_{p^{n-1}}$ on $TR^v(A; p)$ is the restriction of an action by the circle group $\mathbb{T}$, the induced action on the homotopy groups $\mathrm{TR}^v_t(A; p)$ is trivial. Moreover, it follows from [16, Lemma 1.4.2] that the d^2-differential of the spectral sequence is related to Connes' operator d in the following way.

Lemma 5. *Let A be a symmetric ring spectrum. Then, in the spectral sequence*

$$E^2_{s,t} = H_s(C_{p^{n-1}}, \mathrm{TR}^v_t(A; p)) \Rightarrow \mathbb{H}_{s+t}(C_{p^{n-1}}, TR^v(A; p)),$$

the d^2-differential $d^2\colon E^2_{s,t} \to E^2_{s-2,t+1}$ is equal to the map of group homology groups induced by $d + \eta$, if s is congruent to 0 *or* 1 *modulo* 4*, and the map induced by d, if s is congruent to* 2 *or* 3 *modulo* 4.

The Frobenius and Verschiebung maps

$$F\colon \mathbb{H}_q(C_{p^{n-1}}, TR^v(A; p)) \to \mathbb{H}_q(C_{p^{n-2}}, TR^v(A; p))$$
$$V\colon \mathbb{H}_q(C_{p^{n-2}}, TR^v(A; p)) \to \mathbb{H}_q(C_{p^{n-1}}, TR^v(A; p))$$

induce maps of spectral sequences which on the E^2-terms of the corresponding skeleton spectral sequence are given by the transfer and corestriction maps in group homology corresponding to the inclusion of $C_{p^{n-2}}$ in $C_{p^{n-1}}$.

Let $g \in C_{p^{n-1}}$ be the generator $g = \exp(2\pi i/p^{n-1})$, and let $\epsilon\colon W \to \mathbb{Z}$ be the standard resolution which in degree s is a free $\mathbb{Z}[C_{p^{n-1}}]$-module of rank one on a generator x_s with differential $dx_s = Nx_{s-1}$, for s even, and $dx_s = (g-1)x_{s-1}$, for s odd, and with augmentation $\epsilon(x_0) = 1$.

Lemma 6. *Let r and n be positive integers, and let p be a prime number.*

(1) If $r \leqslant n-1$, then

$$H_s(C_{p^{n-1}}, \mathbb{Z}/p^r\mathbb{Z}) = \mathbb{Z}/p^r\mathbb{Z} \cdot z_s,$$

where $z_s = z_s(p,n,r)$ is the class of $Nx_s \otimes 1$.

(2) If $r \geqslant n-1$, then

$$H_s(C_{p^{n-1}}, \mathbb{Z}/p^r\mathbb{Z}) = \begin{cases} \mathbb{Z}/p^r\mathbb{Z} \cdot z_0 & (s=0) \\ \mathbb{Z}/p^{n-1}\mathbb{Z} \cdot z_s & (s \text{ odd}) \\ \mathbb{Z}/p^{n-1}\mathbb{Z} \cdot p^{r-(n-1)}z_s & (s > 0 \text{ and even}) \end{cases}$$

where $z_s = z_s(p,n,r)$ and $p^{r-(n-1)}z_s = p^{r-(n-1)}z_s(p,n,r)$ are the classes of $Nx_s \otimes 1$ and $p^{r-(n-1)}Nx_s \otimes 1$, respectively.

(3) The transfer map

$$F\colon H_s(C_{p^{n-1}}, \mathbb{Z}/p^r\mathbb{Z}) \to H_s(C_{p^{n-2}}, \mathbb{Z}/p^r\mathbb{Z})$$

maps z_s to z_s, if s is odd, maps z_s to pz_s, if $s = 0$ or if $s > 0$ is even and $r \leqslant n-1$, and maps $p^{r-(n-1)}z_s$ to $p^{r-(n-2)}z_s$, if $s > 0$ is even and $r \geqslant n-1$.

(4) The corestriction map

$$V\colon H_s(C_{p^{n-2}}, \mathbb{Z}/p^r\mathbb{Z}) \to H_s(C_{p^{n-1}}, \mathbb{Z}/p^r\mathbb{Z})$$

maps z_s to pz_s, if s is odd, maps z_s to z_s, if $s = 0$ or if $s > 0$ is even and $r \leqslant n-1$, and maps $p^{r-(n-1)}z_s$ to $p^{r-(n-1)}z_s$, if $s > 0$ is even and $r \geqslant n-1$.

Proof. The statements (1) and (2) are readily verified. To prove (3) and (4), we write $\epsilon\colon W \to \mathbb{Z}$ and $\epsilon'\colon W' \to \mathbb{Z}$ for the standard resolutions for the groups $C_{p^{n-1}}$ and $C_{p^{n-2}}$, respectively. Then $\epsilon\colon W \to \mathbb{Z}$ is a resolution of $\mathbb{Z}$ by free $C_{p^{n-2}}$-modules. The map $h\colon W \to W'$ defined by

$$h(g^{dp+r}x_s) = \begin{cases} g'^r x'_s & (s \text{ even}) \\ \delta_{r,p-1} g'^d x'_s & (s \text{ odd}), \end{cases}$$

where $0 \leqslant r < p$ and $0 \leqslant d < n-2$, is a $C_{p^{n-2}}$-linear chain map that lifts the identity map of $\mathbb{Z}$, and the $C_{p^{n-2}}$-linear map $k\colon W' \to W$ defined by

$$k(x'_s) = \begin{cases} x_s & (s \text{ even}) \\ (1+g+\dots g^{p-1})x_s & (s \text{ odd}) \end{cases}$$

is a chain map and lifts the identity of $\mathbb{Z}$. Now the transfer map F is the map of homology groups induced by the composite chain map

$$(W \otimes \mathbb{Z}/p^r\mathbb{Z})^{C_{p^{n-1}}} \hookrightarrow (W \otimes \mathbb{Z}/p^r\mathbb{Z})^{C_{p^{n-2}}} \xrightarrow{h\otimes 1} (W' \otimes \mathbb{Z}/p^r\mathbb{Z})^{C_{p^{n-2}}},$$

where the left-hand map is the canonical inclusion. One verifies readily that this map takes $Nx_{2i} \otimes 1$ to $pN'x'_{2i} \otimes 1$ and $Nx_{2i-1} \otimes 1$ to $N'x'_{2i-1} \otimes 1$. Similarly, the corestriction map V is the map of homology groups induced by the composite chain map

$$(W' \otimes \mathbb{Z}/p^r\mathbb{Z})^{C_{p^{n-2}}} \xrightarrow{k\otimes 1} (W \otimes \mathbb{Z}/p^r\mathbb{Z})^{C_{p^{n-2}}} \xrightarrow{N/N'} (W \otimes \mathbb{Z}/p^r\mathbb{Z})^{C_{p^{n-1}}},$$

where the right-hand map is multiplication by $1 + g + \cdots + g^{p-1}$. This map takes $N'x'_{2i} \otimes 1$ to $Nx_{2i} \otimes 1$ and $N'x'_{2i-1} \otimes 1$ to $pNx_{2i-1} \otimes 1$. □

Lemma 7. *Let n be a positive integer and let p be a prime number. Then*

$$H_s(C_{p^{n-1}}, \mathbb{Z}) = \begin{cases} \mathbb{Z} \cdot z_0 & (s = 0) \\ \mathbb{Z}/p^{n-1}\mathbb{Z} \cdot z_s & (s \text{ odd}) \\ 0 & (s > 0 \text{ even}) \end{cases},$$

where $z_s = z_s(p, n)$ is the class of $Nx_s \otimes 1$. The transfer map

$$F\colon H_s(C_{p^{n-1}}, \mathbb{Z}) \to H_s(C_{p^{n-2}}, \mathbb{Z})$$

maps z_0 to pz_0 and z_s to z_s, for $s > 0$, and the corestriction map

$$V\colon H_s(C_{p^{n-2}}, \mathbb{Z}) \to H_s(C_{p^{n-1}}, \mathbb{Z})$$

maps z_0 to z_0 and z_s to pz_s, for $s > 0$.

Proof. The proof is similar to the proof of Lemma 6. □

5 The Groups $\mathrm{TR}^n_q(\mathbb{S}; 2)$

In this section, we implicitly consider homotopy groups with $\mathbb{Z}_2$-coefficients. The groups $\mathrm{TR}^1_*(\mathbb{S}; 2)$ are the stable homotopy groups of spheres. The group $\mathrm{TR}^1_0(\mathbb{S}; 2)$ is isomorphic to $\mathbb{Z}_2$ generated by the multiplicative unit element $\iota = [1]_1$; the group $\mathrm{TR}^1_1(\mathbb{S}; 2)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ generated by the Hopf class η; the group $\mathrm{TR}^1_2(\mathbb{S}; 2)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ generated by η^2; the group $\mathrm{TR}^1_3(\mathbb{S}; 2)$ is isomorphic to $\mathbb{Z}/8\mathbb{Z}$ generated by the Hopf class ν and $\eta^3 = 4\nu$; the groups $\mathrm{TR}^1_4(\mathbb{S}; 2)$ and $\mathrm{TR}^1_5(\mathbb{S}; 2)$ are zero; the group $\mathrm{TR}^1_6(\mathbb{S}; 2)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ generated by ν^2, and the group $\mathrm{TR}^1_7(\mathbb{S}; 2)$ is isomorphic to $\mathbb{Z}/16\mathbb{Z}$ generated the Hopf class σ.

We consider the skeleton spectral sequence

$$E^2_{s,t} = H_s(C_{2^{n-1}}, \mathrm{TR}^1_t(\mathbb{S};2)) \Rightarrow \mathbb{H}_{s+t}(C_{2^{n-1}}, T(\mathbb{S})).$$

This sequence may be identified with the Atiyah–Hirzebruch spectral sequence that converges to the homotopy groups of the suspension spectrum of the pointed space $(BC_{2^{n-1}})_+$ [14, Proposition 2.4]. Therefore, the edge-homomorphism onto the line $s = 0$ has a retraction, and hence, the differentials $d^r \colon E^r_{r,t} \to E^r_{0,t+r-1}$ are all zero.

Suppose first that $n = 2$. Then the E^2-term for $s + t \leqslant 7$ is takes the form

$$\begin{array}{cccccccc}
\mathbb{Z}/16\mathbb{Z} & & & & & & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & & & & & & \\
0 & 0 & 0 & & & & & \\
0 & 0 & 0 & 0 & & & & \\
\mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 4\mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 4\mathbb{Z}/8\mathbb{Z} & & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \\
\mathbb{Z}_2 & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z},
\end{array}$$

where s is the horizontal coordinate and t the vertical coordinate. The group $E^2_{s,0}$ is generated by the class ιz_s, the group $E^2_{s,1}$ by the class ηz_s, the group $E^2_{s,2}$ by the class $\eta^2 z_s$, the group $E^2_{s,3}$ with $s = 0$ or s an odd positive integer by the class νz_s, the group $E^2_{s,3}$ with s an even positive integer by the class $4\nu z_s$, the group $E^2_{s,6}$ by the class $\nu^2 z_s$, the group $E^2_{s,7}$ with $s = 0$ or s an odd positive integer by the class σz_s, and the group $E^2_{s,7}$ with s an even positive integer by the class $8\sigma z_s$, where z_s are the classes defined in Lemmas 6 and 7. We recall from Lemma 5 that the d^2-differential is given by Connes' operator and by multiplication by η. Since Connes' operator on $\mathrm{TR}^1_*(\mathbb{S};2)$ is zero, we find that the E^3-term begins

$$\begin{array}{cccccccc}
\mathbb{Z}/16\mathbb{Z} & & & & & & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & & & & & & \\
0 & 0 & 0 & & & & & \\
0 & 0 & 0 & 0 & & & & \\
\mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & 4\mathbb{Z}/8\mathbb{Z} & & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & 0 & 0 & \mathbb{Z}/2\mathbb{Z} & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & 0 & 0 & \mathbb{Z}/2\mathbb{Z} & \\
\mathbb{Z}_2 & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & 0 & 0 & 0 & \mathbb{Z}/2\mathbb{Z}.
\end{array}$$

Since the differential $d^3 \colon E^3_{3,0} \to E^3_{0,2}$ is zero, the E^3-term is also the E^4-term. The following result is a consequence of Mosher [26, Proposition 5.2].

Lemma 8. *Let n be a positive integer. Then, in the spectral sequence*

$$E^2_{s,t} = H_s(C_{2^{n-1}}, \mathrm{TR}^1_t(\mathbb{S};2)) \Rightarrow \mathbb{H}_{s+t}(C_{2^{n-1}}, T(\mathbb{S})),$$

the d^4-differential $d^4 \colon E^4_{s,t} \to E^4_{s-4,t+3}$ is equal to the map of sub-quotients induced from the map of group homology groups induced from multiplication by ν, if s is *congruent to* 0, 1, 2, 3, 8, 9, 10, *or* 11 *modulo* 16, *by* 2ν, *if s is congruent to* 6, 7, 12, *or* 13 *modulo* 16, *and by* 0, *if s is congruent to* 4, 5, 14, *or* 15 *modulo* 16. $\square$

In the case at hand, we find that the d^4-differential is zero, for $s + t \leqslant 7$. For degree reasons, the only possible higher non-zero differential all have target on the fiber line $s = 0$. However, we argued above that these differentials are zero. Therefore, for $s + t \leqslant 7$, the E^3-term is also the E^∞-term.

The E^2-term of the skeleton spectral sequence for $\mathbb{H}_q(C_4, T(\mathbb{S}))$ for $s + t \leqslant 7$ is

$$\begin{array}{cccccccc}
\mathbb{Z}/16\mathbb{Z} & & & & & & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & & & & & & \\
0 & 0 & 0 & & & & & \\
0 & 0 & 0 & 0 & & & & \\
\mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/4\mathbb{Z} & 2\mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/4\mathbb{Z} & 2\mathbb{Z}/8\mathbb{Z} & & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \\
\mathbb{Z}_2 & \mathbb{Z}/4\mathbb{Z} & 0 & \mathbb{Z}/4\mathbb{Z} & 0 & \mathbb{Z}/4\mathbb{Z} & 0 & \mathbb{Z}/4\mathbb{Z}.
\end{array}$$

The generators of the groups $E^2_{s,t}$ are as before with exception that the groups $E^3_{s,3}$ and $E^2_{s,7}$ with s an even positive integer are generated by $2\nu z_s$ and $4\sigma z_s$, respectively. We find as before that the E^3-term for $s + t \leqslant 7$ takes the form

$$\begin{array}{cccccccc}
\mathbb{Z}/16\mathbb{Z} & & & & & & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & & & & & & \\
0 & 0 & 0 & & & & & \\
0 & 0 & 0 & 0 & & & & \\
\mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/4\mathbb{Z} & 2\mathbb{Z}/4\mathbb{Z} & \mathbb{Z}/4\mathbb{Z} & 2\mathbb{Z}/8\mathbb{Z} & & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & 0 & 0 & \mathbb{Z}/2\mathbb{Z} & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & 0 & 0 & \mathbb{Z}/2\mathbb{Z} & \\
\mathbb{Z}_2 & \mathbb{Z}/4\mathbb{Z} & 0 & \mathbb{Z}/4\mathbb{Z} & 0 & 2\mathbb{Z}/4\mathbb{Z} & 0 & \mathbb{Z}/4\mathbb{Z}.
\end{array}$$

The only possible non-zero d^3-differential for $s + t \leqslant 7$ is $d^3\colon E^3_{6,1} \to E^3_{3,3}$. Since the corresponding differential in the previous spectral sequence is zero, a comparison by using the Verschiebung map shows that also this differential is zero. The d^4-differentials are given by Lemma 8. Hence, the E^5-term begins

$$\begin{array}{cccccccc}
\mathbb{Z}/16\mathbb{Z} & & & & & & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & & & & & & \\
0 & 0 & 0 & & & & & \\
0 & 0 & 0 & 0 & & & & \\
\mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/4\mathbb{Z} & 2\mathbb{Z}/4\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 2\mathbb{Z}/8\mathbb{Z} & & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & 0 & 0 & \mathbb{Z}/2\mathbb{Z} & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & 0 & 0 & \mathbb{Z}/2\mathbb{Z} & \\
\mathbb{Z}_2 & \mathbb{Z}/4\mathbb{Z} & 0 & \mathbb{Z}/4\mathbb{Z} & 0 & 2\mathbb{Z}/4\mathbb{Z} & 0 & 2\mathbb{Z}/4\mathbb{Z}.
\end{array}$$

We see as before that the E^5-term is also the E^∞-term.

Finally, we consider the skeleton spectral sequence for $\mathbb{H}_q(C_{2^{n-1}}, T(\mathbb{S}))$, where $n \geqslant 4$. The E^2-term for $s + t \leqslant 7$, takes the form

$$\begin{array}{cccccccc}
\mathbb{Z}/16\mathbb{Z} & & & & & & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & & & & & & \\
0 & 0 & 0 & & & & & \\
0 & 0 & 0 & 0 & & & & \\
\mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/8\mathbb{Z} & & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \\
\mathbb{Z}_2 & \mathbb{Z}/2^{n-1}\mathbb{Z} & 0 & \mathbb{Z}/2^{n-1}\mathbb{Z} & 0 & \mathbb{Z}/2^{n-1}\mathbb{Z} & 0 & \mathbb{Z}/2^{n-1}\mathbb{Z}.
\end{array}$$

The generators of the groups $E^2_{s,t}$ are the same as in the skeleton spectral sequence for $\mathbb{H}_q(C_2, T(\mathbb{S}))$ with the exception that the groups $E^2_{s,3}$ and $E^2_{s,7}$ are generated by the classes νz_s and σz_s, respectively, for all $s \geqslant 0$. The d^2-differential is given by Lemma 5. We find that the E^3-term for $s + t \leqslant 7$ becomes

$$\begin{array}{cccccccc}
\mathbb{Z}/16\mathbb{Z} & & & & & & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & & & & & & \\
0 & 0 & 0 & & & & & \\
0 & 0 & 0 & 0 & & & & \\
\mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/4\mathbb{Z} & \mathbb{Z}/4\mathbb{Z} & \mathbb{Z}/8\mathbb{Z} & & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & 0 & 0 & 0 & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & 0 & 0 & \mathbb{Z}/2\mathbb{Z} & \\
\mathbb{Z}_2 & \mathbb{Z}/2^{n-1}\mathbb{Z} & 0 & \mathbb{Z}/2^{n-1}\mathbb{Z} & 0 & 2\mathbb{Z}/2^{n-1}\mathbb{Z} & 0 & \mathbb{Z}/2^{n-1}\mathbb{Z}.
\end{array}$$

A comparison with the previous spectral sequence by using the Verschiebung map shows that the d^3-differential is zero. The d^4-differential is given by Lemma 8. Hence, the E^5-term for $s + t \leqslant 7$ becomes

$$\begin{array}{cccccccc}
\mathbb{Z}/16\mathbb{Z} & & & & & & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & & & & & & \\
0 & 0 & 0 & & & & & \\
0 & 0 & 0 & 0 & & & & \\
\mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/4\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/8\mathbb{Z} & & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & 0 & 0 & 0 & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & 0 & 0 & \mathbb{Z}/2\mathbb{Z} & \\
\mathbb{Z}_2 & \mathbb{Z}/2^{n-1}\mathbb{Z} & 0 & \mathbb{Z}/2^{n-1}\mathbb{Z} & 0 & 2\mathbb{Z}/2^{n-1}\mathbb{Z} & 0 & 2\mathbb{Z}/2^{n-1}\mathbb{Z}
\end{array}$$

and, for $s + t \leqslant 7$, this is also the E^∞-term.

Lemma 9. *(1) There exists unique homotopy classes*

$$\xi_{1,n-1} \in \mathbb{H}_1(C_{2^{n-1}}, T(\mathbb{S})) \qquad (n \geqslant 1)$$

such that $\xi_{1,n-1}$ *represents* $\iota z_1 \in E^\infty_{1,0}$, $F(\xi_{1,n-1}) = \xi_{1,n-2}$, *and* $\xi_{1,0} = \eta$.

(2) There exists unique homotopy classes

$$\xi_{3,n-1} \in \mathbb{H}_3(C_{2^{n-1}}, T(\mathbb{S})) \qquad (n \geqslant 1)$$

such that $\xi_{3,n-1}$ *represents* $\iota z_3 \in E^\infty_{3,0}$, $F(\xi_{3,n-1}) = \xi_{3,n-2}$, *and* $\xi_{3,0} = \nu$.

(3) There exists unique homotopy classes

$$\xi_{5,n-1} \in \mathbb{H}_5(C_{2^{n-1}}, T(\mathbb{S})) \qquad (n \geqslant 1)$$

such that $\xi_{5,n-1}$ *represents* $2\iota z_5 \in E^{\infty}_{5,0}$ *and* $F(\xi_{5,n-1}) = \xi_{5,n-2}$.

Proof. We consider the inverse limit with respect to the Frobenius maps of the skeleton spectral sequences for $\mathbb{H}_q(C_{2^{n-1}}, T(\mathbb{S}))$. By Lemmas 6 and 7, the map of spectral sequences induced by the Frobenius map is given, formally, by $F(z_s) = 2z_s$, if s is even, and $F(z_s) = z_s$, if s is odd. Hence, the E^{∞}-term of the inverse limit spectral sequence for $s + t \leqslant 7$ takes the form

$$\begin{array}{cccccccc}
0 & & & & & & & \\
0 & \mathbb{Z}/2\mathbb{Z} & & & & & & \\
0 & 0 & 0 & & & & & \\
0 & 0 & 0 & 0 & & & & \\
0 & \mathbb{Z}/8\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & 0 & & & \\
0 & \mathbb{Z}/2\mathbb{Z} & 0 & 0 & 0 & 0 & & \\
0 & \mathbb{Z}/2\mathbb{Z} & 0 & 0 & 0 & 0 & 0 & \\
0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & 0 & 2\mathbb{Z}_2 & 0 & 2\mathbb{Z}_2.
\end{array}$$

We now prove the statement (1). There is a unique class

$$\xi_1 = \{\xi_{1,n-1}\} \in \lim_F \mathbb{H}_1(C_{2^{n-1}}, T(\mathbb{S}))$$

such that $\xi_{1,n-1}$ represents the generator $\iota z_1 \in E^{\infty}_{1,0}$, for all $n \geqslant 2$. We can write

$$\xi_{1,n-1} = a_{n-1} d V^{n-1}(1) + b_{n-1} V^{n-1}(\eta),$$

where $a_{n-1} \in \mathbb{Z}/2^{n-1}\mathbb{Z}$ and $b_{n-1} \in \mathbb{Z}/2\mathbb{Z}$. Since the class $\xi_{1,n-1}$ represents ιz_1, the proof of [18, Proposition 4.4.1] shows that $a_{n-1} = 1$. The calculation

$$\xi_{1,n-1} = F(\xi_{1,n}) = F(dV^n(1) + b_{n-1}V^n(\eta)) = dV^{n-1}(1) + V^{n-1}(\eta)$$

shows that also $b_{n-1} = 1$. Finally,

$$\xi_{1,0} = F(\xi_{1,1}) = F(dV(1) + V(\eta)) = \eta,$$

which proves (1). To prove (2), we must show that there is a unique class

$$\xi_3 = \{\xi_{3,n-1}\} \in \lim_F \mathbb{H}_3(C_{2^{n-1}}, T(\mathbb{S}))$$

such that $\xi_{3,n-1}$ represents ιz_3 and such that $\xi_{3,0} = \nu$. There are two classes ξ_3 and ξ_3' that satisfy the first requirement and

$$\xi_{3,n-1} - \xi'_{3,n-1} = dV^{n-1}(\eta^2).$$

Moreover, if $n \geqslant 3$, then $F^{n-1}\colon \mathbb{H}_3(C_{2^{n-1}}, T(\mathbb{S})) \to \mathrm{TR}^1_3(\mathbb{S};2)$ induces a map

$$\overline{F^{n-1}}\colon H_3(C_{2^{n-1}}, \mathrm{TR}^1_0(\mathbb{S};2)) \to \mathrm{TR}^1_3(\mathbb{S};2)/4\,\mathrm{TR}^1_3(\mathbb{S};2).$$

Indeed, $F^{n-1}V^{n-1}(\nu) = 2^{n-1}\nu$ and $F^{n-1}dV^{n-1}(\eta^2) = \eta^3 = 4\nu$. The map $\overline{F^{n-1}}$ is surjective by [36, Table 4]. One readily verifies that it maps the generator ιz_3 to the modulo 4 reduction $\bar{\nu}$ of the Hopf class ν. Hence, F^{n-1} maps one of the classes $\xi_{3,n-1}$ and $\xi'_{3,n-1}$ to ν and the other class to 5ν. The statement (2) follows.

Finally, the statement (3) follows immediately from the inverse limit of the spectral sequences displayed above. □

The group $\mathbb{H}_5(C_8, T(\mathbb{S}))$ is equal to the direct sum of the subgroup generated by the class $\xi_{5,3}$ and a cyclic group. We choose a generator ρ this cyclic group.

Proposition 10. *The groups $\mathbb{H}_q(C_{2^{n-1}}, T(\mathbb{S}))$ with $q \leqslant 5$ are given by*

$$\mathbb{H}_0(C_{2^{n-1}}, T(\mathbb{S})) = \mathbb{Z}_2 \cdot V^{n-1}(\iota),$$

$$\mathbb{H}_1(C_{2^{n-1}}, T(\mathbb{S})) = \begin{cases} \mathbb{Z}/2\mathbb{Z}\cdot\eta & (n=1) \\ \mathbb{Z}/2^{n-1}\mathbb{Z}\cdot\xi_{1,n-1} \oplus \mathbb{Z}/2\mathbb{Z}\cdot V^{n-1}(\eta) & (n \geqslant 2) \end{cases},$$

$$\mathbb{H}_2(C_{2^{n-1}}, T(\mathbb{S})) = \begin{cases} \mathbb{Z}/2\mathbb{Z}\cdot\eta^2 & (n=1) \\ \mathbb{Z}/2\mathbb{Z}\cdot\eta\xi_{1,n-1} \oplus \mathbb{Z}/2\mathbb{Z}\cdot V^{n-1}(\eta^2) & (n \geqslant 2) \end{cases},$$

$$\mathbb{H}_3(C_{2^{n-1}}, T(\mathbb{S})) = \begin{cases} \mathbb{Z}/8\mathbb{Z}\cdot\nu & (n=1) \\ \mathbb{Z}/8\mathbb{Z}\cdot\xi_{3,1} \oplus \mathbb{Z}/8\mathbb{Z}\cdot V(\nu) & (n=2) \\ \mathbb{Z}/2^n\mathbb{Z}\cdot\xi_{3,n-1} \oplus \mathbb{Z}/2\mathbb{Z}\cdot\eta^2\xi_{1,n-1} & (n \geqslant 3) \\ \qquad \oplus\, \mathbb{Z}/8\mathbb{Z}\cdot V^{n-1}(\nu) & \end{cases},$$

$$\mathbb{H}_4(C_{2^{n-1}}, T(\mathbb{S})) = \begin{cases} \mathbb{Z}/2^{n-1}\mathbb{Z}\cdot\nu\xi_{1,n-1} & (n \leqslant 3) \\ \mathbb{Z}/8\mathbb{Z}\cdot\nu\xi_{1,n-1} & (n \geqslant 4) \end{cases},$$

$$\mathbb{H}_5(C_{2^{n-1}}, T(\mathbb{S})) = \begin{cases} 0 & (n \leqslant 2) \\ \mathbb{Z}/4\mathbb{Z}\cdot\xi_{5,2} & (n=3) \\ \mathbb{Z}/2^{n-1}\mathbb{Z}\cdot\xi_{5,n-1} \oplus \mathbb{Z}/2\mathbb{Z}\cdot V^{n-4}(\rho) & (n \geqslant 4) \end{cases}.$$

In addition, $F(\xi_{q,n-1}) = \xi_{q,n-2}$, where $\xi_{1,0} = \eta$ and $\xi_{3,0} = \nu$, and $F(\rho) = 0$.

Proof. We have already evaluated the E^∞-term of the spectral sequence

$$E^2_{s,t} = H_s(C_{2^{n-1}}, \mathrm{TR}^1_t(\mathbb{S};2)) \Rightarrow \mathbb{H}_{s+t}(C_{2^{n-1}}, T(\mathbb{S})),$$

for $s+t \leqslant 7$. We have also defined all the homotopy classes that appear in the statement. Hence, it remains only to prove that these homotopy classes have the indicated order. First, the edge homomorphism of the spectral sequence is the map

$$V^{n-1}\colon \mathrm{TR}^1_t(\mathbb{S};2) \to \mathbb{H}_t(C_{2^{n-1}}, T(\mathbb{S})).$$

Since this map has a retraction, the classes $V^{n-1}(\eta)$ and $V^{n-1}(\eta^2)$ both generate a direct summand $\mathbb{Z}/2\mathbb{Z}$ and the class $V^{n-1}(\nu)$ generates a direct summand $\mathbb{Z}/8\mathbb{Z}$ as stated. This completes the proof for $q \leqslant 2$. Next, the Frobenius map

$$F\colon \mathbb{H}_3(C_2, T(\mathbb{S})) \to \mathrm{TR}^1_3(\mathbb{S}; 2)$$

is surjective by [36, Table 4]. This implies that the class $\xi_{3,1}$ has order 8 and that the group $\mathbb{H}_3(C_2, T(\mathbb{S}))$ is as stated. We note that $4\xi_{3,1}$ is congruent to $dV(\eta^2)$ modulo the image of the edge homomorphism.

Next, we show by induction on $n \geqslant 3$ that the class $\xi_{3,n-1}$ has order 2^n. The class $\xi_{3,2}$ has order either 8 or 16, because $F(\xi_{3,2}) = \xi_{3,1}$ has order 8. If $\xi_{3,2}$ has order 16, then the quotient of $\mathbb{H}_3(C_4, T(\mathbb{S}))$ by the image of the edge homomorphism is equal to $\mathbb{Z}/16\mathbb{Z}$ generated by the image of $\xi_{3,2}$. But then $V(\xi_{3,1})$ has order 8 which contradicts that, modulo the image of the edge homomorphism,

$$4V(\xi_{3,1}) = V(4\xi_{3,1}) \equiv VdV(\eta^2) = 2dV^2(\eta^2) = 0.$$

Hence, $\xi_{3,2}$ has order 8, and the group $\mathbb{H}_3(C_4, T(\mathbb{S}))$ is as stated. So we let $n \geqslant 4$ and assume, inductively, that $\xi_{3,n-2}$ has order 2^{n-1}. The class $2^{n-2}\xi_{3,n-2}$ is represented in the spectral sequence by ηz_2. Now, by Lemma 6(4), we have $V(\eta z_2) = \eta z_2$, which shows that the class $2^{n-2}V(\xi_{3,n-2}) = V(2^{n-2}\xi_{3,n-2})$ is non-zero and represented by ηz_2. This implies that $2^{n-1}\xi_{3,n-1}$ is non-zero, and hence, $\xi_{3,n-1}$ has order 2^n as stated.

Next, we show that, for $n \geqslant 3$, the class $\xi_{5,n-1}$ has order 2^{n-1}. If $n \geqslant 4$, the spectral sequence shows that there is an extension

$$0 \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{H}_5(C_{2^{n-1}}, T(\mathbb{S})) \to \mathbb{Z}/2^{n-2}\mathbb{Z} \to 0.$$

The Verschiebung map induces a map of extensions from the extension for n to the extension for $n+1$, and Lemma 6 shows that the resulting extension of colimits with respect to the Verschiebung maps is an extension

$$0 \to \mathbb{Z}/4\mathbb{Z} \to \operatorname*{colim}_V \mathbb{H}_5(C_{2^{n-1}}, T(\mathbb{S})) \to \mathbb{Q}_2/\mathbb{Z}_2 \to 0.$$

It follows from [25, Lemma 4.4.9] that there is a canonical isomorphism

$$\operatorname{Ext}(\mathbb{Q}_2/\mathbb{Z}_2, \operatorname*{colim}_V \mathbb{H}_5(C_{2^{n-1}}, T(\mathbb{S}))) \xrightarrow{\sim} \lim_F \mathbb{H}_6(C_{2^{n-1}}, T(\mathbb{S}))$$

and, by the proof of Lemma 9, the right-hand group is cyclic of order 2. This implies that the extension for $n \geqslant 4$ is equivalent to the extension

$$0 \to \mathbb{Z}/4\mathbb{Z} \xrightarrow{(1,-2^{n-3})} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2^{n-1}\mathbb{Z} \xrightarrow{2^{n-3}+1} \mathbb{Z}/2^{n-2}\mathbb{Z} \to 0.$$

It follows that, for $n \geqslant 4$, the class $\xi_{5,n-1}$ has order 2^{n-1} as stated. It remains to prove that $\xi_{5,2}$ has order 4. If this is not the case, the map of extensions induced by the Verschiebung map $V\colon \mathbb{H}_5(C_4, T(\mathbb{S})) \to \mathbb{H}_5(C_8, T(\mathbb{S}))$ takes the form

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "{(1,0)}"] \arrow[d, "2"] & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \arrow[r, "0+1"] \arrow[d, "V"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, "2"] & 0 \\
0 \arrow[r] & \mathbb{Z}/4\mathbb{Z} \arrow[r, "{(1,-2)}"] & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \arrow[r, "2+1"] & \mathbb{Z}/4\mathbb{Z} \arrow[r] & 0,
\end{tikzcd}$$

where the middle map V takes $(1,0)$ to $(0,4)$ and $(0,1)$ to either $(1,0)$ or $(1,4)$. The class $\xi_{5,2}$ corresponds to either $(0,1)$ or $(1,1)$ in the top middle group. In either case, we find that the class $V(\xi_{5,2})$ has order 2 and reduces to a generator of the quotient of $\mathbb{H}_5(C_8, T(\mathbb{S}))$ by the subgroup $\mathbb{Z}/8\mathbb{Z} \cdot \xi_{5,3}$. It follows that the class

$$V(\xi_{5,2}) - 2\xi_{5,3} \in \mathbb{H}_5(C_8, T(\mathbb{S}))$$

generates the kernel of the edge homomorphism onto $\mathbb{Z}/4\mathbb{Z} \cdot 2\iota z_5$. Then, Lemma 6 shows that the class $F(V(\xi_{5,2}) - 2\xi_{5,3})$ generates the kernel of the edge homomorphism from $\mathbb{H}_5(C_4, T(\mathbb{S}))$ onto $\mathbb{Z}/2\mathbb{Z} \cdot 2\iota z_5$. But $F(V(\xi_{5,2}) - 2\xi_{5,3}) = 0$ which is a contradiction. We conclude that the group $\mathbb{H}_5(C_4, T(\mathbb{S}))$ is cyclic as stated.

Finally, the Frobenius map $F\colon \mathbb{H}_5(C_8, T(\mathbb{S})) \to \mathbb{H}_5(C_4, T(\mathbb{S}))$ induces a map of extensions which takes the form

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/4\mathbb{Z} \arrow[r, "{(1,-2)}"] \arrow[d, "1"] & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \arrow[r, "2+1"] \arrow[d, "0+1"] & \mathbb{Z}/4\mathbb{Z} \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "2"] & \mathbb{Z}/4\mathbb{Z} \arrow[r, "1"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0.
\end{tikzcd}$$

The class ρ corresponds to one of the elements $(1,0)$ or $(1,4)$ of the top middle group both of which map to zero by the middle vertical map. It follows that $F(\rho)$ is zero as stated. $\square$

We define $\xi_{q,s} \in \mathrm{TR}^n_q(\mathbb{S};2)$ to be the image of $\xi_{q,s} \in \mathbb{H}_q(C_{2^s}, T(\mathbb{S}))$ by the composition of the norm map and the iterated Segal-tom Dieck splitting

$$\mathbb{H}_q(C_{2^s}, T(\mathbb{S})) \to \mathrm{TR}^s_q(\mathbb{S};2) \to \mathrm{TR}^n_q(\mathbb{S};2).$$

Similarly, we define $\rho \in \mathrm{TR}^n_5(\mathbb{S};2)$ to be the image of $\rho \in \mathbb{H}_5(C_8, T(\mathbb{S}))$ by the composition of the norm map and the iterated Segal-tom Dieck splitting

$$\mathbb{H}_5(C_8, T(\mathbb{S})) \to \mathrm{TR}^4_5(\mathbb{S};2) \to \mathrm{TR}^n_5(\mathbb{S};2).$$

Then we have the following result.

Theorem 11. *The groups* $\mathrm{TR}^n_q(\mathbb{S};2)$ *with* $q \leqslant 5$ *are given by*

$$\begin{aligned}
\mathrm{TR}^n_0(\mathbb{S};2) &= \bigoplus_{0\leqslant s<n} \mathbb{Z}_2 \cdot V^s(1)\\
\mathrm{TR}^n_1(\mathbb{S};2) &= \bigoplus_{0\leqslant s<n} \mathbb{Z}/2\mathbb{Z} \cdot V^s(\eta) \oplus \bigoplus_{1\leqslant s<n} \mathbb{Z}/2^s\mathbb{Z} \cdot \xi_{1,s}\\
\mathrm{TR}^n_2(\mathbb{S};2) &= \bigoplus_{0\leqslant s<n} \mathbb{Z}/2\mathbb{Z} \cdot V^s(\eta^2) \oplus \bigoplus_{1\leqslant s<n} \mathbb{Z}/2\mathbb{Z} \cdot \eta\xi_{1,s}\\
\mathrm{TR}^n_3(\mathbb{S};2) &= \bigoplus_{0\leqslant s<n} \mathbb{Z}/8\mathbb{Z} \cdot V^s(\nu) \oplus \bigoplus_{1\leqslant s<n} \mathbb{Z}/2^u\mathbb{Z} \cdot \xi_{3,s} \oplus \bigoplus_{2\leqslant s<n} \mathbb{Z}/2\mathbb{Z} \cdot \eta^2\xi_{1,s}\\
\mathrm{TR}^n_4(\mathbb{S};2) &= \bigoplus_{1\leqslant s<n} \mathbb{Z}/2^v\mathbb{Z} \cdot \nu\xi_{1,s}\\
\mathrm{TR}^n_5(\mathbb{S};2) &= \bigoplus_{2\leqslant s<n} \mathbb{Z}/2^s\mathbb{Z} \cdot \xi_{5,s} \oplus \bigoplus_{3\leqslant s<n} \mathbb{Z}/2\mathbb{Z} \cdot V^{s-3}(\rho)
\end{aligned}$$

where $u = u(s)$ *is the larger of* 3 *and* $s+1$, *and where* $v = v(s)$ *is the smaller of* 3 *and* s. *The restriction map takes* $\xi_{q,s}$ *to* $\xi_{s,q}$, *if* $s < n-1$, *and to zero, if* $s = n-1$, *and takes* ρ *to* ρ, *if* $n \geqslant 5$, *and to zero, if* $n = 4$. *The Frobenius map takes* $\xi_{q,s}$ *to* $\xi_{q,s-1}$, *where* $\xi_{1,0} = \eta$ *and* $\xi_{3,0} = \nu$, *and takes* ρ *to zero. Connes' operator takes* $V^s(1)$ *to* $\xi_{1,s} + V^s(\eta)$, *and takes* $\xi_{1,s}$ *to zero.*

Proof. The Segal-tom Dieck splitting gives a section of the restriction map. Hence, the fundamental long-exact sequence

$$\cdots \to \mathbb{H}_q(C_{p^{n-1}}, T(\mathbb{S})) \xrightarrow{N} \mathrm{TR}^n_q(\mathbb{S};p) \xrightarrow{R} \mathrm{TR}^{n-1}_q(\mathbb{S};p) \xrightarrow{\partial} \mathbb{H}_{q-1}(C_{p^{n-1}}, T(\mathbb{S})) \to \cdots$$

of Proposition 4 breaks into split short-exact sequences and Proposition 10 then shows that the groups $\mathrm{TR}^n_q(\mathbb{S};2)$ are as stated. Since the Frobenius map and the Segal-tom Dieck splitting commute, the formula for the Frobenius also follows form Proposition 10. Finally, from the proof of Proposition 10, we have $\xi_{1,s} = dV^s(1) + V^s(\eta)$. This implies that

$$d\xi_{1,s} = ddV^s(1) + dV^s(\eta) = dV^s(\eta) + dV^s(\eta) = 0$$

as stated. □

Remark 12. We have not determined $\eta\xi_{3,s}$ and $d\xi_{3,s}$.

6 The Groups $\mathrm{TR}^n_q(\mathbb{Z};2)$

In this section, we again implicitly consider homotopy groups with $\mathbb{Z}_2$-coefficients. The groups $\mathrm{TR}^1_q(\mathbb{Z};2)$ were evaluated by Bökstedt [3]; see also [23, Theorem 1.1]. The group $\mathrm{TR}^1_0(\mathbb{Z};2)$ is equal to $\mathbb{Z}_2$ generated by the multiplicative unit element

$\iota = [1]_1$, and for positive integers q, the group $\mathrm{TR}^1_q(\mathbb{Z};2)$ is finite cyclic of order

$$|\,\mathrm{TR}^1_q(\mathbb{Z};2)| = \begin{cases} 2^{v_2(i)} & (q = 2i-1 \text{ odd}) \\ 1 & (q \text{ even}). \end{cases}$$

We choose a generator λ of $K_3(\mathbb{Z})$ such that $2\lambda = \nu$. Then, by [5, Theorem 10.4], the image of λ by the cyclotomic trace map generates the group $\mathrm{TR}^1_3(\mathbb{Z};2)$. We also choose a generator γ of the group $\mathrm{TR}^1_7(\mathbb{Z};2)$. We first derive the following result from Rognes' paper [27].

Proposition 13. *The group* $\mathrm{TR}^n_q(\mathbb{Z};2)$ *is zero, for every positive even integer q and every positive integer n.*

Proof. The group $\mathrm{TR}^n_q(\mathbb{Z};2)$ is finite, for all positive integers q and n. Indeed, this is true, for $n = 1$ by Bökstedt's result that we recalled above and follows, inductively, for $n \geqslant 1$, from the fundamental long-exact sequence of Proposition 4, the skeleton spectral sequence, and the fact that the boundary map

$$\partial\colon \mathrm{TR}^{n-1}_1(\mathbb{Z};2) \to \mathbb{H}_0(C_{2^{n-1}}, T(\mathbb{Z}))$$

in the fundamental long-exact sequence is zero [17, Proposition 3.3]. Moreover, the group $\mathrm{TR}^n_0(\mathbb{Z};2)$ is a free $\mathbb{Z}_2$-module. It follows that, in the strongly convergent whole plane Bockstein spectral sequence

$$E^2_{s,t} = \mathrm{TR}^n_{s+t}(\mathbb{Z};2,2^{-s}\mathbb{Z}/2^{-(s-1)}\mathbb{Z}) \Rightarrow \mathrm{TR}^n_{s+t}(\mathbb{Z};2,\mathbb{Q}_2)$$

induced from the 2-adic filtration of $\mathbb{Q}_2$, all elements of total degree 0 survive to the E^∞-term and all elements of positive total degree are annihilated by differentials. The differentials are periodic in the sense that the isomorphism $2\colon \mathbb{Q}_2 \to \mathbb{Q}_2$ induces an isomorphism of spectral sequences

$$\underline{2}\colon E^r_{s,t} \xrightarrow{\sim} E^r_{s-1,t+1}.$$

We recall from [27, Lemma 9.4] that, for all positive integers n and i,

$$\dim_{\mathbb{F}_2} \mathrm{TR}^n_{2i-1}(\mathbb{Z};2,\mathbb{F}_2) = \dim_{\mathbb{F}_2} \mathrm{TR}^n_{2i}(\mathbb{Z};2,\mathbb{F}_2).$$

Using this result, we show, by induction on $i \geqslant 1$, that every element of total degree $2i-1$ is an infinite cycle and that every non-zero element of total degree $2i$ supports a non-zero differential. The proof of the case $i = 1$ and of the induction step are similar. The statement that every element in total degree $2i-1$ is an infinite cycle follows, for $i = 1$, from the fact that every element of total degree 0 survives to the E^∞-term, and for $i > 1$, from the inductive hypothesis that every non-zero element of total degree $2i-2$ supports a non-zero differential. Since no element of total degree $2i-1$ survives to the E^∞-term, it is hit by a differential supported on an element of total degree $2i$. Since the differentials are periodic and $E^2_{s,2i-1-s}$ and $E^2_{s,2i-s}$ have the same dimension, we find that non-zero every element of total degree $2i$ supports a non-zero differential as stated.

Finally, we consider the strongly convergent left half-plane Bockstein spectral sequence induced from the 2-adic filtration of $\mathbb{Z}_2$,

$$E^2_{s,t} = \mathrm{TR}^n_{s+t}(\mathbb{Z}; 2, 2^{-s}\mathbb{Z}/2^{-(s-1)}\mathbb{Z}) \Rightarrow \mathrm{TR}^n_{s+t}(\mathbb{Z}; 2, \mathbb{Z}_2).$$

The differentials in this spectral sequence are obtained by restricting the differentials in the whole plan Bockstein spectral sequence above. It follows that in this spectral sequence, too, every non-zero element of positive even total degree supports a non-zero differential. This completes the proof. $\square$

Remark 14. The same argument based on Bökstedt and Madsen's paper [5], shows that, for an odd prime p, the groups $\mathrm{TR}^n_q(\mathbb{Z}; p)$ are zero, for every positive even integer q and every positive integer n.

We next consider the skeleton spectral sequence

$$E^2_{s,t} = H_s(C_{2^{n-1}}, \mathrm{TR}^1_t(\mathbb{Z}; 2)) \Rightarrow \mathbb{H}_{s+t}(C_{2^{n-1}}, T(\mathbb{Z})).$$

The E^2-term, for $s + t \leqslant 7$, takes the form

$$\begin{array}{cccccccc}
\mathbb{Z}/4\mathbb{Z} & & & & & & & \\
0 & 0 & & & & & & \\
0 & 0 & 0 & & & & & \\
0 & 0 & 0 & 0 & & & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & & & \\
0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
\mathbb{Z}_2 & \mathbb{Z}/2^{n-1}\mathbb{Z} & 0 & \mathbb{Z}/2^{n-1}\mathbb{Z} & 0 & \mathbb{Z}/2^{n-1}\mathbb{Z} & 0 & \mathbb{Z}/2^{n-1}\mathbb{Z}.
\end{array}$$

The group $E^2_{s,0}$ is generated by ιz_s and the group $E^2_{s,3}$ is generated by λz_s. The group $E^2_{s,7}$ is generated by γz_s, if $s = 0$ or if s is odd of if $n > 1$, and is generated by $2\gamma z_s$, if $n = 1$ and s is positive and even. It follows from [27, Theorem 8.14] that the group $\mathbb{H}_4(C_{2^{n-1}}, T(\mathbb{Z}); \mathbb{F}_2)$ is an $\mathbb{F}_2$-vector space of dimension 1. This implies that $d^4(\iota z_5) = \lambda z_1$. On the other hand, $d^4(\iota z_7) = 0$, since ιz_7 survives to the E^4-term of the skeleton spectral sequence for $\mathbb{H}_q(C_{2^{n-1}}, T(\mathbb{S}))$ and is a d^4-cycle. This shows that the E^5-term for $s + t \leqslant 7$ is given by

$$\begin{array}{cccccccc}
\mathbb{Z}/4\mathbb{Z} & & & & & & & \\
0 & 0 & & & & & & \\
0 & 0 & 0 & & & & & \\
0 & 0 & 0 & 0 & & & & \\
\mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & & & \\
0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
\mathbb{Z}_2 & \mathbb{Z}/2^{n-1}\mathbb{Z} & 0 & \mathbb{Z}/2^{n-1}\mathbb{Z} & 0 & 2\mathbb{Z}/2^{n-1}\mathbb{Z} & 0 & \mathbb{Z}/2^{n-1}\mathbb{Z}.
\end{array}$$

We claim that the differential $d^5\colon E^5_{5,3} \to E^5_{0,7}$ is zero. Indeed, let

$$F^{m-n}\colon {}'E^r_{s,t} \to E^r_{s,t}$$

be the map of spectral sequences induced by the iterated Frobenius map

$$F^{m-n}\colon \mathbb{H}_q(C_{2^{m-1}}, T(\mathbb{Z})) \to \mathbb{H}_q(C_{2^{n-1}}, T(\mathbb{Z})).$$

It follows from Lemma 6 that the map $F^{m-n}\colon {}'E^5_{5,3} \to E^5_{5,3}$ is an isomorphism and that, for $m \geqslant n+2$, the map $F^{m-n}\colon {}'E^5_{5,3} \to E^5_{5,3}$ is zero. Hence, the differential in question is zero as claimed. It follows that the E^5-term of the spectral sequence is also the E^∞-term.

We choose a generator κ of the infinite cyclic group $K_5(\mathbb{Z})$ and recall the generator λ of the group $K_3(\mathbb{Z})$. We continue to write λ and κ for the images of λ and κ in $\mathrm{TR}^n_3(\mathbb{Z};2)$ and $\mathrm{TR}^n_5(\mathbb{Z};2)$ by the cyclotomic trace map. The norm map

$$\mathbb{H}_5(C_2, T(\mathbb{Z})) \to \mathrm{TR}^2_5(\mathbb{Z};2)$$

is an isomorphism, and we will also write κ for the unique class on the left-hand side whose image by the norm map is the class κ on the right-hand side. Finally, we continue to write $\xi_{q,n} \in \mathbb{H}_q(C_{2^{n-1}}, T(\mathbb{Z}))$ for the image by the map induced from the Hurewicz map $\ell\colon \mathbb{S} \to \mathbb{Z}$ of the class $\xi_{q,n} \in \mathbb{H}_q(C_{2^{n-1}}, T(\mathbb{S}))$.

Proposition 15. *The groups $\mathbb{H}_q(C_{2^{n-1}}, T(\mathbb{Z}))$ with $q \leqslant 6$ are given by*

$$\begin{aligned}
\mathbb{H}_0(C_{2^{n-1}}, T(\mathbb{Z})) &= \mathbb{Z}_2 \cdot V^{n-1}(\iota),\\
\mathbb{H}_1(C_{2^{n-1}}, T(\mathbb{Z})) &= \begin{cases} 0 & (n=1)\\ \mathbb{Z}/2^{n-1}\mathbb{Z}\cdot \xi_{1,n-1} & (n \geqslant 2)\end{cases},\\
\mathbb{H}_2(C_{2^{n-1}}, T(\mathbb{Z})) &= 0\\
\mathbb{H}_3(C_{2^{n-1}}, T(\mathbb{Z})) &= \begin{cases} \mathbb{Z}/2\mathbb{Z}\cdot\lambda & (n=1)\\ \mathbb{Z}/2^{n}\mathbb{Z}\cdot \xi_{3,n-1} & (n \geqslant 2)\end{cases},\\
\mathbb{H}_4(C_{2^{n-1}}, T(\mathbb{Z})) &= 0\\
\mathbb{H}_5(C_{2^{n-1}}, T(\mathbb{Z})) &= \begin{cases} 0 & (n=1)\\ \mathbb{Z}/2\mathbb{Z}\cdot\kappa & (n=2)\\ \mathbb{Z}/2^{n-2}\mathbb{Z}\cdot \xi_{5,n-1} \oplus \mathbb{Z}/2\mathbb{Z}\cdot V^{n-2}(\kappa) & (n \geqslant 3)\end{cases},\\
\mathbb{H}_6(C_{2^{n-1}}, T(\mathbb{Z})) &= \begin{cases} 0 & (n=1)\\ \mathbb{Z}/2\mathbb{Z}\cdot dV^{n-2}(\kappa) & (n \geqslant 2)\end{cases}.
\end{aligned}$$

In addition, $F(\xi_{q,n-1}) = \xi_{q,n-2}$, where $\xi_{1,0}$, $\xi_{3,0}$, and $\xi_{5,0}$ are zero.

Proof. The cases $q = 0$ and $q = 1$ follow immediately from the spectral sequence above and from the fact that the map $\mathrm{TR}^1_0(\mathbb{S};2) \to \mathrm{TR}^1_0(\mathbb{Z};2)$ induced by the Hurewicz map is an isomorphism. The cases $q = 2$ and $q = 4$ follow directly from the spectral sequence above. It follows from [27, Theorem 8.14] that $\mathbb{H}_3(C_{2^{n-1}}, T(\mathbb{Z}); \mathbb{F}_2)$ is an $\mathbb{F}_2$-vector space of dimension 1, for all $n \geqslant 1$. The statement for $q = 3$ follows. It also follows from loc. cit. that $\mathbb{H}_5(C_{2^{n-1}}, T(\mathbb{Z}); \mathbb{F}_2)$ is an $\mathbb{F}_2$-vector space of dimension 0, if $n = 1$, dimension 1, if $n = 2$, and dimension 2, if $n \geqslant 3$. Hence, to prove the statement for $q = 5$, it will suffice to show that the group $\mathbb{H}_5(C_2, T(\mathbb{Z}))$ is generated by the class κ, or equivalently, that the composition

$$K_5(\mathbb{Z}) \to \mathrm{TC}_5^2\,\mathbb{Z}; 2) \to \mathrm{TR}_5^2(\mathbb{Z}; 2) \xrightarrow{\sim} \mathrm{TR}_5^2(\mathbb{Z}; 2, \mathbb{F}_2)$$

of the cyclotomic trace map and the modulo 2 reduction map is surjective. But this is the statement that $i_1(\kappa) = \xi_5(0)$ in [30, Proposition 4.2]. (Here $\xi_5(0)$ is name given in loc. cit. to the generator of the right-hand group; it is unrelated to the class $\xi_{5,0}$.) Finally, the statement for $q = 6$ follows from [18, Proposition 4.4.1]. □

Corollary 16. *The cokernel of the map induced by the Hurewicz map*

$$\ell\colon \mathrm{TR}_3^n(\mathbb{S}; 2) \to \mathrm{TR}_3^n(\mathbb{Z}; 2)$$

is equal to $\mathbb{Z}/2\mathbb{Z} \cdot \lambda$.

Proof. The proof is by induction on $n \geqslant 1$. In the case $n = 1$, the Hurewicz map induces the zero map $\mathrm{TR}_q^1(\mathbb{S}; 2) \to \mathrm{TR}_q^1(\mathbb{Z}; 2)$, for all positive integers q. Indeed, the spectrum $\mathrm{TR}^1(\mathbb{Z}; 2)$ is a module spectrum over the Eilenberg–MacLane spectrum for $\mathbb{Z}$ and therefore is weakly equivalent to a product of Eilenberg–MacLane spectra. As we recalled above, $\mathrm{TR}_3^1(\mathbb{Z}; 2) = \mathbb{Z}/2\mathbb{Z}\cdot\lambda$, which proves the case $n = 1$. To prove the induction step, we use that the Hurewicz map induces a map of fundamental long-exact sequences which takes the form

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{H}_3(C_{2^{n-1}}, T(\mathbb{S})) \arrow[r] \arrow[d] & \mathrm{TR}_3^n(\mathbb{S}; 2) \arrow[r] \arrow[d] & \mathrm{TR}_3^{n-1}(\mathbb{S}; 2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{H}_3(C_{2^{n-1}}, T(\mathbb{Z})) \arrow[r] & \mathrm{TR}_3^n(\mathbb{Z}; 2) \arrow[r] & \mathrm{TR}_3^{n-1}(\mathbb{Z}; 2) \arrow[r] & 0.
\end{tikzcd}$$

The zero on the lower right-hand side follows from Proposition 15, and the zero on the lower left-hand side from Proposition 13. Since Propositions 10 and 15 show that the left-hand vertical map is surjective, the induction step follows. □

We owe the proof of the following result to Marcel Bökstedt.

Lemma 17. *The square of homotopy groups with $\mathbb{Z}_2$-coefficients*

$$\begin{tikzcd}
K_5(\mathbb{S}; \mathbb{Z}_2) \arrow[r] \arrow[d] & K_5(\mathbb{S}_2; \mathbb{Z}_2) \arrow[d] \\
K_5(\mathbb{Z}; \mathbb{Z}_2) \arrow[r] & K_5(\mathbb{Z}_2; \mathbb{Z}_2),
\end{tikzcd}$$

where the vertical maps are induced by the Hurewicz maps and the horizontal maps are induced by the completion maps, takes the from

$$\begin{tikzcd}
\mathbb{Z}_2 \cdot 8\kappa \arrow[r] \arrow[d] & \mathbb{Z}_2 \cdot (4\kappa + \tau) \arrow[d] \\
\mathbb{Z}_2 \cdot \kappa \arrow[r] & \mathbb{Z}_2 \cdot \kappa \oplus \mathbb{Z}/2\mathbb{Z} \cdot \tau
\end{tikzcd}$$

Proof. It was proved in [30, Proposition 4.2] that the group $K_5(\mathbb{Z}_2; \mathbb{Z}_2)$ is the direct sum of a free $\mathbb{Z}_2$-module of rank one generated by κ and a torsion subgroup of order 2; the class τ is the unique generator of the torsion subgroup. Moreover, [31, Theorem 5.8] shows that the group $K_5(\mathbb{S}; \mathbb{Z}_2)$ is a free $\mathbb{Z}_2$-module of rank one, and [31, Theorem 2.11] and [4, Theorem 5.17] show that the group $K_5(\mathbb{S}_2; \mathbb{Z}_2)$ is a free $\mathbb{Z}_2$-module of rank one. To complete the proof of the lemma, it remains to show that the left-hand vertical map in the diagram in the statement is equal to the inclusion of a subgroup of index 8. This is essentially proved in [2] as we now explain. In op. cit., Bökstedt constructs a homotopy commutative diagram of pointed spaces

$$\begin{tikzcd}
G/O \arrow[r, "e"] \arrow[d] & \mathrm{Fib}(s) \arrow[d] & \mathrm{Fib}(t) \arrow[l, "g"'] \arrow[d] \\
BSO \arrow[r, equal] \arrow[d] & BSO \arrow[r, equal] \arrow[d, "s"] & BSO \arrow[d, "t"] \\
BSG \arrow[r, "\eta"] & SG & SJ \arrow[l]
\end{tikzcd}$$

in which the columns are fibration sequences. The induced diagram of fourth homotopy groups with $\mathbb{Z}_2$-coefficients is isomorphic to the diagram

$$\begin{tikzcd}
\mathbb{Z}_2 \arrow[r, "8"] \arrow[d, "8"] & \mathbb{Z}_2 \arrow[d, "\mathrm{id}"] & \mathbb{Z}_2 \arrow[l, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"] \\
\mathbb{Z}_2 \arrow[r, equal] \arrow[d] & \mathbb{Z}_2 \arrow[r, equal] \arrow[d] & \mathbb{Z}_2 \arrow[d] \\
\mathbb{Z}/8\mathbb{Z} \arrow[r] & 0 & 0. \arrow[l]
\end{tikzcd}$$

We compare this diagram to the following diagram constructed by Waldhausen.

$$\begin{tikzcd}
& G/O \arrow[r, "f"] \arrow[d, "e"] & \Omega\,\mathrm{Wh}^{\mathrm{Diff}}(*) \arrow[d] & \\
\mathrm{Fib}(t) \arrow[r, "g"] & \mathrm{Fib}(s) \arrow[r] & \Omega K(\mathbb{Z}) \arrow[r] & \Omega JK(\mathbb{Z}).
\end{tikzcd}$$

It is proved in [2, p. 30] that the composition of the lower horizontal maps in this diagram becomes a weak equivalence after 2-completion. Moreover, it is proved in [31, Theorem 7.5] that the upper horizontal map induces an isomorphism of homotopy groups with $\mathbb{Z}_2$-coefficients in degrees less than or equal to 8. Hence, the induced diagram of fourth homotopy groups with $\mathbb{Z}_2$-coefficients is isomorphic to the diagram

$$\begin{tikzcd} & \mathbb{Z}_2 \arrow[r, "\mathrm{id}"] \arrow[d, "8"] & \mathbb{Z}_2 \arrow[d, "8"] & \\ \mathbb{Z}_2 \arrow[r, "\mathrm{id}"] & \mathbb{Z}_2 \arrow[r, "\mathrm{id}"] & \mathbb{Z}_2 \arrow[r, "\mathrm{id}"] & \mathbb{Z}_2. \end{tikzcd}$$

The right-hand vertical map in this diagram is induced by the composition

$$\mathrm{Wh}^{\mathrm{Diff}}(*) \to K(\mathbb{S}) \to K(\mathbb{Z})$$

of the canonical section of the canonical map $K(\mathbb{S}) \to \mathrm{Wh}^{\mathrm{Diff}}(*)$ and the map induced by the Hurewicz map. The left-hand map induces an isomorphism of fifth homotopy groups with $\mathbb{Z}_2$-coefficients because $\pi_5(\mathbb{S}; \mathbb{Z}_2)$ is zero. This completes the proof that the map induced by the Hurewicz map $K_5(\mathbb{S}; \mathbb{Z}_2) \to K_5(\mathbb{Z}; \mathbb{Z}_2)$ is the inclusion of an index eighth subgroup. The lemma follows. □

We define the class $\xi_{q,s} \in \mathrm{TR}^n_q(\mathbb{Z}; 2)$ to be the image of the class $\xi_{q,s} \in \mathrm{TR}^n_q(\mathbb{S}; 2)$ by the map induced by the Hurewicz map.

Theorem 18. *The groups* $\mathrm{TR}^n_q(\mathbb{Z}; 2)$ *with* $q \leqslant 6$ *are given by*

$$\begin{aligned}
\mathrm{TR}^n_0(\mathbb{Z};2) &= \bigoplus_{0 \leqslant s < n} \mathbb{Z}_2 \cdot V^s(1),\\
\mathrm{TR}^n_1(\mathbb{Z};2) &= \bigoplus_{1 \leqslant s < n} \mathbb{Z}/2^s\mathbb{Z} \cdot \xi_{1,s},\\
\mathrm{TR}^n_2(\mathbb{Z};2) &= 0,\\
\mathrm{TR}^n_3(\mathbb{Z};2) &= \begin{cases} \mathbb{Z}/2\mathbb{Z} \cdot \lambda & (n = 1)\\ \mathbb{Z}/8\mathbb{Z} \cdot \lambda \oplus \bigoplus_{2 \leqslant s < n} \mathbb{Z}/2^{s+1}\mathbb{Z} \cdot \xi_{3,s} & (n \geqslant 2) \end{cases},\\
\mathrm{TR}^n_4(\mathbb{Z};2) &= 0,\\
\mathrm{TR}^n_5(\mathbb{Z};2) &= \mathbb{Z}/2^{n-1}\mathbb{Z} \cdot \kappa \oplus \bigoplus_{2 \leqslant s < n} \mathbb{Z}/2^{s-1}\mathbb{Z} \cdot (\xi_{5,s} + \cdots + \xi_{5,n-1} + 4u\kappa),\\
\mathrm{TR}^n_6(\mathbb{Z};2) &= 0,
\end{aligned}$$

where $u \in \mathbb{Z}_2^*$ *is a unit.*

Proof. The map induced by the Hurewicz map is an isomorphism, for $q = 0$, so the statement for the group $\mathrm{TR}_0^n(\mathbb{Z};2)$ follows from Theorem 11. The statement for $q = 1$ follows from Proposition 15 and from the fact that the generator $\xi_{1,s}$ is annihilated by 2^s. For $q = 3$, the case $n = 1$ was recalled at the beginning of the section, so suppose that $n \geqslant 2$. We know from Proposition 15 that the two sides of the statement are groups of the same order. We also know that both groups are the direct sum of $n-1$ cyclic groups. Indeed, this is trivial, for the right-hand side, and is proved in [27, Lemma 9.4], for the left-hand side. Now, it follows from Theorem 11 that $\xi_{3,s}$ is annihilated by 2^{s+1}, so it suffices to show that λ is annihilated by 8. We have a commutative diagram

$$\begin{tikzcd} K_3(\mathbb{Z};\mathbb{Z}_2) \arrow[r] \arrow[d] & \mathrm{TR}_3^n(\mathbb{Z};2,\mathbb{Z}_2) \arrow[d] \\ K_3(\mathbb{Z}_2;\mathbb{Z}_2) \arrow[r] & \mathrm{TR}_3^n(\mathbb{Z}_2;2,\mathbb{Z}_2) \end{tikzcd}$$

where the horizontal maps are the cyclotomic trace maps, where the vertical maps are induced by the completion maps, and where we have explicitly indicated that we are considering the homotopy groups with $\mathbb{Z}_2$-coefficients. The right-hand vertical map is an isomorphism by [17, Addendum 6.2]. Therefore, it suffices to show that the image of λ in $K_3(\mathbb{Z}_2;\mathbb{Z}_2)$ has order 8. But this is proved in [30, Proposition 4.2].

It remains to prove the statement for $q = 5$. We first show that $\mathrm{TR}_5^n(\mathbb{Z};2)$ is generated by the classes $\kappa, \xi_{5,2}, \dots, \xi_{5,n-1}$, or equivalently, that the group

$$\mathrm{TR}_5^n(\mathbb{Z};2)/2\,\mathrm{TR}_5^n(\mathbb{Z};2) \xrightarrow{\sim} \mathrm{TR}_5^n(\mathbb{Z};2,\mathbb{F}_2)$$

is generated by the images of the classes $\kappa, \xi_{5,2}, \dots, \xi_{5,n-1}$. We prove this by induction on $n \geqslant 2$. The case $n = 2$ is true, so we assume the statement for $n-1$ and prove it for n. The fundamental long-exact sequence takes the form

$$\mathbb{H}_5(C_{2^{n-1}}, T(\mathbb{Z});\mathbb{F}_2) \xrightarrow{N} \mathrm{TR}_5^n(\mathbb{Z};2,\mathbb{F}_2) \xrightarrow{R} \mathrm{TR}_5^{n-1}(\mathbb{Z};2,\mathbb{F}_2) \to 0.$$

Inductively, the right-hand group is generated by the classes κ, $\xi_{5,2}, \dots, \xi_{5,n-2}$, which are the images by the restriction map of the classes $\kappa, \xi_{5,2}, \dots, \xi_{5,n-2}$ in the middle group. Moreover, Proposition 15 shows that the left-hand group is generated by the classes $V^{n-2}(\kappa)$ and $\xi_{5,n-1}$. Hence, it will suffice to show that, for $n \geqslant 3$, the image of the class $V^{n-2}(\kappa)$ in $\mathrm{TR}_5^n(\mathbb{Z};2,\mathbb{F}_2)$ is zero. This follows from [27, Theorem 8.14] as we now explain. We have the commutative diagram with exact rows

$$\begin{tikzcd} \mathrm{TR}_{q+1}^{n-1}(\mathbb{Z};2,\mathbb{F}_2) \arrow[r, "\partial"] \arrow[d, "\hat{\Gamma}"] & \mathbb{H}_q(C_{2^{n-1}}, T(\mathbb{Z});\mathbb{F}_2) \arrow[r, "N"] \arrow[d, equal] & \mathrm{TR}_q^n(\mathbb{Z};2,\mathbb{F}_2) \arrow[d, "\Gamma"] \\ \hat{\mathbb{H}}^{-q-1}(C_{2^{n-1}}, T(\mathbb{Z});\mathbb{F}_2) \arrow[r, "\partial^h"] & \mathbb{H}_q(C_{2^{n-1}}, T(\mathbb{Z});\mathbb{F}_2) \arrow[r, "N^h"] & \mathbb{H}^{-q}(C_{2^{n-1}}, T(\mathbb{Z});\mathbb{F}_2) \end{tikzcd}$$

considered first in [5, (6.1)]. It is follows from [27, Theorems 0.2, 0.3] that the left-hand vertical map $\hat{\Gamma}$ is an isomorphism, for all integers $q+1 \geqslant 0$ and $n \geqslant 1$. Hence, it suffices to show that the class $V^{n-2}(\kappa)$ in the lower middle group is in the image of the lower left-hand horizontal map ∂^h. The lower left-hand group is the abutment of the strongly convergent, upper half-plan Tate spectral sequence

$$\hat{E}^2_{s,t} = \hat{H}^{-s}(C_{2^{n-1}}, \mathrm{TR}^1_t(\mathbb{Z}; 2, \mathbb{F}_2)) \Rightarrow \hat{\mathbb{H}}^{-s-t}(C_{2^{n-1}}, T(\mathbb{Z}); \mathbb{F}_2),$$

and the middle groups are the abutment of the strongly convergent, first quadrant skeleton spectral sequence

$$E^2_{s,t} = H_s(C_{2^{n-1}}, \mathrm{TR}^1_t(\mathbb{Z}; 2, \mathbb{F}_2)) \Rightarrow \mathbb{H}_{s+t}(C_{2^{n-1}}, T(\mathbb{Z}); \mathbb{F}_2).$$

Moreover, the map ∂^h induces a map of spectral sequences

$$\partial^{h,r} \colon \hat{E}^r_{s,t} \to E^r_{s-1,t},$$

which is an isomorphism, for $r = 2$ and $s \geqslant 1$. Suppose that the homotopy class $\tilde{x} \in \mathbb{H}_q(C_{2^{n-1}}, T(\mathbb{Z}); \mathbb{F}_2)$ is represented by the infinite cycle $x \in E^2_{s,t}$, and let $y \in \hat{E}^2_{s+1,t}$ be the unique element with $\partial^{h,2}(y) = x$. Then, if y is an infinite cycle, there exists a homotopy class $\tilde{y} \in \hat{\mathbb{H}}^{-q-1}(C_{2^{n-1}}, T(\mathbb{Z}); \mathbb{F}_2)$ represented by y such that $\partial^h(\tilde{y}) = \tilde{x}$; compare [5, Theorem 2.5]. We now return to [27, Theorem 8.14]. The homotopy class $V^{n-2}(\kappa)$ is represented by the unique generator of $E^2_{2,3}$ which, in turn, is the image by the map $\partial^{h,2}$ of the unique generator of $\hat{E}^{h,2}_{3,3}$. In loc. cit., the latter generator is given the name $u_{n-1}t^{-2}e_3$ and proved to be an infinite cycle for $n \geqslant 3$. This shows that the image of the class $V^{n-2}(\kappa)$ by the norm map

$$N \colon \mathbb{H}_5(C_{2^{n-1}}, T(\mathbb{Z}); \mathbb{F}_2) \to \mathrm{TR}^n_5(\mathbb{Z}; 2, \mathbb{F}_2)$$

is zero as stated. We conclude that $\kappa, \xi_{5,2}, \dots, \xi_{5,n-1}$ generate $\mathrm{TR}^n_5(\mathbb{Z}; 2)$.

We know from Theorem 11 that $\xi_{5,s}$ is annihilated by 2^s and further claim that κ is annihilated by 2^{n-1} and that, for some unit $u \in \mathbb{Z}_2^*$,

$$2 \cdot (\xi_{5,2} + \cdots + \xi_{5,n-1} + 4u\kappa) = 0.$$

This implies the statement of the theorem for $q = 5$. Indeed, the abelian group generated by $\kappa, \xi_{5,2}, \dots, \xi_{5,n-1}$ and subject to the relations above is equal to

$$\mathbb{Z}/2^{n-1}\mathbb{Z} \cdot \kappa \oplus \bigoplus_{2 \leqslant s < n} \mathbb{Z}/2^{s-1}\mathbb{Z} \cdot (\xi_{5,s} + \cdots + \xi_{5,n-1} + 4u\kappa)$$

and subjects onto $\mathrm{TR}^n_5(\mathbb{Z}; 2)$. Hence, it suffices to show that the two groups have the same order. But this follows by an induction argument based on the exact sequence

$$0 \to \mathbb{H}_5(C_{2^{n-1}}, T(\mathbb{Z})) \to \mathrm{TR}_5^n(\mathbb{Z};2) \to \mathrm{TR}_5^{n-1}(\mathbb{Z};2) \to 0$$

and Proposition 15.

It remains to prove the claim. We first show that the class $2^{n-1} \cdot \kappa$ is zero by induction on $n \geqslant 2$. The case $n = 2$ is true, so we assume the statement for $n-1$ and prove it for n. We again use the exact sequence

$$0 \to \mathbb{H}_5(C_{2^{n-1}}, T(\mathbb{Z})) \to \mathrm{TR}_5^n(\mathbb{Z};2) \to \mathrm{TR}_5^{n-1}(\mathbb{Z};2) \to 0$$

and the calculation of the left-hand group in Proposition 15. The inductive hypothesis implies that the image of the class $2^{n-2} \cdot \kappa$ by the right-hand map is zero, and hence, this class is in the image of the left-hand map. It follows that we can write

$$2^{n-2} \cdot \kappa = a \cdot \xi_{5,n-1} + b \cdot V^{n-2}(\kappa)$$

with $a \in \mathbb{Z}/2^{n-2}\mathbb{Z}$ and $b \in \mathbb{Z}/2\mathbb{Z}$. We apply the Frobenius map

$$F\colon \mathrm{TR}_5^n(\mathbb{Z};2) \to \mathrm{TR}_5^{n-1}(\mathbb{Z};2)$$

to this equation. The image of the left-hand side is zero, by induction, and the image of the right-hand side is $\bar{a} \cdot \xi_{5,n-2}$, where $\bar{a} \in \mathbb{Z}/2^{n-3}\mathbb{Z}$ is reduction of a modulo 2^{n-3}. It follows that $\bar{a}$ is zero, or equivalently, that $a \in 2^{n-3}\mathbb{Z}/2^{n-2}\mathbb{Z}$. This shows that $2^{n-1} \cdot \kappa$ is zero as desired.

Finally, to prove the relation $2 \cdot (\xi_{5,2} + \dots \xi_{5,n-1} + 4u\kappa) = 0$, we consider the following long-exact sequence

$$\cdots \to \mathrm{TR}_6(\mathbb{S};2) \xrightarrow{1-F} \mathrm{TR}_6(\mathbb{S};2) \xrightarrow{\partial} K_5(\mathbb{S}_2;\mathbb{Z}_2) \xrightarrow{\mathrm{tr}} \mathrm{TR}_5(\mathbb{S};2) \xrightarrow{1-F} \mathrm{TR}_5(\mathbb{S};2) \to \cdots$$

We know from Lemma 17 that the group $K_5(\mathbb{S}_2;\mathbb{Z}_2)$ is a free $\mathbb{Z}_2$-module of rank one generated by the class $4\kappa + \tau$. Moreover, it follows from Theorem 11 that the left-hand map $1-F$ is surjective and that the kernel of the right-hand map $1-F$ is isomorphic to a free $\mathbb{Z}_2$-module of rank one generated by the element $\Delta = (\Delta^{(n)})$ with $\Delta^{(n)} = \xi_{5,2} + \cdots + \xi_{5,n-1}$. It follows that there exists a unit $u \in \mathbb{Z}_2^*$ such that $\Delta + u(4\kappa + \tau) = 0$ in $\mathrm{TR}_5(\mathbb{S};2)$. But then $2(\Delta + 4u\kappa) = 0$, since $2(4\kappa + \tau) = 8\kappa$. This completes the proof. □

Corollary 19. *The cokernel of the map induced by the Hurewicz map*

$$\ell\colon \mathrm{TR}_5^n(\mathbb{S};2) \to \mathrm{TR}_5^n(\mathbb{Z};2)$$

is equal to $\mathbb{Z}/2^v\mathbb{Z} \cdot \kappa$, *where* $v = v(n-1)$ *is the smaller of* 3 *and* $n-1$.

Proof. It follows immediately from Theorem 18 that the cokernel of the map ℓ is generated by the class of κ. Moreover, since the class

$$\xi_{5,1} + \cdots + \xi_{5,n-1} + 4u\kappa$$

has order 2, it is also clear that the cokernel of the map ℓ is annihilated by multiplication by 8. Hence, it will suffice to show that, for $n = 4$, the cokernel of the map ℓ is not annihilated by 4, or equivalently, that the map ℓ takes the class $\rho \in \mathrm{TR}_5^4(\mathbb{S}; 2)$ to zero. But this follows immediately from the structure of the spectral sequences that abuts $\mathbb{H}_5(C_8, T(\mathbb{S}))$ and $\mathbb{H}_5(C_8, T(\mathbb{Z}))$. $\square$

7 The Groups $\mathrm{TR}_q^n(\mathbb{S}, I; 2)$

We again implicitly consider homotopy groups with $\mathbb{Z}_2$-coefficients. The Hurewicz map from the sphere spectrum $\mathbb{S}$ to the Eilenberg MacLane spectrum $\mathbb{Z}$ for the ring of integers induces a map of topological Hochschild $\mathbb{T}$-spectra

$$\ell \colon T(\mathbb{S}) \to T(\mathbb{Z}).$$

In [5, Appendix], Bökstedt and Madsen constructs a sequence of cyclotomic spectra

$$T(\mathbb{S}, I) \xrightarrow{i} T(\mathbb{S}) \xrightarrow{\ell} T(\mathbb{Z}) \xrightarrow{\partial} \Sigma T(\mathbb{S}, I)$$

such that the underlying sequence of $\mathbb{T}$-spectra is a cofibration sequence. As a consequence, the equivariant homotopy groups

$$\mathrm{TR}_q^n(\mathbb{S}, I; p) = [S^q \wedge (\mathbb{T}/C_{p^{n-1}})_+, T(\mathbb{S}, I)]_{\mathbb{T}}$$

come equipped with maps

$$\begin{aligned}
R &\colon \mathrm{TR}_q^n(\mathbb{S}, I; p) \to \mathrm{TR}_q^{n-1}(\mathbb{S}, I; p) && \text{(restriction)}\\
F &\colon \mathrm{TR}_q^n(\mathbb{S}, I; p) \to \mathrm{TR}_q^{n-1}(\mathbb{S}, I; p) && \text{(Frobenius)}\\
V &\colon \mathrm{TR}_q^{n-1}(\mathbb{S}, I; p) \to \mathrm{TR}_q^n(\mathbb{S}, I; p) && \text{(Verschiebung)}\\
d &\colon \mathrm{TR}_q^n(\mathbb{S}, I; p) \to \mathrm{TR}_{q+1}^n(\mathbb{S}, I; p) && \text{(Connes' operator)}
\end{aligned}$$

and all maps in the long-exact sequence of equivariant homotopy groups induced by the cofibration sequence above,

$$\cdots \to \mathrm{TR}_q^n(\mathbb{S}, I; 2) \xrightarrow{i} \mathrm{TR}_q^n(\mathbb{S}; 2) \xrightarrow{\ell} \mathrm{TR}_q^n(\mathbb{Z}; 2) \xrightarrow{\partial} \mathrm{TR}_{q-1}^n(\mathbb{S}, I; 2) \to \cdots,$$

are compatible with restriction maps, Frobenius maps, Verschiebung maps, and Connes' operator. Moreover, this is a sequence of graded modules over the graded ring $\mathrm{TR}_*^n(\mathbb{S}; p)$.

Lemma 20. *The following sequence is exact, for all $n \geqslant 1$.*

$$0 \to \mathbb{H}_3(C_{2^{n-1}}, T(\mathbb{S}, I)) \xrightarrow{N} \mathrm{TR}_3^n(\mathbb{S}, I; 2) \xrightarrow{R} \mathrm{TR}_3^{n-1}(\mathbb{S}, I; 2) \to 0.$$

Proof. From the proof of Corollary 16 we have a map of short-exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{H}_3(C_{2^{n-1}}, T(\mathbb{S})) \arrow[r] \arrow[d] & \mathrm{TR}_3^n(\mathbb{S}; 2) \arrow[r] \arrow[d] & \mathrm{TR}_3^{n-1}(\mathbb{S}; 2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{H}_3(C_{2^{n-1}}, T(\mathbb{Z})) \arrow[r] & \mathrm{TR}_3^n(\mathbb{Z}; 2) \arrow[r] & \mathrm{TR}_3^{n-1}(\mathbb{Z}; 2) \arrow[r] & 0
\end{tikzcd}$$

and that the left-hand vertical map is surjective. Moreover, Lemma 13 and Proposition 15 identify the sequence of the statement with the sequence of kernels of the vertical maps in this diagram. This completes the proof. □

Corollary 21. *The restriction map*

$$R\colon \mathrm{TR}_q^n(\mathbb{S}, I; 2) \to \mathrm{TR}_q^{n-1}(\mathbb{S}, I; 2)$$

is surjective, for all $q \leqslant 4$ *and all* $n \geqslant 1$.

We recall that for $n = 1$, the map ℓ is an isomorphism, if $q = 0$, and the zero map, if $q > 0$. It follows that the groups $\mathrm{TR}_0^1(\mathbb{S}, I; 2)$, $\mathrm{TR}_4^1(\mathbb{S}, I; 2)$, and $\mathrm{TR}_5^1(\mathbb{S}, I; 2)$ are zero, that $\mathrm{TR}_1^1(\mathbb{S}, I; 2)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ generated by the unique class $\tilde{\eta}$ with $i(\tilde{\eta}) = \eta$, that $\mathrm{TR}_2^1(\mathbb{S}, I; 2)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ generated by $\eta\tilde{\eta}$ and by the class $\bar{\lambda} = \partial(\lambda)$, and that $\mathrm{TR}_3^1(\mathbb{S}, I; 2)$ is isomorphic to $\mathbb{Z}/8\mathbb{Z}$ generated by the unique class $\tilde{\nu}$ with $i(\tilde{\nu}) = \nu$. We note that $\eta\bar{\lambda} = 0$, since $\mathrm{TR}_4^1(\mathbb{Z}; 2)$ is zero, while $\eta^2\tilde{\eta} = 4\tilde{\nu}$. We consider the skeleton spectral sequences

$$E_{s,t}^2 = H_s(C_{2^{n-1}}, \mathrm{TR}_t^1(\mathbb{S}, I; 2)) \Rightarrow \mathbb{H}_{s+t}(C_{2^{n-1}}, T(\mathbb{S}, I)).$$

In the case $n = 2$, the E^2-term for $s + t \leqslant 5$ takes the form

$$\begin{array}{cccccc}
0 & & & & & \\
0 & 0 & & & & \\
\mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 4\mathbb{Z}/8\mathbb{Z} & & & \\
(\mathbb{Z}/2\mathbb{Z})^2 & (\mathbb{Z}/2\mathbb{Z})^2 & (\mathbb{Z}/2\mathbb{Z})^2 & (\mathbb{Z}/2\mathbb{Z})^2 & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \\
0 & 0 & 0 & 0 & 0 & 0.
\end{array}$$

The group $E_{s,1}^2$ is generated by the class $\tilde{\eta}z_s$, the group $E_{s,2}^2$ by the classes $\eta\tilde{\eta}z_s$ and $\bar{\lambda}z_s$, the group $E_{s,3}^2$ with $s = 0$ or s an odd positive integer by the class νz_s, and the group $E_{s,3}^2$ with s an even positive integer by the class $4\nu z_s$. We claim that $d^2(\tilde{\eta}z_2) = \bar{\lambda}z_0$, or equivalently, that Connes' operator maps

$$d\tilde{\eta} = \bar{\lambda}.$$

We show that the class $V(\bar{\lambda}) \in \mathbb{H}_2(C_2, T(\mathbb{S}, I))$ represented by $\bar{\lambda}z_0$ is zero. By Lemma 20, we may instead show that the image $V(\bar{\lambda}) \in \mathrm{TR}_2^2(\mathbb{S}, I; 2)$ by the norm map is zero. Now, $V(\bar{\lambda}) = V(\partial(\lambda)) = \partial(V(\lambda))$, and by Proposition 18, the class $V(\lambda) \in \mathrm{TR}^2_3(\mathbb{Z};2)$ is either zero or equal to 4λ. But $\partial(4\lambda) = 4\partial(\lambda) = 4\bar{\lambda}$ which is zero, by Corollary 16. This proves the claim. The d^2-differential is now given by Lemma 5. We find that the E^3-term for $s+t \leqslant 5$ takes the form

$$
\begin{array}{cccccc}
0 & & & & & \\
0 & 0 & & & & \\
\mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

and, for degree reasons, this is also the E^∞-term. The group $E^3_{s,2}$ is generated by the class of $\eta\tilde{\eta}z_s$. The class of $\bar{\lambda}z_s$ in $E^3_{s,2}$ is equal to zero, if s is congruent to 0 or 1 modulo 4, and is equal to the class of $\eta\tilde{\eta}z_s$, if s is congruent to 2 or 3 modulo 4.

The spectral sequences for $n \geqslant 3$ are similar with the only difference being the groups $E^r_{s,3}$ with $s > 0$. In the case $n = 3$, the E^∞-term for $s+t \leqslant 5$ takes the form

$$
\begin{array}{cccccc}
0 & & & & & \\
0 & 0 & & & & \\
\mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/4\mathbb{Z} & 2\mathbb{Z}/4\mathbb{Z} & & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

and in the case $n \geqslant 4$, it takes the form

$$
\begin{array}{cccccc}
0 & & & & & \\
0 & 0 & & & & \\
\mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/4\mathbb{Z} & & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0.
\end{array}
$$

We define $\epsilon \in \mathbb{H}_5(C_4, T(\mathbb{S}, I))$ and $\tilde{\rho} \in \mathbb{H}_5(C_8, T(\mathbb{S}, I))$ to be the unique homotopy classes that represent $2\tilde{\nu}z_2$ and $\tilde{\nu}z_2$, respectively. We note that $V(\epsilon) = 2\tilde{\rho}$. We further define $\bar{\kappa} = \partial(\kappa) \in \mathbb{H}_4(C_2, T(\mathbb{S}, I))$.

Proposition 22. *The groups $\mathbb{H}_q(C_{2^{n-1}}, T(\mathbb{S}, I))$ with $q \leqslant 5$ are given by*

$$
\begin{aligned}
\mathbb{H}_0(C_{2^{n-1}}, T(\mathbb{S}, I)) &= 0,\\
\mathbb{H}_1(C_{2^{n-1}}, T(\mathbb{S}, I)) &= \mathbb{Z}/2\mathbb{Z} \cdot V^{n-1}(\tilde{\eta}),\\
\mathbb{H}_2(C_{2^{n-1}}, T(\mathbb{S}, I)) &= \mathbb{Z}/2\mathbb{Z} \cdot dV^{n-1}(\tilde{\eta}) \oplus \mathbb{Z}/2\mathbb{Z} \cdot V^{n-1}(\eta\tilde{\eta}),\\
\mathbb{H}_3(C_{2^{n-1}}, T(\mathbb{S}, I)) &= \mathbb{Z}/2\mathbb{Z} \cdot dV^{n-1}(\eta\tilde{\eta}) \oplus \mathbb{Z}/8\mathbb{Z} \cdot V^{n-1}(\tilde{\nu}),\\
\mathbb{H}_4(C_{2^{n-1}}, T(\mathbb{S}, I)) &= \begin{cases} 0 & (n=1) \\ \mathbb{Z}/2^v\mathbb{Z} \cdot dV^{n-1}(\tilde{\nu}) \oplus \mathbb{Z}/2\mathbb{Z} \cdot V^{n-2}(\bar{\kappa}) & (n \geqslant 2), \end{cases}
\end{aligned}
$$

$$\mathbb{H}_5(C_{2^{n-1}}, T(\mathbb{S}, I)) = \begin{cases} 0 & (n = 1) \\ \mathbb{Z}/2\mathbb{Z} \cdot d\bar{\kappa} & (n = 2) \\ \mathbb{Z}/2\mathbb{Z} \cdot dV(\bar{\kappa}) \oplus \mathbb{Z}/2\mathbb{Z} \cdot \epsilon & (n = 3) \\ \mathbb{Z}/2\mathbb{Z} \cdot dV^{n-2}(\bar{\kappa}) \oplus \mathbb{Z}/4\mathbb{Z} \cdot V^{n-4}(\tilde{\rho}) & (n \geqslant 4), \end{cases}$$

where $v = v(n-1)$ is the smaller of 3 *and $n - 1$.*

Proof. The statement for $q \leqslant 3$ follows immediately from the spectral sequence above since the generators given in the statement have the indicated orders. To prove the statement for $q = 4$, we first note that $dV^{n-1}(\tilde{\nu})$ has order $v(n-1)$. Indeed, the class $\tilde{\nu}$ has order 8 and $d\tilde{\nu} = 0$. Moreover, the image of the map

$$\ell \colon \mathbb{H}_5(C_{2^{n-1}}, T(\mathbb{S})) \to \mathbb{H}_5(C_{2^{n-1}}, T(\mathbb{Z}))$$

does not contain the class $V^{n-2}(\kappa)$. Indeed, this follows immediately from the induced map of spectral sequences. It follows that $V^{n-2}(\bar{\kappa})$ is a non-zero class of order 2 which is represented by the element $\eta\tilde{\eta}z_2 = \bar{\lambda}z_2$ in the E^{∞}-term of the spectral sequence above. This proves the statement for $q = 4$. It remains to prove the statement for $q = 5$. It follows from [18, Proposition 4.4.1] that the element $\eta\tilde{\eta}z_3 = \bar{\lambda}z_3$ in the E^{∞}-term of the spectral sequence above represents the class $dV^{n-2}(\bar{\kappa})$. Hence, this class is non-zero and has order 2. Moreover, the spectral sequence shows that the subgroup of $\mathbb{H}_5(C_{2^{n-1}}, T(\mathbb{S}, I))$ generated by $dV^{n-2}(\bar{\kappa})$ is a direct summand. This completes the proof. □

The following result was proved by Costeanu in [7, Proposition 2.6].

Lemma 23. *The map*

$$\ell \colon \mathrm{TR}_1^n(\mathbb{S}; 2) \to \mathrm{TR}_1^n(\mathbb{Z}; 2)$$

takes the class $\eta = \eta \cdot [1]_n$ to the class $\xi_{1,1}$.

Proof. We temporarily write $[1_{\mathbb{S}}]_n$ and $[1_{\mathbb{Z}}]_n$ for the multiplicative unit elements of the graded rings $\mathrm{TR}_*^n(\mathbb{S}; 2)$ and $\mathrm{TR}_*^n(\mathbb{Z}; 2)$, respectively. By [17, Proposition 2.7.1], the map ℓ is a map of graded algebras over the graded ring given by the stable homotopy groups of spheres. Hence, it takes the class $\eta \cdot [1_{\mathbb{S}}]_n$ to the class $\eta \cdot [1_{\mathbb{Z}}]_n$. Similarly, it is proved in [12, Corollary 6.4.1] that the cyclotomic trace map

$$\mathrm{tr} \colon K_*(\mathbb{Z}) \to \mathrm{TR}_*^n(\mathbb{Z}; 2)$$

is a map of graded algebras over the graded ring given by the stable homotopy groups of spheres. Hence, the class $\eta \cdot [1_{\mathbb{Z}}]_n$ is equal to the image by the cyclotomic trace map of the class $\eta \cdot 1_{\mathbb{Z}} \in K_1(\mathbb{Z})$. The latter class is known to be equal to the generator $\{-1\} \in K_1(\mathbb{Z})$. It is proved in [18, Lemma 2.3.3] that the image by the cyclotomic trace map of the generator $\{-1\}$ is equal to the class

$$d\log[-1]_n \in \mathrm{TR}_1^n(\mathbb{Z}; 2).$$

To evaluate this class, we recall from [17, Theorem F] that the ring $\mathrm{TR}_0^n(\mathbb{Z};2)$ is canonically isomorphic to the ring of Witt vectors $W_n(\mathbb{Z})$. One readily verifies that

$$[-1]_n = -[1]_n + V([1]_{n-1})$$

by evaluating the ghost coordinates. It follows that $d[-1]_n = dV([1]_{n-1})$, and since the class $[-1]_n$ is a square root of 1, we find

$$\begin{aligned} d\log[-1]_n &= [-1]_n d[-1]_n = (-[1]_n + V([1]_{n-1})) \cdot dV([1]_{n-1}) \\ &= dV([1]_{n-1}) + V(FdV([1]_{n-1})) = dV([1]_{n-1}) + V(\eta \cdot [1]_{n-1}). \end{aligned}$$

But by Theorem 11, this is the class $\xi_{1,1}$ as stated. □

Remark 24. It follows from Lemma 23 that $\ell(V^s(\eta)) = \sum_{t \geqslant s} 2^{t-1}\xi_{1,t}$, if $s \geqslant 2$.

At present, we do not know the precise value of the map

$$\ell\colon \mathrm{TR}_q^n(\mathbb{S};2) \to \mathrm{TR}_q^n(\mathbb{Z};2)$$

for $q \geqslant 3$. However, we have the following result. We define $\tilde{\eta} \in \mathrm{TR}_1^n(\mathbb{S}, I;2)$ to be the unique class such that $i(\tilde{\eta}) = \eta - \xi_{1,1}$. The class $\tilde{\nu}$ that appears in the statement will be defined in the course of the proof. It would be desirable to better understand this class. In particular, we do not know the values of $\eta^2\tilde{\eta}$ or $Fd\tilde{\nu}$.

Theorem 25. *The groups* $\mathrm{TR}_q^n(\mathbb{S}, I;2)$ *with* $q \leqslant 3$ *are given by*

$$\begin{aligned} \mathrm{TR}_0^n(\mathbb{S}, I;2) &= 0, \\ \mathrm{TR}_1^n(\mathbb{S}, I;2) &= \bigoplus_{0 \leqslant s < n} \mathbb{Z}/2\mathbb{Z} \cdot V^s(\tilde{\eta}), \\ \mathrm{TR}_2^n(\mathbb{S}, I;2) &= \bigoplus_{0 \leqslant s < n} \big(\mathbb{Z}/2\mathbb{Z} \cdot V^s(\eta\tilde{\eta}) \oplus \mathbb{Z}/2\mathbb{Z} \cdot dV^s(\tilde{\eta})\big), \\ \mathrm{TR}_3^n(\mathbb{S}, I;2) &= \bigoplus_{0 \leqslant s < n} \mathbb{Z}/8\mathbb{Z} \cdot V^s(\tilde{\nu}) \oplus \bigoplus_{1 \leqslant s < n} \mathbb{Z}/2\mathbb{Z} \cdot dV^s(\eta\tilde{\eta}), \end{aligned}$$

and the group $\mathrm{TR}_4^n(\mathbb{S}, I;2)$ *is generated by* $dV^s(\tilde{\nu})$ *with* $0 \leqslant s < n$. *Moreover, the restriction map takes* $\tilde{\eta}$ *to* $\tilde{\eta}$ *and* $\tilde{\nu}$ *to* $\tilde{\nu}$, *and the Frobenius map takes both* $\tilde{\eta}$ *and* $\tilde{\nu}$ *to zero and takes* $d\tilde{\eta}$ *to* $d\tilde{\eta}$. *The class* $d(\eta\tilde{\eta}) = \eta d(\tilde{\eta})$ *is zero.*

Proof. The statement for $q = 0$ follows immediately from Theorem 11 and Proposition 18. For $q = 1$, Lemma 13 shows that the map $i\colon \mathrm{TR}_1^n(\mathbb{S}, I;2) \to \mathrm{TR}_1^n(\mathbb{S};2)$ is injective, and Lemma 23 shows that the class $\eta - \xi_{1,1}$ is in the image. As said above, we define $\tilde{\eta} \in \mathrm{TR}_1^n(\mathbb{S}, I;2)$ to be the unique class with $i(\tilde{\eta}) = \eta - \xi_{1,1}$. The statement for $q = 1$ now follows immediately from Theorem 11 and Proposition 18. For $q = 2$, a similar argument shows that the group $\mathrm{TR}_2^n(\mathbb{S}, I;2)$ contains the subgroup

$$\mathrm{TR}_2^n(\mathbb{S}, I; 2)' = \bigoplus_{0 \leqslant s < n} \mathbb{Z}/2\mathbb{Z} \cdot V^s(\eta\tilde{\eta}) \oplus \bigoplus_{1 \leqslant s < n} \mathbb{Z}/2\mathbb{Z} \cdot dV^s(\tilde{\eta}),$$

which maps isomorphically onto the image of $i\colon \mathrm{TR}_2^n(\mathbb{S}, I; 2) \to \mathrm{TR}_2^n(\mathbb{S}; 2)$, and Lemma 16 shows that the kernel of the latter map is $\mathbb{Z}/2\mathbb{Z} \cdot \bar{\lambda}$. Therefore, to prove the statement for $q = 2$, it remains to prove that $d\tilde{\eta} = \bar{\lambda}$. We have already proved this equality, for $n = 1$, in the discussion preceeding Proposition 22. It follows that the iterated restriction map $R^{n-1}\colon \mathrm{TR}_2^n(\mathbb{S}, I; 2) \to \mathrm{TR}_2^1(\mathbb{S}, I; 2)$ takes the class $d\tilde{\eta}$ to the class $\bar{\lambda}$. Since the kernel of this map is equal to the subgroup $\mathrm{TR}_2^n(\mathbb{S}, I; 2)'$, it suffices to show that the class $i(d\tilde{\eta} - \bar{\lambda}) \in \mathrm{TR}_2^n(\mathbb{S}; 2)$ is zero. We have

$$i(d\tilde{\eta} - \bar{\lambda}) = i(d\tilde{\eta}) = d(i(\tilde{\eta})) = d\eta - d\xi_{1,1}.$$

The class $d\eta$ is zero, since η is in the image of the cyclotomic trace map, and we proved in Theorem 11 that $d\xi_{1,1}$ is zero. Th statement for $q = 2$ follows. It also follows that $F(d\tilde{\eta}) = d\tilde{\eta}$, since $d\tilde{\eta} = \partial(\lambda)$ and λ is in the image of the cyclotomic trace map.

We next prove the statement for $q = 3$. By Lemma 20, the sequences

$$0 \to \mathbb{H}_3(C_{2^{n-1}}, T(\mathbb{S}, I)) \xrightarrow{N} \mathrm{TR}_3^n(\mathbb{S}, I; 2) \xrightarrow{R} \mathrm{TR}_3^{n-1}(\mathbb{S}, I; 2) \to 0$$

are exact. The left-hand group was evaluated in Proposition 22. To complete the proof, we inductively construct classes

$$\tilde{\nu} = \tilde{\nu}_n \in \mathrm{TR}_3^n(\mathbb{S}, I; 2) \qquad (n \geqslant 1)$$

such that $R(\tilde{\nu}_n) = \tilde{\nu}_{n-1}$ and $F(\tilde{\nu}_n) = 0$, and such that $\tilde{\nu}_1$ is the class $\tilde{\nu}$ already defined. By Proposition 13 and Corollary 16, we have a short-exact sequence

$$0 \to \mathrm{TR}_3^n(\mathbb{S}, I; 2) \xrightarrow{i} \mathrm{TR}_3^n(\mathbb{S}; 2) \xrightarrow{\ell} \mathrm{TR}_3^n(\mathbb{Z}; 2)' \to 0,$$

where the right-hand group is the index two subgroup of $\mathrm{TR}_3^n(\mathbb{Z}; 2)$ defined by

$$\mathrm{TR}_3^n(\mathbb{Z}; 2)' = \bigoplus_{1 \leqslant s < n} \mathbb{Z}/2^{s+1}\mathbb{Z} \cdot \xi_{3,s}.$$

To define the class $\tilde{\nu}_2$, we first note that $\ell(\nu) = a_1\xi_{3,1}$, where $a_1 \in (\mathbb{Z}/4\mathbb{Z})^*$ is a unit, and choose a unit $\tilde{a}_1 \in (\mathbb{Z}/8\mathbb{Z})^*$ whose reduction modulo 4 is a_1. Then, we have $\ell(\nu - \tilde{a}_1\xi_{3,1}) = 0$ and $F(\nu - \tilde{a}_1\xi_{3,1}) = (1 - \tilde{a}_1)\nu$. We choose $b_1 \in \mathbb{Z}/8\mathbb{Z}$ such that $2b_1 = \tilde{a}_1 - 1$ and define $\tilde{\nu}_2$ to be the unique class such that

$$i(\tilde{\nu}_2) = \nu - \tilde{a}_1\xi_{3,1} + b_1V(\nu).$$

Then $R(\tilde{\nu}_2) = \tilde{\nu}_1$ and $F(\tilde{\nu}_2) = 0$ as desired.

We next define the class $\tilde{\nu}_3$. The image of $\tilde{\nu}_2$ by the composition

$$\mathrm{TR}^2_3(\mathbb{S}, I; 2) \xrightarrow{i} \mathrm{TR}^2_3(\mathbb{S}; 2) \xrightarrow{S} \mathrm{TR}^3_3(\mathbb{S}; 2) \xrightarrow{\ell} \mathrm{TR}^3_3(\mathbb{Z}; 2)'$$

is equal to $a_2\xi_{3,2}$, for some $a_2 \in \mathbb{Z}/8\mathbb{Z}$. We claim that, in fact, $a_2 \in 4\mathbb{Z}/8\mathbb{Z}$. Indeed, since $F(\tilde{\nu}_2) = 0$, we have $F(a_2\xi_{3,2}) = 0$. But $F(\xi_{3,2}) = \xi_{3,1}$ which shows that the modulo 4 reduction of a_2 is zero as claimed. We let $b_2 \in \mathbb{Z}/2\mathbb{Z}$ be the unique element such that $4b_2 = a_2$ and define $\tilde{\nu}_3$ to be the unique class such that

$$i(\tilde{\nu}_3) = \begin{cases} S(i(\tilde{\nu}_2)) + b_2(4\xi_{3,2} + dV^2(\eta^2) + 2V^2(\nu)) & \text{if } 4\xi_{3,1} = dV(\eta^2) \\ S(i(\tilde{\nu}_2)) + b_2(4\xi_{3,2} + dV^2(\eta^2)) & \text{if } 4\xi_{3,1} = \eta^2\xi_{1,1}. \end{cases}$$

The sum on the right-hand side is in the kernel of ℓ, since both $\ell(\eta^2) \in \mathrm{TR}^1_2(\mathbb{Z}; 2)$ and $\ell(\nu) \in \mathrm{TR}^1_3(\mathbb{Z}; 2)$ are zero. We also have $R(\tilde{\nu}_3) = \tilde{\nu}_2$ and $F(\tilde{\nu}_3) = 0$ as desired. Indeed, if $4\xi_{3,1} = dV(\eta^2)$, then

$$\begin{aligned} i(F(\tilde{\nu}_3)) &= F(S(i(\tilde{\nu}_2)) + b_2(4\xi_{3,2} + dV^2(\eta^2) + 2V^2(\nu))) \\ &= S(i(F(\tilde{\nu}_2))) + b_2(4\xi_{3,1} + dV(\eta^2) + V(\eta^3) + 4V(\nu)), \end{aligned}$$

and if $4\xi_{3,1} = \eta^2\xi_{3,1}$, then

$$\begin{aligned} i(F(\tilde{\nu}_3)) &= F(S(i(\tilde{\nu}_2)) + b_2(4\xi_{3,2} + dV^2(\eta^2))) \\ &= S(i(F(\tilde{\nu}_2))) + b_2(4\xi_{3,1} + dV(\eta^2) + V(\eta^3)), \end{aligned}$$

and in either case, the sum is zero.

Finally, we let $n \geqslant 4$ and assume that the class $\tilde{\nu}_{n-1}$ has been defined. We find as before that the image of the class $\tilde{\nu}_{n-1}$ by the composition

$$\mathrm{TR}^{n-1}_3(\mathbb{S}, I; 2) \xrightarrow{i} \mathrm{TR}^{n-1}_3(\mathbb{S}; 2) \xrightarrow{S} \mathrm{TR}^n_3(\mathbb{S}; 2) \xrightarrow{\ell} \mathrm{TR}^n_3(\mathbb{Z}; 2)'$$

is equal to $a_{n-1}\xi_{3,n-1}$ with $a_{n-1} \in 2^{n-1}\mathbb{Z}/2^n\mathbb{Z}$ and define $\tilde{\nu}_n$ to be the unique class whose image by the map i is equal to

$$i(\tilde{\nu}_n) = S(i(\tilde{\nu}_{n-1})) - a_{n-1}\xi_{3,n-1}.$$

Then $R(\tilde{\nu}_n) = \tilde{\nu}_{n-1}$ and $F(\tilde{\nu}_n) = 0$, since $2^{n-1}\xi_{3,n-2} = 0$, for $n \geqslant 4$.

It remains to prove that the group $\mathrm{TR}^n_4(\mathbb{S}, I; 2)$ is generated by the homotopy classes $dV^s(\tilde{\nu})$ with $0 \leqslant s < n$. The sequence

$$\mathbb{H}_4(C_{2^{n-1}}, T(\mathbb{S}, I)) \to \mathrm{TR}^n_4(\mathbb{S}, I; 2) \to \mathrm{TR}^{n-1}_4(\mathbb{S}, I; 2) \to 0,$$

which is exact by Corollary 21, together with Proposition 22 show that $\mathrm{TR}^n_4(\mathbb{S}, I; 2)$ is generated by the classes $dV(\tilde{\nu})$, $1 \leqslant s < n$, and $\bar{\kappa}$. Indeed, since the boundary map

$$\partial\colon \mathrm{TR}^n_5(\mathbb{Z};2) \to \mathrm{TR}^n_4(\mathbb{S},I;2)$$

commutes with the Verschiebung, it follows that $V^{n-1}(\bar{\kappa}) = c\bar{\kappa}$, for some $c \in 2^{n-1}\mathbb{Z}$. Hence, it suffices to show that there exists a class $x_n \in \mathrm{TR}^n_3(\mathbb{S},I;2)$ with $dx_n = \bar{\kappa}$. The statement for $n = 1$ is trivial, since the group $\mathrm{TR}^1_4(\mathbb{S},I;2)$ is zero. We postpone the proof of the statement for $n = 2$ to Lemma 26 and here prove the induction step. So we let $n \geqslant 3$ and assume that there exists a class $x_{n-1} \in \mathrm{TR}^{n-1}_3(\mathbb{S},I;2)$ with $dx_{n-1} = \bar{\kappa}$. We use Corollary 21 to choose a class $x'_n \in \mathrm{TR}^n_3(\mathbb{S},I;2)$ with $R(x'_n) = x_{n-1}$. Then the exact sequence above and Proposition 22 show that

$$dx'_n = \bar{\kappa} + adV^{n-1}(\tilde{\nu}) + bV^{n-1}(\kappa) = adV^{n-1}(\tilde{\nu}) + (1+bc)\bar{\kappa},$$

for some integers a and b. Since $1 + bc$ is a 2-adic unit, the class

$$x_n = (1+bc)^{-1}(x'_n - aV^{n-1}(\tilde{\nu}))$$

is well-defined and satisfies $dx_n = \bar{\kappa}$ as desired. □

One wonders whether the class $\tilde{\nu}$, which was defined in the proof, satisfies that $d\tilde{\nu} = \bar{\kappa}$. This would imply that $Fd\tilde{\nu} = d\tilde{\nu}$, since κ is in the image of the cyclotomic trace map.

The following result was used in the proof of Theorem 25.

Lemma 26. *Connes' operator*

$$d\colon \mathrm{TR}^2_3(\mathbb{S},I;2) \to \mathrm{TR}^2_4(\mathbb{S},I;2)$$

is surjective.

Proof. The groups $\mathrm{TR}^2_q(\mathbb{S},I;2)$ for $q \leqslant 5$ are given by

$$\begin{aligned}
&\mathrm{TR}^2_0(\mathbb{S},I;2) = 0,\\
&\mathrm{TR}^2_1(\mathbb{S},I;2) = \mathbb{Z}/2\mathbb{Z}\cdot\tilde{\eta} \oplus \mathbb{Z}/2\mathbb{Z}\cdot V(\tilde{\eta}),\\
&\mathrm{TR}^2_2(\mathbb{S},I;2) = \mathbb{Z}/2\mathbb{Z}\cdot d\tilde{\eta} \oplus \mathbb{Z}/2\mathbb{Z}\cdot dV(\tilde{\eta}) \oplus \mathbb{Z}/2\mathbb{Z}\cdot \eta\tilde{\eta} \oplus \mathbb{Z}/2\mathbb{Z}\cdot V(\eta\tilde{\eta}),\\
&\mathrm{TR}^2_3(\mathbb{S},I;2) = \mathbb{Z}/2\mathbb{Z}\cdot dV(\eta\tilde{\eta}) \oplus \mathbb{Z}/8\mathbb{Z}\cdot\tilde{\nu} \oplus \mathbb{Z}/8\mathbb{Z}\cdot V(\tilde{\nu}),\\
&\mathrm{TR}^2_4(\mathbb{S},I;2) = \mathbb{Z}/2\mathbb{Z}\cdot\bar{\kappa} \oplus \mathbb{Z}/2\mathbb{Z}\cdot dV(\tilde{\nu}),\\
&\mathrm{TR}^2_5(\mathbb{S},I;2) = 0.
\end{aligned}$$

Hence, the lemma is equivalent to the statement that in the spectral sequence

$$E^2_{s,t} = H_s(C_2, \mathrm{TR}^2_t(\mathbb{S},I;2)) \Rightarrow \mathbb{H}_{s+t}(C_2, TR^2(\mathbb{S},I)),$$

the d^2-differential $d^2\colon E^2_{3,3} \to E^2_{1,4}$ is surjective. We first argue that this is equivalent to the statement that $\mathbb{H}_5(C_2, TR^2(\mathbb{S},I))$ has order 4. The elements $\bar{\kappa}z_0$ and $dV(\tilde{\nu})z_0$

in $E^2_{0,4}$ are infinite cycles and represent the homotopy classes $V^2(\bar{\kappa})$ and $2dV^2(\tilde{\nu})$ of $\mathbb{H}_4(C_2, TR^2(\mathbb{S}, I))$. We claim that these classes are non-zero and generate a subgroup of order 4. To see this, we consider the norm maps from Proposition 4,

$$\mathbb{H}_4(C_2, TR^2(\mathbb{S}, I)) \xrightarrow{N_2} \mathrm{TR}^3_4(\mathbb{S}, I; 2) \xleftarrow{N_1} \mathbb{H}_4(C_4, T(\mathbb{S}, I)).$$

It will suffice to show that the subgroup of the middle group generated by the images of the classes $V^2(\bar{\kappa})$ and $dV^2(\tilde{\nu})$ has order 4. This subgroup is equal to the subgroup generated by the images of the classes $V^2(\bar{\kappa})$ and $dV^2(\tilde{\nu})$ of the right-hand group. The right-hand map is injective, since $\mathrm{TR}^2_5(\mathbb{S}, I; 2)$ is zero. (The left-hand map is also injective, since $\mathrm{TR}^1_5(\mathbb{S}, I; 2)$ is zero.) Hence, it suffices to show that the subgroup of the right-hand group generated by the classes $V^2(\bar{\kappa})$ and $dV^2(\tilde{\nu})$ has order 4. But this is proved in Proposition 22. The claim follows. We conclude that in the spectral sequence under consideration, the differentials

$$d^r \colon E^r_{r,5-r} \to E^r_{0,4}$$

are zero, for all $r \geqslant 2$. It follows that the groups $E^\infty_{0,5}$, $E^\infty_{2,3}$, $E^\infty_{3,2}$, $E^\infty_{4,1}$, and $E^\infty_{5,0}$ have orders 0, 2, 2, 0, and 0, respectively, and that for all $r \geqslant 3$, the differentials

$$d^r \colon E^r_{r+1,4-r} \to E^r_{1,4}$$

are zero. We conclude that the differential $d^2 \colon E^2_{3,3} \to E^2_{1,4}$ is surjective if and only if the group $\mathbb{H}_5(C_2, TR^2(\mathbb{S}, I))$ has order 4.

The order of the group $\mathbb{H}_5(C_2, TR^2(\mathbb{S}, I)$ is divisible by 4 and to show that it is equal to 4, we consider the following diagram with exact rows and columns:

$$\begin{tikzcd}
\mathrm{TR}^1_7(\mathbb{S}; 2) \arrow[r, "0"] \arrow[d] & \mathbb{H}_6(C_2, TR^2(\mathbb{S}; 2)) \arrow[r] \arrow[d, "\ell"] & \mathrm{TR}^3_6(\mathbb{S}; 2) \arrow[d] \\
\mathrm{TR}^1_7(\mathbb{Z}; 2) \arrow[r, "\delta''"] \arrow[d] & \mathbb{H}_6(C_2, TR^2(\mathbb{Z}; 2)) \arrow[r] \arrow[d, "\partial"] & \mathrm{TR}^3_6(\mathbb{Z}; 2) \arrow[d] \\
\mathrm{TR}^1_6(\mathbb{S}, I; 2) \arrow[r, "\delta'"] \arrow[d] & \mathbb{H}_5(C_2, TR^2(\mathbb{S}, I)) \arrow[r] \arrow[d] & \mathrm{TR}^3_5(\mathbb{S}, I; 2) \arrow[d] \\
\mathrm{TR}^1_6(\mathbb{S}; 2) \arrow[r, "0"] & \mathbb{H}_5(C_2, TR^2(\mathbb{S}; 2)) \arrow[r] & \mathrm{TR}^3_5(\mathbb{S}; 2).
\end{tikzcd}$$

It follows from Theorems 11 and 18 that the group $\mathrm{TR}^3_5(\mathbb{S}, I; 2)$ is equal to $\mathbb{Z}/2\mathbb{Z} \cdot 2\xi_{5,2}$. Hence, it will suffice to show that the image of the map δ' has order at most 2. Since the lower left-hand horizontal map in the diagram is zero, we conclude that the image of the map δ' is contained in the image of the map ∂. Therefore, it suffices to show that the image of the map ∂ has order at most 2.

The group $\mathrm{TR}^3_6(\mathbb{Z};2)$ is zero by Proposition 13 and the group $\mathrm{TR}^1_7(\mathbb{Z};2)$ is cyclic of order 4. It follows that the group $\mathbb{H}_6(C_2, TR^2(\mathbb{Z};2))$ is cyclic and has order either 0, 2, or 4. If the order is either 0 or 2, we are done, so assume that the order is 4. We must show that 2 times a generator is contained in the image of the map ℓ in the diagram. To this end, we consider the diagram

$$\begin{tikzcd}
\mathbb{H}_6(C_4, TR^2(\mathbb{S};2)) \arrow[r, "F"] \arrow[d, "\ell"] & \mathbb{H}_6(C_2, TR^2(\mathbb{S};2) \arrow[d, "\ell"] \\
\mathbb{H}_6(C_4, TR^2(\mathbb{Z};2)) \arrow[r, "F"] & \mathbb{H}_6(C_2, TR^2(\mathbb{Z};2).
\end{tikzcd}$$

We first show that the lower horizontal map F is surjective. The assumption that the lower right-hand group has order 4 implies that a generator of this group is represented in the spectral sequence

$$E^2_{s,t} = H_s(C_2, \mathrm{TR}^2_t(\mathbb{Z};2)) \Rightarrow \mathbb{H}_{s+t}(C_2, TR^2(\mathbb{Z};2))$$

by the element $\lambda z_3 \in E^2_{3,3}$. This element is the image by the map of spectral sequences induced by the map F of the element $\lambda z_3 \in E^2_{3,3}$ in the spectral sequence

$$E^2_{s,t} = H_s(C_4, \mathrm{TR}^2_t(\mathbb{Z};2)) \Rightarrow \mathbb{H}_{s+t}(C_4, TR^2(\mathbb{Z};2)).$$

We must show that the latter element λz_3 is an infinite cycle. For degree reasons, the only possible non-zero differential is $d^3\colon E^3_{3,3} \to E^3_{0,5}$. The target group is equal to $\mathbb{Z}/2\mathbb{Z} \cdot \kappa z_0$, and the generator κz_0 represents the homotopy class $V^2(\kappa)$ in $\mathbb{H}_5(C_4, TR^2(\mathbb{Z};2))$. To see that this class is non-zero, we consider the norm maps

$$\mathbb{H}_4(C_4, TR^2(\mathbb{Z};2)) \xrightarrow{N_2} \mathrm{TR}^4_5(\mathbb{Z};2) \xleftarrow{N_1} \mathbb{H}_4(C_8, T(\mathbb{Z})).$$

We may instead prove that the image of the class $V^2(\kappa)$ by the left-hand map is non-zero. This image class, in turn, is equal to the image of the class $V^2(\kappa)$ by the right-hand map which is injective since $\mathrm{TR}^3_6(\mathbb{Z};2)$ is zero. Now, Proposition 15 shows that the class $V^2(\kappa)$ in the right-hand group is non-zero. We conclude that the lower horizontal map F in the square diagram is surjective as stated.

Finally, we show that the image of the left-hand vertical map ℓ in the square diagram contains two times the homotopy class represented by the element λz_3. In fact, the image of the composition

$$\mathbb{H}_5(C_4, T(\mathbb{S})) \xrightarrow{S} \mathbb{H}_5(C_4, TR^2(\mathbb{S};2)) \xrightarrow{\ell} \mathbb{H}_5(C_4, TR^2(\mathbb{Z};2))$$

of the Segal-tom Dieck splitting and the map ℓ contains 2 times the class represented by λz_3. Indeed, by Proposition 10, the element $\nu z_3 \in E^2_{3,3}$ of the spectral sequence

$$E^2_{s,t} = H_s(C_4, \mathrm{TR}^1_t(\mathbb{S};2)) \Rightarrow \mathbb{H}_{s+t}(C_4, T(\mathbb{S}))$$

is an infinite cycle whose image by the map of spectral sequence induced by the composition of the maps S and ℓ is equal $2\lambda z_3 \in E^2_{3,3} = \mathbb{Z}/4\mathbb{Z}\cdot\lambda z_3$. This completes the proof. $\square$

8 The Groups $\mathrm{Wh}^{\mathrm{Top}}_q(S^1)$ for $q \leqslant 3$

In this section, we complete the proof of Theorem 1 of the Introduction. It follows from [15, Theorem 1.2] that the odd-primary torsion subgroup of $\mathrm{Wh}^{\mathrm{Top}}_q(S^1)$ is zero, for $q \leqslant 3$. Hence, it suffices to consider the homotopy groups with $\mathbb{Z}_2$-coefficients. We implicitly consider homotopy groups with $\mathbb{Z}_2$-coefficients.

As we explained in the introduction, there is a long-exact sequence

$$\cdots \to \mathrm{Wh}^{\mathrm{Top}}_q(S^1) \to \widetilde{\mathrm{TR}}_q(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}];2) \xrightarrow{1-F} \widetilde{\mathrm{TR}}_q(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}];2) \to \cdots$$

where the middle and on the right-hand terms are the cokernel of the assembly map

$$\alpha\colon \mathrm{TR}_q(\mathbb{S};I;2) \oplus \mathrm{TR}_{q-1}(\mathbb{S};I;2) \to \mathrm{TR}_q(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}];2).$$

Moreover, since the groups $\mathrm{TR}^n_q(\mathbb{S}, I;2)$ are finite, for all integers q and $n \geqslant 1$, the limit system $\{\mathrm{TR}^n_q(\mathbb{S}, I;2)\}$ satisfies the Mittag-Leffler condition, and Corollary 3 then shows that the same holds for the limit system $\{\widetilde{\mathrm{TR}}^n_q(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}];2)\}$. It follows that, for all integers q, the canonical map

$$\widetilde{\mathrm{TR}}_q(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}];2) \to \lim_n \widetilde{\mathrm{TR}}^n_q(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}];2)$$

is an isomorphism. Finally, Theorem 2 expresses the right-hand side in terms of the groups $\mathrm{TR}^m_q(\mathbb{S}, I;2)$ which we evaluated in Theorem 25, for $q \leqslant 3$.

Theorem 27. *The groups* $\mathrm{Wh}^{\mathrm{Top}}_0(S^1)$ *and* $\mathrm{Wh}^{\mathrm{Top}}_1(S^1)$ *are zero.*

Proof. We first note that, as an immediate consequence of Theorems 2 and 25, the group $\widetilde{\mathrm{TR}}_0(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}];2)$ is zero. Moreover, we showed in Theorem 25 that the Frobenius map $F\colon \mathrm{TR}^m_1(\mathbb{S}, I;2) \to \mathrm{TR}^{m-1}_1(\mathbb{S}, I;2)$ is zero, and hence,

$$1 - F\colon \widetilde{\mathrm{TR}}_1(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}];2) \to \widetilde{\mathrm{TR}}_1(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}];2),$$

is the identity map. This shows that the group $\mathrm{Wh}_0^{\mathrm{Top}}(S^1)$ is zero as stated. To prove that $\mathrm{Wh}_1^{\mathrm{Top}}(S^1)$ is zero, it remains to prove that the map

$$1 - F \colon \widetilde{\mathrm{TR}}_2(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}]; 2) \to \widetilde{\mathrm{TR}}_2(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}]; 2),$$

is surjective. So let $\omega = (\omega^{(n)})$ be an element on the right-hand side. We find an element $\omega' = (\omega'^{(n)})$ such that $(R-F)(\omega') = \omega$. By Theorem 2, we can write $\omega^{(n)}$ uniquely as a sum

$$\sum_{j \in \mathbb{Z} \smallsetminus \{0\}} \left(a_{0,j}^{(n)}[x]_n^j + b_{0,j}^{(n)}[x]_n^j d\log[x]_n\right) + \sum_{\substack{1 \leqslant s < n \\ j \in \mathbb{Z} \smallsetminus 2\mathbb{Z}}} \left(V^s(a_{s,j}^{(n)}[x]_{n-s}^j) + dV^s(b_{s,j}^{(n)}[x]_{n-s}^j)\right)$$

with $a_{s,j}^{(n)} \in \mathrm{TR}_2^{n-s}(\mathbb{S}, I; 2)$ and $b_{s,j}^{(n)} \in \mathrm{TR}_1^{n-s}(\mathbb{S}, I; 2)$. We first consider the four types of summands separately.

First, if $\omega^{(n)} = V^s(a^{(n)}[x]^j)$ with $s \geqslant 1$, we let $\omega' = \omega$. Then

$$(R-F)(\omega'^{(n+1)}) = (R-F)(V^s(a^{(n+1)}[x]^j)) = V^s(a^{(n)}[x]^j),$$

since $FV = 2$ and $2a^{(n)} = 0$. We note that here j may be any integer.

Second, if $\omega^{(n)} = dV^s(b^{(n)}[x]^j)$, where j and $s \geqslant 1$ are integers, we define

$$\omega'^{(n)} = -\sum_{s \leqslant r < n-1} dV^{r+1}(b^{(n-1-r+s)}[x]^j) - \sum_{s \leqslant r < n} V^r(\eta b^{(n-r+s)}[x]^j).$$

Then we have $R(\omega'^{(n+1)}) = \omega'^{(n)}$ and

$$\begin{aligned}(R-F)(\omega'^{(n+1)}) &= -\sum_{s \leqslant r < n-1} dV^{r+1}(b^{(n-1-r+s)}[x]^j) - \sum_{s \leqslant r < n} V^r(\eta b^{(n-r+s)}[x]^j) \\ &\quad + \sum_{s \leqslant r < n} dV^r(b^{(n-r+s)}[x]^j) + \sum_{s \leqslant r < n} V^r(\eta b^{(n-r+s)}[x]^j) \\ &= dV^s(b^{(n)}[x]^j)\end{aligned}$$

as desired.

Third, if $\omega^{(n)} = b^{(n)}[x]^j d\log[x]$, we let $\omega' = \omega$. Then $(R-F)(\omega'^{(n+1)}) = \omega^{(n)}$, since $F(b^{(n)}) = 0$.

Fourth, we consider the case $\omega^{(n)} = a^{(n)}[x]^j$. Then $a^{(n)} \in \mathrm{TR}_2^n(\mathbb{S}, I; 2)$ and we showed in Theorem 25 that this group is an $\mathbb{F}_2$-vector space with a basis given by the classes $V^s(\eta\tilde{\eta})$ and $dV^s(\eta\tilde{\eta})$, where $0 \leqslant s < n$. If $a^{(n)} = V^s(\eta\tilde{\eta})$ with $0 \leqslant s < n$, then we let $\omega' = \omega$. Then $(R-F)(\omega'^{(n+1)}) = \omega^{(n)}$, since $F(\tilde{\eta}) = 0$. Next, suppose that $a^{(n)} = dV^s(\tilde{\eta})$ with $1 \leqslant s < n$. Then

$$dV^s(\tilde{\eta})[x]^j = dV^s(\tilde{\eta}[x]^{2^s j}) - jV^s(\tilde{\eta})[x]^j d\log[x].$$

and we have already considered the two terms on the right-hand side. Hence, also in this case, there exists ω' such that $(R - F)(\omega'^{(n+1)}) = \omega^{(n)}$. Similarly, in the remaining case $\omega^{(n)} = (d\tilde{\eta})[x]^j$, the calculation

$$\begin{aligned}(R-F)(dV(\tilde{\eta}[x]^j)) &= dV(\tilde{\eta}[x]^j) - d(\tilde{\eta}[x]^j) - \eta\tilde{\eta}[x]^j \\ &= dV(\tilde{\eta}[x]^j) - (d\tilde{\eta})[x]^j + j\tilde{\eta}[x]^j d\log[x] - \eta\tilde{\eta}[x]^j\end{aligned}$$

shows that there exists ω' such that $(R - F)(\omega'^{(n+1)}) = \omega^{(n)}$. Indeed, we have already considered $dV(\tilde{\eta}[x]^j)$, $\tilde{\eta}[x]^j d\log[x]$, and $\eta\tilde{\eta}[x]^j$.

Finally, we can write every element $\omega = (\omega^{(n)})$ of $\widetilde{\mathrm{TR}}_2(\mathbb{S}[x]^{\pm 1}, I[x^{\pm 1}]; 2)$ as a series $\omega = \sum_{i\in I}\omega_i$, where each ω_i is an element of the one of the four types considered above, and where, for every $n \geqslant 1$, all but finitely many of the $\omega_i^{(n)}$ are zero. Now, for every $i \in I$, we have constructed an element $\omega_i' = (\omega_i'^{(n)})$ such that $(R - F)(\omega_i') = \omega_i$. Moreover, the element ω_i' has the property that, if $\omega_i^{(n)} = 0$, then also $\omega_i'^{(n)} = 0$. It follows that, for all $n \geqslant 1$, all but finitely many of the $\omega_i'^{(n)}$. Hence, the series $\omega' = \sum_{i\in I}\omega_i'$ defines an element with $(R - F)(\omega') = \omega$ as desired. □

Theorem 28. *There is a canonical isomorphism*

$$\mathrm{Wh}_2^{\mathrm{Top}}(S^1) \xrightarrow{\sim} \bigoplus_{r\geqslant 1}\ \bigoplus_{j\in\mathbb{Z}\smallsetminus 2\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}.$$

Proof. We first evaluate the kernel of the map $1-F$ in the long-exact sequence at the beginning of the section. Let $\omega = (\omega^{(n)})$ be an element of $\widetilde{\mathrm{TR}}_2(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}]; 2)$. Then ω lies in the kernel of $1 - F$ if and only if the coefficients

$$\begin{aligned} a_{s,j}^{(n)} &= a_{s,j}(\omega^{(n)}) \in \mathrm{TR}_2^{n-s}(\mathbb{S}, I; 2) \\ b_{s,j}^{(n)} &= b_{s,j}(\omega^{(n)}) \in \mathrm{TR}_1^{n-s}(\mathbb{S}, I; 2)\end{aligned}$$

satisfy the equations of Corollary 3. In the case at hand, the equations imply that the coefficients above are determined by the coefficients $b_{1,j}^{(n)}$. Indeed, if we write j as $2^u j'$ with j' odd, then we have

$$a_{s,j}^{(n)} = \begin{cases} F^u(db_{1,j'}^{(n+1+u)} + \eta b_{1,j'}^{(n+1+u)}) & (s = 0) \\ \eta b_{1,j}^{(n+1-s)} & (1 \leqslant s < n)\end{cases}$$

$$b_{s,j}^{(n)} = \begin{cases} 0 & (s = 0 \text{ and } j \text{ even}) \\ -jb_{1,j}^{(n+1)} & (s = 0 \text{ and } j \text{ odd}) \\ b_{1,j}^{(n+1-s)} & (1 \leqslant s < n).\end{cases}$$

The coefficients $b_{1,j}^{(n)}$, however, are not unrestricted, since for every $n \geqslant 1$, all but finitely many of the coefficients $a_{s,j}^{(n)}$ and $b_{s,j}^{(n)}$ are zero. We write

$$b_{1,j}^{(n)} = \sum_{0 \leqslant r < n-1} c_{r,j} V^r(\tilde{\eta})$$

and consider the coefficients

$$c_{r,j} = c_{r,j}(\omega) \in \mathbb{Z}/2\mathbb{Z}.$$

Since $R(b_{1,j}^{(n+1)}) = b_{1,j}^{(n)}$ and $R(\tilde{\eta}) = \tilde{\eta}$, the coefficients $c_{r,j}$ depend only on the integers $r \geqslant 0$ and $j \in \mathbb{Z} \smallsetminus \mathbb{Z}$ and not on n. They determine and are determined by the coefficients $a_{s,j}^{(n)}$ and $b_{s,j}^{(n)}$.

The requirement that for all $n \geqslant 1$, all but finitely many of the $b_{s,j}^{(n)}$ be zero implies that there exists a finite subset $I = I(\omega) \subset \mathbb{Z}/2\mathbb{Z}$ such that $c_{r,j}$ is zero, unless $j \in I$. We fix $j \in I$ and consider $a_{0,2^u j}^{(n)}$, with $u \geqslant 0$. We calculate

$$\begin{aligned}
a_{0,2^u j}^{(n)} &= F^u(db_j^{(n+1+u)} + \eta b_j^{(n+1+u)}) \\
&= \sum_{0 \leqslant r < n+u} c_{r,j} F^u(dV^r(\tilde{\eta}) + V^r(\eta\tilde{\eta})) \\
&= \sum_{0 \leqslant r < u} c_{r,j} F^{u-r}(d\tilde{\eta} + \eta\tilde{\eta}) \quad + \sum_{u \leqslant r < u+n} c_{r,j}(dV^{r-u}(\tilde{\eta}) + V^{r-u}(\eta\tilde{\eta})) \\
&= \sum_{0 \leqslant r < u} c_{r,j} d\tilde{\eta} \quad + \sum_{u \leqslant r < u+n} c_{r,j}(dV^{r-u}(\tilde{\eta}) + V^{r-u}(\eta\tilde{\eta})).
\end{aligned}$$

Now, for all $n \geqslant 1$, there exists $N^{(n)} = N^{(n)}(\omega)$ such that for all $j \in I$ and all $u \geqslant N^{(n)}$, the coefficient $a_{0,2^u j}^{(n)}$ is zero. We assume that $N^{(n)}$ is chosen minimal. Since

$$R\colon \mathrm{TR}_2^n(\mathbb{S}, I; 2) \to \mathrm{TR}_2^{n-1}(\mathbb{S}, I; 2)$$

is surjective and takes $a_{0,2^u j}^{(n)}$ to $a_{0,2^u j}^{(n-1)}$, we have $N^{(n)} \geqslant N^{(n-1)}$. Considering the coefficients of $d\tilde{\eta}$ and $\eta\tilde{\eta}$ in the sum above, we find that for all $u \geqslant N^{(n)}$,

$$\begin{aligned}
\sum_{0 \leqslant r < u+1} c_{r,j} &= 0 \qquad \text{(coefficient of } d\tilde{\eta}) \\
c_{u,j} &= 0 \qquad \text{(coefficients of } \eta\tilde{\eta}).
\end{aligned}$$

But these equations are satisfied also for $u \geqslant N^{(n-1)}$ which implies that we also have $N^{(n)} \leqslant N^{(n-1)}$. We conclude that there exists an integer $N = N(\omega) \geqslant 0$ independent of n such that $c_{u,j} = 0$, for $u \geqslant N$, and that the coefficient $c_{0,j}$ is equal to the sum of the coefficients $c_{r,j}$ with $r \geqslant 1$. Conversely, suppose we are given coefficients $c_{r,j}$ all but finitely many of which are zero. Then, for every $n \geqslant 1$, all but finitely many of the corresponding coefficients $a_{s,j}^{(n)}$ and $b_{s,j}^{(n)}$ are zero. This shows that the map

$$\ker(1-F\colon \widetilde{\mathrm{TR}}_2(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}];2) \to \widetilde{\mathrm{TR}}_2(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}];2)) \to \bigoplus_{r\geqslant 1}\bigoplus_{j\in\mathbb{Z}\smallsetminus\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$$

that to ω assigns $(c_{r,j}(\omega))$ is an isomorphism.

It remains to show that the map

$$1-F\colon \widetilde{\mathrm{TR}}_3(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}];2) \to \widetilde{\mathrm{TR}}_3(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}];2)$$

is surjective. Given the element $\omega = (\omega^{(n)})$ on the right-hand side, we find an element $\omega' = (\omega'^{(n)})$ on the left-hand side such that $(R-F)(\omega') = \omega$. As in the proof of Theorem 27, we first consider several cases separately.

First, if $\omega^{(n)} = dV^s(b^{(n)}[x]^j)$, where j and $s \geqslant 1$ are integers, we define

$$\omega'^{(n)} = -\sum_{s\leqslant r<n-1} dV^{r+1}(b^{(n-1-r+s)}[x]^j) - \sum_{s\leqslant r<n} V^r(\eta b^{(n-r+s)}[x]^j).$$

Then we find that $R(\omega'^{(n+1)}) = \omega'^{(n)}$ and $(R-F)(\omega'^{(n+1)}) = \omega^{(n)}$ by calculations entirely similar to the ones in the proof of Theorem 27.

Second, if $\omega^{(n)} = b^{(n)}[x]^j d\log[x]$, we consider three cases separately. In the case $\omega^{(n)} = V^s(\eta\tilde{\eta})[x]^j d\log[x]$ with $0 \leqslant s < n$, we let $\omega' = \omega$. Then $(R-F)(\omega') = \omega$ since $F(\tilde{\eta}) = 0$. In the case $\omega^{(n)} = dV^s(\tilde{\eta})[x]^j d\log[x]$, where $1 \leqslant s < n$, we note that $\omega^{(n)} = dV^s(\tilde{\eta}[x]^{2^s j} d\log[x])$ and define

$$\omega'^{(n)} = -\sum_{s\leqslant r<n-1} dV^{r+1}(\tilde{\eta}[x]^{2^s j} d\log[x]) - \sum_{s\leqslant r<n} V^r(\eta\tilde{\eta}[x]^{2^s j} d\log[x]).$$

Then $R(\omega'^{(n+1)}) = \omega'^{(n)}$ and $(R-F)(\omega'^{(n+1)}) = \omega^{(n)}$ as before. In the remaining case $\omega^{(n)} = (d\tilde{\eta})[x]^j d\log[x]$, the calculation

$$\begin{aligned}&(R-F)(dV(\tilde{\eta}[x]^j d\log[x]))\\&\quad = dV(\tilde{\eta}[x]^j d\log[x]) - (d\tilde{\eta})[x]^j d\log[x] - \eta\tilde{\eta}[x]^j d\log[x]\end{aligned}$$

shows that there exists ω' with $(R-F)(\omega') = \omega$. Indeed, we have already considered $dV(\tilde{\eta}[x]^j d\log[x])$ and $\eta\tilde{\eta}[x]^j d\log[x]$.

Third, if $\omega^{(n)} = a^{(n)}[x]^j$, we consider two cases separately. In the first case, we have $\omega^{(n)} = V^s(\tilde{\nu})[x]^j$ with $0 \leqslant s < n$ and define

$$\omega'^{(n)} = V^s(\tilde{\nu})[x]^j + F(V^s(\tilde{\nu})[x]^j) + F^2(V^s(\tilde{\nu})[x]^j).$$

Then $R(\omega'^{(n+1)}) = \omega'^{(n)}$ and $(R-F)(\omega'^{(n+1)}) = \omega^{(n)}$ because $F^3V^s(\tilde{\nu}) = 0$. In the second case, $\omega^{(n)} = dV^s(\eta\tilde{\eta})[x]^j$, we calculate

$$dV^s(\eta\tilde{\eta})[x]^j = dV^s(\eta\tilde{\eta}[x]^{2^s j}) - jV^s(\eta\tilde{\eta})[x]^j d\log[x].$$

Since we have already considered the two terms on the right-hand side, it follows that there exists ω' with $(R-F)(\omega')=\omega$.

Finally, we consider $\omega^{(n)} = V^s(a^{(n)}[x]^j)$ with $1 \leqslant s < n$. For $s \geq 3$, we define

$$\omega'^{(n)} = V^s(a^{(n)}[x]^j) + FV^s(a^{(n+1)}[x]^j) + F^2V^s(a^{(n+2)}[x]^j).$$

Then $R(\omega'^{(n+1)}) = \omega'^{(n)}$ and $(R-F)(\omega'^{(n+1)}) = \omega^{(n)}$ since $8a^{(n+3)} = 0$. For $s=0$ and $s=1$, the calculation

$$\begin{aligned} (R-F)(V(a^{(n+1)}[x]^j)) &= V(a^{(n)}[x]^j) - 2a^{(n+1)}[x]^j \\ (R-F)(V^2(a^{(n+1)}[x]^j) + FV^2(a^{(n+2)}[x]^j)) &= V^2(a^{(n)}[x]^j) - 4a^{(n+2)}[x]^j, \end{aligned}$$

shows that there exists ω' with $(R-F)(\omega') = \omega$. Indeed, we have already considered $2a^{(n+1)}[x]^j$ and $4a^{(n+2)}[x]^j$ above.

The elements ω' with $(R-F)(\omega') = \omega$ which we constructed above have the property that, if $\omega^{(n)}$ is zero, then $\omega'^{(n)}$ is zero. It follows as in the proof of Theorem 27 that the map $1-F$ in question is surjective. □

Theorem 29. *There is a canonical isomorphism*

$$\mathrm{Wh}_3^{\mathrm{Top}}(S^1) \xrightarrow{\sim} \bigoplus_{r \geqslant 0} \bigoplus_{j \in \mathbb{Z} \smallsetminus 2\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{r \geqslant 1} \bigoplus_{j \in \mathbb{Z} \smallsetminus 2\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}.$$

Proof. We first show that the kernel of the map $1-F$ in the long-exact sequence at the beginning of the section is canonically isomorphic to the group that appears on the right-hand side in the statement. So we let $\omega = (\omega^{(n)})$ be an element of $\widetilde{\mathrm{TR}}_3(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}]; 2)$ that lies in the kernel of $1-F$. The equations of Corollary 3 again show that the coefficients

$$\begin{aligned} a_{s,j}^{(n)} &= a_{s,j}(\omega^{(n)}) \in \mathrm{TR}_3^{n-s}(\mathbb{S}, I; p) \\ b_{s,j}^{(n)} &= b_{s,j}(\omega^{(n)}) \in \mathrm{TR}_2^{n-s}(\mathbb{S}, I; p) \end{aligned}$$

are completely determined by the coefficients $b_{1,j}^{(n)}$. Indeed, we find

$$\begin{aligned} a_{s,j}^{(n)} &= \begin{cases} F^u(db_{1,j'}^{(n+1+u)} + \eta b_{1,j'}^{(n+1+u)}) & (s=0) \\ \eta b_{1,j}^{(n+1-s)} & (1 \leqslant s < n) \end{cases} \\ b_{s,j}^{(n)} &= \begin{cases} jF^u(b_{1,j'}^{(n+1+u)}) & (s=0) \\ b_{1,j}^{(n+1-s)} & (1 \leqslant s < n), \end{cases} \end{aligned}$$

where $j = 2^u j'$ with j' odd. For example, if $1 \leqslant s < n$, then

$$\begin{aligned}
a_{s,j}^{(n)} &= 2a_{s+1,j}^{(n+1)} + \eta b_{s+1,j}^{(n+1)} = 2(2a_{s+2,j}^{(n+2)} + \eta b_{s+2,j}^{(n+2)}) + \eta b_{s+1,j}^{(n+1)} \\
&= 2(2(2a_{s+3,j}^{(n+3)} + \eta b_{s+3,j}^{(n+3)}) + \eta b_{s+2,j}^{(n+2)}) + \eta b_{s+1,j}^{(n+1)} \\
&= \eta b_{s+1,j}^{(n+1)} = \eta b_{1,j}^{(n+1-s)}
\end{aligned}$$

since $\mathrm{TR}_3^{n-s}(\mathbb{S}, I; 2)$ is annihilated by 8. We now write

$$b_{1,j}^{(n)} = \sum_{0 \leqslant r < n-1} c_{r,j} V^r(\eta\tilde{\eta}) + \sum_{0 \leqslant r < n-1} c'_{r,j} dV^r(\tilde{\eta}),$$

where the coefficients $c_{r,j} = c_{r,j}(\omega)$ and $c'_{r,j} = c'_{r,j}(\omega)$ are independent on n. It is clear that the $c_{r,j}$ and $c'_{r,j}$ are non-zero for only finitely many values of the odd integer j. We fix such a j and evaluate the coefficients $a_{0,2^u j}^{(n)}$ and $b_{0,2^u j}^{(n)}$ for $u \geqslant 1$ as functions of the coefficients $c_{r,j}$ and $c'_{r,j}$.

$$\begin{aligned}
a_{0,2^u j}^{(n)} &= F^u(db_j^{(n+1+u)} + \eta b_j^{(n+1+u)}) \\
&= \sum_{0 \leqslant r < n+u} c_{r,j}(F^u dV^r(\eta\tilde{\eta}) + \eta F^u V^r(\eta\tilde{\eta})) \\
&\quad + \sum_{0 \leqslant r < n+u} c'_{r,j}(F^u ddV^r(\tilde{\eta}) + \eta F^u dV^r(\tilde{\eta})) \\
&= \sum_{u \leqslant r < n+u} c_{r,j}(dV^{r-u}(\eta\tilde{\eta}) + V^{r-u}(\eta^2\tilde{\eta})) \\
b_{0,2^u j}^{(n)} &= jF^u(b_j^{(n+1+u)}) \\
&= \sum_{0 \leqslant r < n+u} jc_{r,j} F^u V^r(\eta\tilde{\eta}) \;\; + \sum_{0 \leqslant r < n+u} jc'_{r,j} F^u dV^r(\tilde{\eta}) \\
&= \sum_{0 \leqslant r < u} jc'_{r,j} d\tilde{\eta} \;\; + \sum_{u \leqslant r < n+u} jc'_{r,j}(dV^{r-u}(\tilde{\eta}) + V^{r-u}(\eta\tilde{\eta})).
\end{aligned}$$

We claim that the elements $dV^{r-u}(\eta\tilde{\eta})$ and $V^{r-u}(\eta^2\tilde{\eta})$ with $u < r < n+u$ form a linearly independent set. Indeed, the map

$$i_*\colon \mathrm{TR}_3^{n+u}(\mathbb{S}, I; 2) \to \mathrm{TR}_3^{n+u}(\mathbb{S}; 2)$$

is injective by Proposition 13, and Lemma 23 shows that

$$\begin{aligned}
i_*(dV^{r-u}(\eta\tilde{\eta}) &= dV^{r-u}(\eta^2) + dV^{r-u+1}(\eta^2) \\
i_*(V^{r-u}(\eta^2\tilde{\eta}) &= V^{r-u}(\eta^3) + V^{r-u+1}(\eta^3) = 4V^{r-u}(\nu) + 4V^{r-u+1}(\nu).
\end{aligned}$$

The claim then follows from Theorem 11. We now conclude as in the proof of Theorem 28 that the map that to ω assigns $((c_{r,j}(\omega)), (c'_{r,j}(\omega)))$ defines an isomorphism

$$\begin{aligned}&\ker(1-F\colon \widetilde{\mathrm{TR}}_3(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}];2)\to \widetilde{\mathrm{TR}}_3(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}];2))\\ &\quad\xrightarrow{\sim}\bigoplus_{r\geqslant 0}\bigoplus_{j\in\mathbb{Z}\smallsetminus\mathbb{Z}}\mathbb{Z}/2\mathbb{Z}\oplus\bigoplus_{r\geqslant 1}\bigoplus_{j\in\mathbb{Z}\smallsetminus\mathbb{Z}}\mathbb{Z}/2\mathbb{Z}\end{aligned}$$

Finally, we argue as in the proof of Theorem 27 that the map

$$1-F\colon \widetilde{\mathrm{TR}}_4(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}];2)\to \widetilde{\mathrm{TR}}_4(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}];2)$$

is surjective. Given $\omega=(\omega^{(n)})$ on the right-hand side, we find $\omega'=(\omega'^{(n)})$ on the left-hand side with $(R-F)(\omega')=\omega$.

First, if $\omega^{(n)}=dV^s(b^{(n)}[x]^j)$, where $1\leqslant s<n$ and j are integers, we define

$$\omega'^{(n)}=-\sum_{s\leqslant r<n-1}dV^{r+1}(b^{(n-1-r+s)}[x]^j)-\sum_{s\leqslant r<n}V^r(\eta b^{(n-r+s)}[x]^j).$$

Then we have $R(\omega'^{(n+1)})=\omega'^{(n)}$ and $(R-F)(\omega'^{(n+1)})=\omega^{(n)}$ as desired.

Second, if $\omega^{(n)}=b^{(n)}[x]^j\,d\log[x]$, we consider two cases separately. In the case $\omega^{(n)}=dV^s(\eta\tilde{\eta})[x]^j\,d\log[x]$ with $1\leqslant s<n$, we write $\omega^{(n)}=dV^s(\eta\tilde{\eta}[x]^{2^s j})d\log[x]$ and define

$$\omega'^{(n)}=-\sum_{s\leqslant r<n-1}dV^{r+1}(\eta\tilde{\eta}[x]^j\,d\log[x])-\sum_{s\leqslant r<n}V^r(\eta^2\tilde{\eta}[x]^j\,d\log[x]).$$

Then $R(\omega'^{(n+1)})=\omega'^{(n)}$ and $(R-F)(\omega'^{(n+1)})=\omega^{(n)}$ as before. In the case where $\omega^{(n)}=V^s(\tilde{\nu})[x]^j\,d\log[x]$ with $0\leqslant s<n$, we define

$$\omega'^{(n)}=V^s(\tilde{\nu})[x]^j\,d\log[x]+F(V^s(\tilde{\nu})[x]^j\,d\log[x])+F^2(V^s(\tilde{\nu})[x]^j\,d\log[x]).$$

Then $R(\omega'^{(n+1)})=\omega'^{(n)}$ and $(R-F)(\omega'^{(n+1)})=\omega^{(n)}$ since $8\tilde{\nu}$ and $F\tilde{\nu}$ are zero.

Finally, we consider $\omega^{(n)}=dV^s(\tilde{\nu})[x]^j$ with $0\leqslant s<n$. For $s\geqslant 1$,

$$dV^s(\tilde{\nu})[x]^j=dV^s(\tilde{\nu}[x]^{2^s j})-jV^s(\tilde{\nu})[x]^j\,d\log[x]$$

and the two terms on the right-hand side were considered above. It follows that there exists ω' with $(R-F)(\omega')=\omega$. For $s=0$, we calculate

$$\begin{aligned}&(R-F)(dV(\tilde{\nu}[x]^j))=dV(\tilde{\nu}[x]^j)-(d\tilde{\nu})[x]^j+j\tilde{\nu}[x]^j\,d\log[x]-\eta\tilde{\nu}[x]^j\\ &\qquad(R-F)(\eta\tilde{\nu}[x]^j)=\eta\tilde{\nu}[x]^j.\end{aligned}$$

This shows that also for $\omega^{(n)}=(d\tilde{\nu})[x]^j$, there exists ω' such that $(R-F)(\omega')=\omega$. Indeed, we have already considered the remaining classes on the right-hand side.

The elements ω' with $(R-F)(\omega')=\omega$ which we constructed above have the property that, if $\omega^{(n)}$ is zero, then $\omega'^{(n)}$ is zero. It follows as in the proof of Theorem 27 that the map $1 - F$ in question is surjective. This completes the proof. □

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Cocycle Categories

J.F. Jardine

Abstract The cocycle category $H(X, Y)$ is defined for objects X and Y in a model category, and it is shown that the set of homotopy category morphisms $[X, Y]$ is isomorphic to the set of path components of $H(X, Y)$, provided that the ambient model category is right proper, and if weak equivalences are closed under finite products. Various applications of this result are displayed, including the homotopy classification of torsors, abelian sheaf cohomology groups, group extensions and gerbes. The older classification results have simple new proofs involving canonically defined cocycles. Cocycle methods are also used to show that the algebraic K-theory presheaf of spaces is a simplicial stack associated to a suitably defined parabolic groupoid.

1 Introduction

Suppose that G is a sheaf of groups on a space X and that $U_\alpha \subset X$ is an open covering. Then a cocycle for the covering is traditionally defined to be a family of elements $g_{\alpha\beta} \in G(U_\alpha \cap U_\beta)$ such that $g_{\alpha\alpha} = e$, and $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$ when all elements are restricted to the group $G(U_\alpha \cap U_\beta \cap U_\gamma)$.

A more compact way of saying this is to assert that such a cocycle is a map of simplicial sheaves $C(U) \to BG$ on the space X, where $C(U)$ is the Čech resolution associated to the covering family $\{U_\alpha\}$. The canonical map $C(U) \to *$ is a local weak equivalence of simplicial sheaves, and is a fibration in each section since $C(U)$ is actually the nerve of a groupoid – the map $C(U) \to *$ is therefore a hypercover, which is most properly defined to be a map of simplicial sheaves (or presheaves) which is a Kan fibration and a weak equivalence in each stalk. Every cocycle in the traditional sense therefore determines a picture of simplicial sheaf morphisms

$$* \leftarrow C(U) \to BG.$$

where the canonical map $C(U) \to *$ is a hypercover.

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More generally, it has been known since the mid 1980s [4, 10] that the locally fibrant simplicial sheaves (i.e. simplicial sheaves which are Kan complexes in each stalk) have a partial homotopy theory which leads to a calculus of fractions approach to formally inverting local weak equivalences, and therefore specifies a construction of the homotopy category for simplicial sheaves. Specifically, one defines morphisms $[X, Y]$ in the homotopy category by setting

$$[X, Y] = \varinjlim_{[\pi]:V \to X} \pi(V, Y), \tag{1}$$

where the filtered colimit is indexed over simplicial homotopy classes of hypercovers $\pi : V \to X$, and $\pi(V, Y)$ denotes simplicial homotopy classes of maps $V \to Y$. Thus, morphisms in the homotopy category are represented by pictures

$$X \xleftarrow{\pi} V \to Y, \tag{2}$$

where π is a hypercover. The relation (1) is usually called the Generalized Verdier Hypercovering Theorem, and it is the historical starting point for the homotopy theory of simplicial sheaves.

Much has transpired in the intervening years. We now know that there is a plethora of Quillen model structures for simplicial sheaves and simplicial presheaves on all small Grothendieck sites [15], all of which determine the same homotopy category. In the first of these structures [11, 21], the monomorphisms are the cofibrations and the weak equivalences are the local weak equivalences (i.e. stalkwise weak equivalences in the presence of enough stalks), and then the homotopy categories for simplicial sheaves and presheaves (which are equivalent) are constructed by methods introduced by Quillen. The morphisms $[X, Y]$ in the homotopy category of simplicial sheaves coincide with those specified by (1) if X and Y are locally fibrant. These basic model structures also inherit many of the good properties of simplicial sets, including right properness, which means that weak equivalences are stable under pullback along fibrations. Weak equivalences of simplicial presheaves and sheaves are also closed under finite products.

One still needs a cocycle or hypercover theory, because it is involved in the proofs of many of the standard homotopy classification theorems for sheaf cohomology, such as the identification of sheaf cohomology with morphisms $[*, K(A, n)]$ in the homotopy category. It turns out, however, that the traditional requirement that the map π in (2) should be something like a fibration in formal manipulations based on the generalized Verdier hypercover theorem is usually quite awkward in practice. This has been especially apparent in all attempts to convert n-types to more algebraic objects.

The basic point of this paper is that there is a better approach to defining and manipulating cocycles, which essentially starts with removing the fibration condition on the weak equivalence π in (2).

A cocycle from X to Y is defined in this paper to be a pair of maps

$$X \xleftarrow{f} Z \xrightarrow{g} Y \tag{3}$$

where f is a weak equivalence. A morphism of cocycles is the obvious thing, namely a commutative diagram

$$\begin{tikzcd}
& Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd] & \\
X & & Y \\
& Z' \arrow[ul, "f'"] \arrow[ur, "g'"'] &
\end{tikzcd}$$

The cocycle category is denoted by $H(X, Y)$. Then the basic point is this: there is an obvious function

$$\phi : \pi_0 H(X, Y) \to [X, Y]$$

taking values in morphisms in the homotopy category, which is defined by sending the cocycle (3) to the morphism gf^{-1} in the homotopy category. Then Theorem 1 of this paper says that this function ϕ is a bijection.

Cocycle categories and the function ϕ are defined in great generality. The fact that ϕ is a bijection (Theorem 1) holds in the rather common setting of a right proper closed model structure in which weak equivalences are preserved by finite products. The formal definition of cocycle categories and their basic properties appear in the first section of this paper. The overall theory is easy to demonstrate.

The subsequent sections of this paper are taken up with a tour of applications. Some of these are well known, and are given here with simple new proofs. This general approach to cocycles is also implicated in several recent results in non-abelian cohomology theory, which are displayed here.

Section 3 gives a quick (and hypercover-free) demonstration of the homotopy classification theorem for G-torsors for a sheaf of groups G – this is Theorem 2. It is also shown that the ideas behind the proof of Theorem 2 admit great generalization: examples include the definition and homotopy classification of torsors for a presheaf of categories enriched in simplicial sets (Theorem 3), the explicit constructions of the stack completion of a sheaf of groupoids in terms of both torsors and cocycles given in Section 3.3, and the calculation of the morphism set $[*, \underrightarrow{\mathrm{holim}}_G X]$ for a diagram X of simplicial presheaves defined on a presheaf of groupoids G which appears in Theorem 5.

Section 4 gives a new demonstration of homotopy classification theorem for abelian sheaf cohomology (Corollary 3). Section 5 gives a new description of the homotopy classification theorem for group extensions in terms of cocycles taking values in 2-groupoids (Theorem 6), and in Section 6 we discuss but do not prove a classification theorem for gerbes up to local equivalence as path components of a suitable cocycle category. This last result is Theorem 7 – it is proved in [19]. The cocycles appearing in the proofs of Theorems 2, 6 and 7 are canonically defined.

In Section 7, the methods of this paper are used to show that the algebraic K-theory presheaf of spaces K^1 can be constructed as a simplicial stack for the Zariski topology which is associated to a simplicial sheaf of groupoids made up of parabolic subgroups of general linear groups. The argument for the proof of the main result of the section (Theorem 8) specializes locally to a proof of an old result of Schechtman [26] on the representability of the K-theory space, and the local result implies Theorem 8, so that Schechtman's theorem is essentially equivalent to Theorem 8.

The real interest in Theorem 8 is in possible generalizations of its proof. It should be the case, for example, that there is a simplicial groupoid made up of "parabolic" subgroups of the orthogonal and/or symplectic groups that represents a Q-construction for hermitian K-theory, in the sense that an associated simplicial stack defines Karoubi's ${}_{\epsilon}L^1$-spaces.

2 Cocycles

Suppose that $\mathcal{M}$ is a right proper closed model category, and suppose further that the weak equivalences of $\mathcal{M}$ are closed under finite products.

The assertion that $\mathcal{M}$ is a closed model category means that $\mathcal{M}$ has all finite limits and colimits (**CM1**), and that the class of morphisms of $\mathcal{M}$ contain three subclasses, namely weak equivalences, fibrations and cofibrations which satisfy some properties. These properties include the two of three condition for weak equivalences (**CM2**: if any two of the maps f, g or $g \cdot f$ are weak equivalences, then so is the third), the requirement that all three classes of maps are closed under retraction (**CM3**), and the factorization axiom (**CM5**) which asserts that any map f has factorizations $f = qj = pi$ where q is a trivial fibration and j is a cofibration, and p is a fibration and i is a trivial cofibration. Note, for example, that a trivial fibration is a fibration and a weak equivalence – "trivial" things are always weak equivalences. Finally, $\mathcal{M}$ should satisfy the lifting axiom (**CM4**) which says that in any solid arrow diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur] & Y \end{tikzcd}$$

where p is a fibration and i is a cofibration, the dotted arrow exists making the diagram commute if either i or p is trivial.

A model category $\mathcal{M}$ is said to be right proper if weak equivalences are closed under pullback along fibrations.

Not every model category is right proper, but right proper model structures are fairly common: examples include topological spaces, simplicial sets, spectra, simplicial presheaves, presheaves of spectra, and certain "good" localizations such as the motivic and motivic stable model categories. In all of these examples as well, weak equivalences are closed under finite products, meaning that if $g : X \to Y$ is a weak equivalence and Z is any other object, then the map $g \times 1 : X \times Z \to Y \times Z$ is a weak equivalence.

Weak equivalences are closed under finite products for any f-local model structure for simplicial presheaves (such as the motivic model structure), because one can show that if Z is f-local and A is an arbitrary simplicial presheaf, then the internal function complex $\mathbf{Hom}(A, X)$ is an f-local simplicial presheaf. See [7] for the relevant definitions.

Suppose that X and Y are objects of $\mathcal{M}$. Let $H(X, Y)$ be the category whose objects are all pairs of maps (f, g)

$$X \xleftarrow{f} Z \xrightarrow{g} Y$$

such that f is a weak equivalence. A morphism $\alpha : (f, g) \to (f', g')$ of $H(X, Y)$ is a commutative diagram

$$\begin{tikzcd}
& Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "\alpha"] & \\
X & & Y \\
& Z' \arrow[ul, "f'"] \arrow[ur, "g'"'] &
\end{tikzcd}$$

One says that $H(X, Y)$ is the *category of cocycles* from X to Y.

Example 1. Suppose that $V \to *$ is a sheaf epimorphism (possibly arising from a covering) and that G is a sheaf of groups. Traditional cocycles for the underlying site with coefficients in G can be interpreted as simplicial sheaf maps

$$* \leftarrow C(V) \to BG,$$

where $C(V)$ is the Čech resolution for the cover, and the canonical map $C(V) \to *$ is a local weak equivalence of simplicial sheaves. All cocycles of this type therefore represent objects of the cocycle category $H(*, BG)$ in simplicial sheaves. This is a motivating example for the present definition of a cocycle.

More generally, it has been known for some time [10] that morphisms $X \to Y$ in the homotopy category of simplicial sheaves can be represented by pairs of morphisms

$$X \xleftarrow{\pi} U \to Y$$

where π is a hypercover, or locally trivial fibration, provided that X and Y are locally fibrant. Such pictures are members of the cocycle category $H(X, Y)$ in simplicial sheaves.

Write $\pi_0 H(X,Y)$ for the class of path components of $H(X,Y)$, and let $[X,Y]$ be the set of morphisms from X to Y in the homotopy category $\mathrm{Ho}(\mathcal{M})$ of $\mathcal{M}$. Recall that the category $\mathrm{Ho}(\mathcal{M})$ is constructed from the model category $\mathcal{M}$ by formally inverting the weak equivalences. Then one sees that there is a function

$$\phi : \pi_0 H(X,Y) \to [X,Y]$$

which is defined by $(f,g) \mapsto g \cdot f^{-1}$.

Theorem 1. *Suppose that $\mathcal{M}$ is a right proper closed model category for which the class of weak equivalences is closed under finite products. Then the function $\phi : \pi_0 H(X,Y) \to [X,Y]$ is a bijection for all X and Y.*

Remark 1. Cocycle categories have appeared before in the literature. In particular, it is known from the Dwyer–Kan theory of hammock localizations that if Y is a fibrant object in a model category $\mathcal{M}$ then the nerve $BH(X,Y)$ of the category $H(X,Y)$ (which is a variant of the "moduli category" for the theory) is a model for the function complex of morphisms from X to Y, and the conclusion of Theorem 1 that there is a bijection

$$\pi_0 H(X,Y) \cong [X,Y]$$

holds in that case. See [1,6], and more recently [5]. The assumptions on the model category $\mathcal{M}$ in the statement of Theorem 1 give a way to avoid the technical issues around hammock localizations – the proof which follows below is really quite simple. It is also vital for the applications of Theorem 1 which appear below that there cannot be a fibrancy condition on the target object Y.

Suppose that the maps $f,g : X \to Y$ are left homotopic. Then there is a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "1"'] \arrow[d, "d_0"] \arrow[dr, "f"] & \\ X & X \otimes I \arrow[l, "s"'] \arrow[r, "h"] & Y \\ & X \arrow[ul, "1"] \arrow[u, "d_1"'] \arrow[ur, "g"'] & \end{tikzcd}$$

for some choice of cylinder object $X \otimes I$, in which s is a weak equivalence and h is the homotopy. Then there are morphisms

$$(1_X, f) \to (s,h) \leftarrow (1_X, g)$$

in $H(X,Y)$, so that $(1_X, f)$ and $(1_X, g)$ are in the same path component of $H(X,Y)$. It follows that the assignment $f \mapsto [(1_X, f)]$ defines a function

$$\psi : \pi(X, Y) \to \pi_0 H(X, Y)$$

where $\pi(X, Y)$ denotes left homotopy classes of maps from X to Y.

Theorem 1 is a formal consequence of the following two results:

Lemma 1. *Suppose that $\alpha : X \to X'$ and $\beta : Y \to Y'$ are weak equivalences. Then the induced function*

$$(\alpha, \beta)_* : \pi_0 H(X, Y) \to \pi_0 H(X', Y')$$

is a bijection.

Lemma 2. *Suppose that X is cofibrant and Y is fibrant. Then the function*

$$\phi : \pi_0 H(X, Y) \to [X, Y]$$

is a bijection.

Proof (Proof of Lemma 1). Identify the object $(f, g) \in H(X', Y')$ with a map $(f, g) : Z \to X' \times Y'$ of $\mathcal{M}$ such that f is a weak equivalence.

There is a factorization

$$\begin{tikzcd} Z \arrow[r, "j"] \arrow[dr, "(f,g)"'] & W \arrow[d, "(p_{X'}, p_{Y'})"] \\ & X' \times Y' \end{tikzcd}$$

such that j is a trivial cofibration and $(p_{X'}, p_{Y'})$ is a fibration. Observe that $p_{X'}$ is a weak equivalence.

Form the pullback

$$\begin{tikzcd} W_* \arrow[r, "(\alpha\times\beta)_*"] \arrow[d, "(p_X^*, p_Y^*)"'] & W \arrow[d, "(p_{X'}, p_{Y'})"] \\ X \times Y \arrow[r, "\alpha\times\beta"'] & X' \times Y' \end{tikzcd}$$

Then (p_X^*, p_Y^*) is a fibration and $(\alpha \times \beta)_*$ is a weak equivalence. The map p_X^* is also a weak equivalence.

The assignment $(f, g) \mapsto (p_X^*, p_Y^*)$ defines a function

$$\pi_0 H(X', Y') \to \pi_0 H(X, Y)$$

which is inverse to $(\alpha, \beta)_*$. □

Proof (Proof of Lemma 2). The canonical function $\pi(X,Y) \to [X,Y]$ is a bijection since X is cofibrant and Y is fibrant, and there is a commutative diagram

$$\begin{tikzcd} \pi(X,Y) \arrow[r, "\psi"] \arrow[dr, "\cong"'] & \pi_0 H(X,Y) \arrow[d, "\phi"] \\ & {[X,Y]} \end{tikzcd}$$

It suffices to show that ψ is surjective, or that any object $X \xleftarrow{f} Z \xrightarrow{g} Y$ is in the path component of some pair $X \xleftarrow{1} X \xrightarrow{k} Y$ for some map k. Form the diagram

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, "j"] & \\ X & V \arrow[l, "p"] \arrow[r, dotted, "\theta"'] & Y \end{tikzcd}$$

where j is a trivial cofibration and p is a fibration; the map θ exists because Y is fibrant. The object X is cofibrant, so the trivial fibration p has a section σ, and there is a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "1"'] \arrow[dr, "\theta\sigma"] \arrow[d, "\sigma"] & \\ X & V \arrow[l, "p"] \arrow[r, dotted, "\theta"'] & Y \end{tikzcd}$$

The composite $\theta\sigma$ is the required map k. □

We conclude this section with the following strengthening of Lemma 1, which holds in many cases of interest.

Lemma 3. *Suppose that, in addition to the hypotheses of Theorem 1, the model structure $\mathcal{M}$ has functorial factorizations. Suppose that $\alpha : X \to X'$ and $\beta : Y \to Y'$ are weak equivalences of $\mathcal{M}$. Then composition with $\alpha \times \beta$ induces a weak equivalence*

$$(\alpha,\beta)_* : BH(X,Y) \to BH(X',Y').$$

Proof. In the notation of the proof of Lemma 1, the assignment

$$(f,g) \mapsto (p_X^*, p_Y^*) = \psi(f,g)$$

defines a functor $\psi : H(X', Y') \to H(X, Y)$. Then there are natural transformations

$$(f, g) \to (p_{X'}, p_{Y'}) \leftarrow (\alpha, \beta)_* \psi(f, g).$$

Similarly, for members (h, k) of the cocycle category $H(X, Y)$, there is a natural transformation

$$(h, k) \to \psi(\alpha, \beta)_*(h, k).$$

It follows that the map

$$(\alpha, \beta)_* : BH(X, Y) \to BH(X', Y').$$

is a homotopy equivalence. $\square$

3 Torsors

3.1 Torsors for Sheaves of Groups

Suppose that G is a sheaf of groups on a small Grothendieck site $\mathcal{C}$.

A G-torsor is usually defined to be a sheaf X with a free G-action such that the map $X/G \to *$ is an isomorphism in the sheaf category.

The G-action on X is free if and only if the canonical simplicial sheaf map $EG \times_G X \to X/G$ is a local weak equivalence. One sees this by noting that the fundamental groups of the Borel construction $EG \times_G X$ are isotropy subgroups for the G-action. Further, $EG \times_G X$ is the nerve of a groupoid so there are no higher homotopy groups.

It follows that a sheaf X with G-action is a G-torsor if and only if the canonical simplicial sheaf map $EG \times_G X \to *$ is a local weak equivalence. Write $G -$ **tors** for the groupoid of G-torsors and G-equivariant maps.

Suppose given a cocycle

$$* \xleftarrow{\simeq} Y \xrightarrow{\alpha} BG$$

in the category of simplicial sheaves. Form pullback

$$\begin{tikzcd} \mathrm{pb}(Y) \arrow[r] \arrow[d] & Y \arrow[d, "\alpha"] \\ EG \arrow[r, "\pi"'] & BG \end{tikzcd} \tag{1}$$

Then the simplicial sheaf $\mathrm{pb}(Y)$ inherits a G-action from the G-action on EG, and the induced map $EG \times_G \mathrm{pb}(Y) \to Y$ is a weak equivalence. The square (1) is locally homotopy cartesian, and it follows that the map $\mathrm{pb}(Y) \to \tilde{\pi}_0\,\mathrm{pb}(Y)$ is a

G-equivariant weak equivalence. Here, $\tilde{\pi}_0\,\mathrm{pb}(Y)$ is the sheaf of path components of the simplicial sheaf $\mathrm{pb}(Y)$, in this case identified with a constant simplicial sheaf.

It follows that the maps

$$EG \times_G \tilde{\pi}_0\,\mathrm{pb}(Y) \leftarrow EG \times_G \mathrm{pb}(Y) \to Y \simeq *$$

are weak equivalences, so that $\tilde{\pi}_0\,\mathrm{pb}(Y)$ is a G-torsor. A functor

$$H(*, BG) \to G - \mathbf{tors} \tag{2}$$

is therefore defined by sending the cocycle $* \xleftarrow{\simeq} Y \to BG$ to the torsor $\tilde{\pi}_0\,\mathrm{pb}(Y)$.

A functor

$$G - \mathbf{tors} \to H(*, BG) \tag{3}$$

is defined by sending a G-torsor X to the cocycle $* \xleftarrow{\simeq} EG \times_G X \to BG$.

Theorem 2. *Suppose that $\mathcal{C}$ is a small Grothendieck site, and that G is a sheaf of groups on $\mathcal{C}$. Then the functors (2) and (3) induce bijections*

$$[*, BG] \cong \pi_0 H(*, BG) \cong \pi_0(G - \mathbf{tors}) = H^1(\mathcal{C}, G).$$

Proof. The functors (2) and (3) induce functions $\pi_0 H(*, BG) \to \pi_0(G - \mathbf{tors})$ and $\pi_0(G - \mathbf{tors}) \to \pi_0 H(*, BG)$ which are inverse to each other. To see this, there are two statements to verify, only one of which is non-trivial.

To show that the composite

$$\pi_0(G - \mathbf{tors}) \to \pi_0 H(*, G) \to \pi_0(G - \mathbf{tors})$$

is the identity, one uses the fact that the diagram

$$\begin{array}{ccc} X & \longrightarrow & EG \times_G X \\ \downarrow & & \downarrow \\ * & \longrightarrow & BG \end{array} \tag{4}$$

is sectionwise homotopy cartesian. □

The identification of the non-abelian cohomology invariant $H^1(\mathcal{C}, G)$ with morphisms $[*, BG]$ in the homotopy category of simplicial sheaves of Theorem 2 is, at this writing, almost twenty years old [12]. Unlike the original proof, the demonstration given here contains no references to hypercovers or pro objects.

Suppose that $U \to *$ is a sheaf epimorphism, and let $C(U)$ denote the corresponding Čech resolution. Recall that $C(U)$ is the nerve of a sheaf of groupoids whose object sheaf is U and whose morphism sheaf is $U \times U$, with source and target given by the two projections $U \times U \to U$. It's possibly a bit confusing, but I also use the notation $C(U)$ to denote this underlying groupoid.

I say that a G-torsor Y trivializes over U if there is a sheaf map $\sigma : U \to Y$; the sheaf map σ is called a trivialization.

Since Y is a G-torsor, the isomorphism $G \times Y \xrightarrow{\cong} Y \times Y$ defined by $(g, y) \mapsto (y, gy)$ determines an isomorphism

$$C(Y) \cong EG \times_G Y.$$

Thus, a trivialization $\sigma : U \to Y$ determines a composite

$$\sigma_* : C(U) \xrightarrow{\sigma} C(Y) \cong EG \times_G Y \to BG,$$

which is a member of the cocycle category $H(*, BG)$. This is the classical method of associating a cocycle to a torsor Y.

More generally, any two trivializations $\sigma, \sigma' : U \to Y$ determine a canonical map

$$U \times U \xrightarrow{\sigma \times \sigma'} Y \times Y \cong G \times Y.$$

and hence a unique element $(\sigma'(v), \sigma(u))_* \in G$ such that

$$(\sigma'(v), \sigma(u))_* \sigma(u) = \sigma'(v)$$

for all sections u, v of U. In particular, the element $(\sigma(v), \sigma(u))_*$ is the image of the morphism (u, v) under the cocycle σ_* associated to σ. It also follows that σ and σ' together determine commutative diagrams

$$\begin{tikzcd}
* \arrow[r, "(\sigma(v),\sigma(u))_*"] \arrow[d, "(\sigma'(u),\sigma(u))_*"'] & * \arrow[d, "(\sigma'(v),\sigma(v))"] \\
* \arrow[r, "(\sigma'(v),\sigma'(u))"'] & *
\end{tikzcd}$$

so that there is a canonical homotopy $C(U) \times \Delta^1 \to BG$ from the cocycle σ_* to the cocycle σ'_*. This homotopy is a member of the groupoid $G^{C(U)}$ of functors $C(U) \to G$ and natural isomorphisms $C(U) \times \mathbf{1} \to G$ between them; the nerve of this groupoid is the function complex $\mathbf{hom}(C(U), BG)$.

If $\theta : Y \to Y'$ is a morphism of G-torsors and $\sigma : U \to Y$ is a trivialization of Y, then the cocycles σ_* and $(\theta\sigma)_*$ coincide, since the map θ is G-equivariant.

Write $G - \mathbf{tors}_U$ for the full subcategory of the groupoid of G-torsors, whose objects are the torsors Y admitting a trivialization $U \to Y$ over U. For each such torsor Y, make a fixed choice of trivialization $\sigma_Y : U \to Y$. Then a morphism of torsors $\theta : Y \to Y'$ induces a unique natural isomorphisms

$$\sigma_{Y*} = (\theta\sigma_Y)_* \xrightarrow{\theta_*} \sigma_{Y'*}.$$

We have therefore defined a functor

$$\omega_U : G - \mathbf{tors}_U \to G^{C(U)},$$

with $\omega_U(Y) = \sigma_{Y*}$.

A different choice of family of trivializations $\{\sigma'_Y\}$ determines a functor ω'_U which is canonically homotopic to ω_U.

Lemma 4. *The functor $\omega_U : G - \mathbf{tors}_U \to G^{C(U)}$ is a weak equivalence of groupoids.*

Proof. Let $\gamma : C(U) \to BG$ be a cocycle, and write $Y_\gamma = \tilde{\pi}_0 \operatorname{pb} C(U)$ for the torsor associated to γ. Then there is a coequalizer diagram of sheaves

$$U \times U \times G \underset{d_1}{\overset{d_0}{\rightrightarrows}} U \times G \xrightarrow{\pi} y_\gamma$$

where $d_0((u,v),h) = (v, \gamma(u,v)h)$ and $d_1((u,v),h) = (u,h)$. Write $\sigma(u) = \pi(u,e)$ to define a trivialization $\sigma : U \to Y_f$. Then one finds that

$$\gamma(u,v)\sigma(u) = \gamma(u,v)\pi(u,e) = \pi(u, \gamma(u,v)^{-1}) = \pi(v.e) = \sigma(v),$$

so that the cocycle σ_* determined by the trivialization σ coincides with the original cocycle γ. It follows that the morphism ω_U is surjective on path components.

We can choose the trivialization for the trivial torsor G to be the global section $e : * \to G$ determined by the identity. The associated cocycle e_* is the unique functor $* \to G$. The automorphisms of the trivial torsor G are the maps $\cdot k : G \to G$ defined by right multiplication by global sections k of G, while the homotopies $\Delta^1 \to BG$ are again the global sections of G. One then shows that the functor ω_U induces an isomorphism

$$G - \mathbf{tors}_U(G,G) \xrightarrow{\cong} G^{C(U)}(e_*, e_*).$$

A descent argument shows that the functor ω_U induces an isomorphism

$$G - \mathbf{tors}_U(Y,Y) \xrightarrow{\cong} G^{C(U)}(\sigma_{Y*}, \sigma_{Y*})$$

for all torsors Y which are trivialized over U. □

Lemma 4 has been known in one form or another for a long time, but a question one could now ask is this: which part of the cocycle category $H(*, BG)$ corresponds to the torsors $G - \mathbf{tors}_U$ which trivialize over U?

The answer is essentially obvious. Let $H(*, BG)_U$ be the union of the path components of all cocycles $C(U) \to BG$ inside $H(*, BG)$. Then one checks that the functors defined in (2) and (3) restrict to functors $H(*, BG)_U \to G - \mathbf{tors}_U$ and $G - \mathbf{tors}_U \to H(*, BG)_U$, respectively, as do the arguments for Theorem 2. We therefore have the following:

Proposition 1. *Suppose that $\mathcal{C}$ is a small Grothendieck site and that G is a sheaf of groups on $\mathcal{C}$. Suppose that $U \to *$ is a sheaf epimorphism. Then the functors (2) and (3) induce a bijection*

$$\pi_0 H(*, BG)_U \cong \pi_0(G - \mathbf{tors}_U).$$

3.2 Diagrams and Torsors

There is a local model structure for simplicial presheaves on a small site $\mathcal{C}$ which is Quillen equivalent to the local model structure for simplicial sheaves [11] – this has been known since the 1980s. Just recently [17], it has been shown that both the torsor concept and the homotopy classification result Theorem 2 admit substantial generalizations, to the context of diagrams of simplicial presheaves defined on presheaves of categories enriched in simplicial sets.

To explain, when I say that A is a presheaf of categories enriched in simplicial sets I mean that A consists of a presheaf $\mathrm{Ob}(A)$ and a simplicial presheaf $\mathrm{Mor}(A)$, together with source and target maps $s, t : \mathrm{Mor}(A) \to \mathrm{Ob}(A)$, a map $e : \mathrm{Ob}(A) \to \mathrm{Mor}(A)$ which is a section for both s and t, and an associative law of composition

$$\mathrm{Mor}(A) \times_{t,s} \mathrm{Mor}(A) \to \mathrm{Mor}(A)$$

for which the map e is a two-sided identity. Here, the simplicial presheaf

$$\mathrm{Mor}(A) \times_{t,s} \mathrm{Mor}(A)$$

is defined by the pullback

$$\begin{tikzcd}
\mathrm{Mor}(A) \times_{t,s} \mathrm{Mor}(A) \arrow[r] \arrow[d] & \mathrm{Mor}(A) \arrow[d, "s"] \\
\mathrm{Mor}(A) \arrow[r, "t"'] & \mathrm{Ob}(A)
\end{tikzcd}$$

and describes composable pairs of morphisms in A. To say it a different way, A is a category object in simplicial presheaves with simplicially discrete objects.

An A-diagram X (expressed internally [2, p. 325], [24, p. 240]) consists of a simplicial presheaf map $\pi : X \to \mathrm{Ob}(A)$, together with an action

$$\begin{tikzcd}
\mathrm{Mor}(A) \times_{s,\pi} X \arrow[r, "m"] \arrow[d] & X \arrow[d, "\pi"] \\
\mathrm{Mor}(A) \arrow[r, "t"'] & \mathrm{Ob}(A)
\end{tikzcd} \tag{1}$$

of the morphism object $\mathrm{Mor}(A)$ which is associative and respects identities.

The proof of Theorem 2 (specifically, the assertion that the diagram (4) is homotopy cartesian) uses Quillen's Theorem B, which can be interpreted as saying that if $Y : I \to s\mathbf{Set}$ is an ordinary diagram of simplicial sets defined on a small category I such that every morphism $i \to j$ induces a weak equivalence $Y_i \to Y_j$, then the pullback diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\pi"'] & \underrightarrow{\mathrm{holim}}_I Y \arrow[d] \\ \mathrm{Ob}(I) \arrow[r] & BI \end{tikzcd}$$

is homotopy cartesian. Here, we write $Y = \bigsqcup_{i \in \mathrm{Ob}(I)} Y_i$. This result automatically holds when the index category I is a groupoid, but for more general index categories one has to be more careful.

Say that $Y : I \to s\mathbf{Set}$ is a *diagram of equivalences* if all induced maps $Y_i \to Y_j$ are weak equivalences of simplicial sets, and observe that this is equivalent to the requirement that the corresponding action

$$\begin{tikzcd} \mathrm{Mor}(I) \times_{s,\pi} Y \arrow[r, "m"] \arrow[d] & Y \arrow[d, "\pi"] \\ \mathrm{Mor}(I) \arrow[r, "t"'] & \mathrm{Ob}(I) \end{tikzcd}$$

is homotopy cartesian. This is the formulation that works in general: given a presheaf of categories A enriched in simplicial sets, I say that an A-diagram X is a diagram of equivalences if the action diagram (1) is homotopy cartesian for the local model structure on simplicial presheaves.

The Borel construction $EG \times_G Z$ for a sheaf (or presheaf) Z having an action by a sheaf of groups G is the homotopy colimit for the action, thought of as a diagram defined on G. In general, say that an A-diagram X is an *A-torsor* if:

1. X is a diagram of equivalences.
2. The canonical map $\underrightarrow{\mathrm{holim}}_A X \to *$ is a local weak equivalence.

A map $X \to Y$ of A-torsors is just a natural transformation, meaning a simplicial presheaf map

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & \mathrm{Ob}(A) & \end{tikzcd}$$

over $\mathrm{Ob}(A)$ which respects actions in the obvious sense. Write $A - \mathbf{Tors}$ for the corresponding category of A-torsors. This category of torsors is not a groupoid in general, but one can show that every map of A-torsors is a local weak equivalence.

Theorem 3. *Suppose that A is a presheaf of categories enriched on simplicial sets on a small Grothendieck site $\mathcal{C}$. Then the homotopy colimit functor induces a bijection*

$$\pi_0(A - \mathbf{Tors}) \cong \pi_0 H(*, BA) \cong [*, BA].$$

This result is proved in [17], by a method which generalizes the proof of Theorem 2. This same collection of ideas is also strongly implicated in the homotopy invariance results for stack cohomology which appear in [16].

The definition of A-torsor and the homotopy classification result Theorem 3 have analogues in localized model categories of simplicial presheaves, provided that those model structures are right proper (so that Theorem 1 applies) – this is proved in [17]. The motivic model category of Morel and Voevodsky [MV] is an example of a localized model structure for which this result holds.

3.3 Stack Completion

The overall technique displayed in Section 3.2 specializes to give an explicit model for the stack associated to a sheaf of groupoids G, as in [18]. In general, the stack associated to G has global sections with objects given by the "discrete" G-torsors. This construction of the stack associated to a groupoid G is a direct generalization of the classical observation that the groupoid of H-torsors forms the stack associated to a sheaf of groups H.

More explicitly, a discrete G-torsor is a G-torsor X as above, with the extra requirement that X is simplicially constant on a sheaf of vertices. Alternatively, since G is a sheaf of groupoids, one could say that X is a G-functor taking values in sheaves such that the map $\underrightarrow{\mathrm{holim}}_G X \to *$ is a weak equivalence.

A map $X \to Y$ of discrete G-torsors is a natural transformation, or equivalently (compare induced fibre sequences over BG) it is a sheaf isomorphism

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & \mathrm{Ob}(G) & \end{tikzcd}$$

fibred over $\mathrm{Ob}(G)$ which respects the actions. Write $G{-}\mathbf{Tors}_d$ for the corresponding groupoid of discrete G-torsors and their morphisms.

Note that if H is a sheaf of groups then the category $H - \mathbf{Tors}_d$ of discrete H-torsors coincides with the classical category $H - \mathbf{tors}$ of H-torsors discussed in Section 3.1.

Suppose that $* \xleftarrow{\simeq} Y \to BG$ is a cocycle, and form the G-diagram $\mathrm{pb}(Y)$ (of simplicial sets) by the pullbacks

$$\begin{tikzcd} \mathrm{pb}(Y)(U)_x \arrow[r] \arrow[d] & Y(U) \arrow[d] \\ BG(U)/x \arrow[r] & BG(U) \end{tikzcd}$$

Then by formal nonsense (and Quillen's Theorem B) there are weak equivalences

$$\underrightarrow{\mathrm{holim}}_G\, \tilde{\pi}_0\, \mathrm{pb}(Y) \simeq \underrightarrow{\mathrm{holim}}_G\, \mathrm{pb}(Y) \xrightarrow{\simeq} Y \simeq *$$

and so the G-diagram $\tilde{\pi}_0\,\mathrm{pb}(Y)$ is a discrete G-torsor. This is the discrete G-torsor which is associated to the cocycle $Y \to BG$. This construction is functorial in cocycles.

Suppose that x is a global section of the object sheaf $\mathrm{Ob}(G)$ of G. Then the map $B(G/x) \to B(G)$ is a cocycle of BG, and the associated discrete torsor is isomorphic to the G-diagram $G(\ ,x)$ which is defined in sections by $a \mapsto G(a,x)$, and which sends a morphism $\alpha : a \to b$ to precomposition with α^{-1}. Any morphism $x \to y$ in global sections of G induces an morphism $G(\ ,x) \to G(\ ,y)$ of discrete G-torsors in the obvious way.

Every discrete G-torsor X on $\mathcal{C}$ restricts to a $G|_U$-torsor $X|_U$ on the site $\mathcal{C}/U$. It follows that there is a presheaf of groupoids $G-\mathbf{Tors}_d$ which is defined by the assignment $U \mapsto G|_U - \mathbf{Tors}_d$. The assignment

$$x \mapsto j(x) = G|_U(\ ,x)$$

for $x \in \mathrm{Ob}(G)(U)$ defines a morphism

$$j : G \to G - \mathbf{Tors}_d$$

of presheaves of groupoids.

The homotopy colimit and pullback functors determine adjoint functors [8, VI.4.6]

$$\mathrm{pb} : H(*, BG) \leftrightarrows G - \mathbf{Tors} : \underrightarrow{\mathrm{holim}}\,.$$

The inclusion

$$G - \mathbf{Tors}_d \subset G - \mathbf{Tors}$$

has a left adjoint, defined by $X \mapsto \tilde{\pi}_0 X$, where $\tilde{\pi}_0 X$ is the sheaf associated to the presheaf $\pi_0 X$ of path components of X. It follows that there are weak equivalences

$$B(G - \mathbf{Tors}_d) \simeq B(G - \mathbf{Tors}) \simeq BH(*, BG) \tag{1}$$

provided that one bounds the size of the objects in $G - \mathbf{Tors}$ and cocycles in $H(*, BG)$ suitably.

There is such a bound, because the discrete G-torsors are bounded above in sections by cardinals bigger than the sections of G. In effect, global sections $\alpha \in X(x)$

for a discrete G-torsor X are classified by isomorphisms of torsors $H(\ ,x) \to X$, so that all non-empty sets of sections $X(U)$ are isomorphic to some $G|_U(\ ,x)(*)$ after restriction to $\mathcal{C}/U$.

The corresponding weak equivalences

$$B(G|_U - \mathbf{Tors}_d) \simeq B(G|_U - \mathbf{Tors}) \simeq BH(*, BG|_U)$$

respect composition with all functors $\mathcal{C}/V \to \mathcal{C}/U$ induced by morphisms $V \to U$ of $\mathcal{C}$, and determine sectionwise equivalences of simplicial presheaves.

Write $\mathbf{H}(*, BG)$ for the presheaf of categories defined by

$$\mathbf{H}(*, BG)(U) = H(*, BG|_U).$$

Observe that the composite

$$BG(U) \xrightarrow{j_*} B(G|_U - \mathbf{Tors}_d) \simeq B(G|_U - \mathbf{tors}) \simeq BH(*, BG|_U)$$

is induced by the functor $k : G(U) \to H(*, BG|_U)$ which associates to an object $x \in G(U)$ the cocycle $B(G|_U/x) \to BG|_U$ on the site $\mathcal{C}/U$.

Theorem 4. *Suppose that G is a sheaf of groupoids. Then the map $k_* : BG \to B\mathbf{H}(*, BG)$ of simplicial presheaves which is induced by the functor $k : G \to \mathbf{H}(*, BG)$ is a local weak equivalence, and the simplicial presheaf $B\mathbf{H}(*, BG)$ satisfies descent.*

Recall that a simplicial presheaf X satisfies descent if any local weak equivalence $j : X \to Z$ with Z globally fibrant (aka. a globally fibrant model for X) is a sectionwise equivalence in the sense that all simplicial set maps $X(U) \to Z(U)$ are weak equivalences.

Proof (Proof of Theorem 4). One uses Theorem 1 (or just Lemma 2) to show that the map $k_* : BG(U) \to BH(*, BG|_U)$ induces an isomorphism in path components if G is a stack. The morphism $j : G \to G{-}\mathbf{Tors}_d$ induces isomorphisms $\mathrm{Aut}(x,x) \to \mathrm{Aut}(j(x), j(x))$ in automorphism groups for all sections x of $\mathrm{Ob}(G)$. It follows that the maps j_* and k_* are sectionwise equivalences if G is a stack.

More generally, let $j : G \to H$ be a fibrant model (or stack completion) for a sheaf of groupoids H. Then the induced maps

$$BH(*, BG|_U) \to BH(*, BH|_U)$$

are weak equivalences of simplicial sets by Lemma 3, so that the map

$$j_* : B\mathbf{H}(*, BG) \to B\mathbf{H}(*, BH)$$

is a sectionwise weak equivalence.

Thus, in the homotopy commutative diagram

$$\begin{tikzcd} BG \arrow[r, "k_*"] \arrow[d, "j_*"'] & B\mathbf{H}(*, BG) \arrow[d, "\simeq"] \\ BH \arrow[r, "\simeq", "k_*"'] & B\mathbf{H}(*, BH) \end{tikzcd}$$

the indicated maps are sectionwise equivalences, while j_* is a local weak equivalence. It follows that $k_* : BG \to B\mathbf{H}(*, BG)$ is a local weak equivalence. We also see that $B\mathbf{H}(*, BG)$ is sectionwise equivalent to a globally fibrant object, namely BH, and it follows that $B\mathbf{H}(*, BG)$ satisfies descent. □

Corollary 1. *Suppose that G is a sheaf of groupoids on a small Grothendieck site $\mathcal{C}$. Then the simplicial presheaf $B(G - \mathbf{Tors}_d)$ satisfies descent, and the simplicial presheaf map*

$$j : BG \to B(G - \mathbf{Tors}_d)$$

is a local weak equivalence.

In other words, the discrete G-torsors form a model for the stack completion of a sheaf of groupoids G, as does the presheaf of categories $\mathbf{H}(*, BG)$. Corollary 1 is the main result of [18], and appears there with a different proof. Theorem 4 is equivalent to Corollary 1, but the reformulation given here in terms of cocycle categories is new.

Finally, observe that the presheaf of categories $\mathbf{H}(*, BG)$ and the functor $k : G \to \mathbf{H}(*, BG)$ are both defined for presheaves of groupoids G, and that k commutes with morphisms of presheaves of groupoids up to homotopy. Theorem 4 therefore has a presheaf version:

Corollary 2. *Suppose that G is a presheaf of groupoids. Then the map $k_* : BG \to B\mathbf{H}(*, BG)$ of simplicial presheaves which is induced by the functor $k : G \to \mathbf{H}(*, BG)$ is a local weak equivalence, and the simplicial presheaf $B\mathbf{H}(*, BG)$ satisfies descent.*

Example 2. Suppose that F is a presheaf of sets on a Grothendieck site $\mathcal{C}$, and identify F with a discrete presheaf of groupoids. It is well known that the associated sheaf map $\eta : F \to \tilde{F}$ is a globally fibrant model for F, or equivalently that the discrete groupoid on the sheaf $\tilde{F}$ is the stack associated to F. It follows from Corollary 2 that there is a sectionwise isomorphism of presheaves

$$\pi_0 B\mathbf{H}(*, F) \cong \tilde{F} \tag{2}$$

so that the assignment $F \mapsto \pi_0 B\mathbf{H}(*, F)$ constructs the associated sheaf $\tilde{F}$. It also follows that the simplicial presheaf $B\mathbf{H}(*, F)$ is a sectionwise resolution of $\tilde{F}$ in the sense that we have the isomorphism (2), and all presheaves of higher homotopy groups of $B\mathbf{H}(*, F)$ are trivial.

I would like to thank a referee for pointing out the existence of the isomorphism (2). In isolation, the proof of this isomorphism does not require anything like Theorem 4 or Corollary 2; it is a straightforward consequence of Theorem 1.

3.4 Homotopy Colimits

Suppose that G is a presheaf of groupoids.

Write $s\,\mathrm{Pre}(\mathcal{C})^G$ for the category of G-diagrams in simplicial presheaves, defined for $G = A$ as above. Following [16], this category of G-diagrams has an injective model structure for which the weak equivalences (respectively cofibrations) are those maps of G-diagrams

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & \mathrm{Ob}(G) & \end{tikzcd}$$

for which the simplicial presheaf map $f : X \to Y$ is a local weak equivalence (respectively cofibration).

Write $s\,\mathrm{Pre}(\mathcal{C})/BG$ for the category of simplicial presheaf maps $X \to BG$, with morphisms

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr] & & Y \arrow[dl] \\ & BG & \end{tikzcd}$$

In a standard way, this category inherits a model structure from simplicial presheaves, for which a map as above is a weak equivalence (respectively cofibration, fibration) if and only if the simplicial presheaf map $g : X \to Y$ is a local weak equivalence (respectively cofibration, global fibration) of simplicial presheaves.

The homotopy colimit construction defines a functor

$$\underrightarrow{\mathrm{holim}}_G : s\,\mathrm{Pre}(\mathcal{C})^G \to s\,\mathrm{Pre}(\mathcal{C})/BG.$$

Since G is a presheaf of groupoids, this functor has a left adjoint

$$\mathrm{pb} : s\,\mathrm{Pre}(\mathcal{C})/BG \to s\,\mathrm{Pre}(\mathcal{C})^G,$$

which is defined at $X \to BG$ in sections for $U \in \mathcal{C}$ by pulling back the map $X(U) \to BG(U)$ over the canonical maps $BG(U)/x \to BG(U)$. The functor pb preserves cofibrations and weak equivalences (this by Quillen's Theorem B), but more is true: the functors pb and $\underrightarrow{\mathrm{holim}}_G$ define a Quillen equivalence

$$\mathrm{pb} : s\,\mathrm{Pre}(\mathcal{C})/BG \leftrightarrows s\,\mathrm{Pre}(\mathcal{C})^G : \underrightarrow{\mathrm{holim}}_G$$

relating the model structures for these categories. This result is Lemma 18 of [16].

Suppose now that X is a fixed G-diagram in simplicial presheaves, and write $G - \mathbf{Tors}/X$ for the category with objects consisting of G-diagram morphisms $A \to X$ with A a G-torsor. The morphisms of $G - \mathbf{Tors}/X$ are commutative diagrams

$$\begin{tikzcd} A \arrow[dd, "\theta"'] \arrow[dr] & \\ & X \\ B \arrow[ur] & \end{tikzcd}$$

of morphisms of G-diagrams. One can show that the map θ must be a weak equivalence of G-diagrams in all such pictures. The homotopy colimit construction defines a functor

$$\underrightarrow{\mathrm{holim}}_G : G - \mathbf{Tors}/X \to H(*, \underrightarrow{\mathrm{holim}}_G X)$$

by sending the G-diagram morphism $A \to X$ to the cocycle

$$* \xleftarrow{\simeq} \underrightarrow{\mathrm{holim}}_G A \to \underrightarrow{\mathrm{holim}}_A X.$$

On the other hand, given a cocycle $* \xleftarrow{\simeq} Y \to \underrightarrow{\mathrm{holim}}_G X$ the adjoint $\mathrm{pb}(Y) \to X$ is an object of $G - \mathbf{Tors}/X$, so the adjunction defines a functor

$$\mathrm{pb} : H(*, \underrightarrow{\mathrm{holim}}_G X) \to G - \mathbf{Tors}/X.$$

These functors are adjoint and therefore define inverse functions in path components, so we have the following:

Theorem 5. *Suppose that G is a presheaf of groupoids and that X is a G-diagram in simplicial presheaves. Then there are natural bijections*

$$\pi_0(G - \mathbf{Tors}/X) \cong \pi_0 H(*, \underrightarrow{\mathrm{holim}}_G X) \cong [*, \underrightarrow{\mathrm{holim}}_G X].$$

The identification of $\pi_0 H(*, \underrightarrow{\mathrm{holim}}_G X)$ with morphisms $[*, \underrightarrow{\mathrm{holim}}_G X]$ in the homotopy category of simplicial presheaves in the statement of Theorem 5 is a consequence of Theorem 1.

Theorem 5 is a generalization of Theorem 16 of [14], which deals with the case where G is a sheaf of groups and X is a sheaf (i.e. constant simplicial sheaf) with G-action.

4 Abelian Sheaf Cohomology

Suppose that A is a sheaf of abelian groups, and let $A \to J$ be an injective resolution of A, thought of as a $\mathbb{Z}$-graded chain complex and concentrated in negative degrees. Identify A with a chain complex concentrated in degree 0, and consider the shifted chain map $A[-n] \to J[-n]$. In particular, $A[-n]$ is the chain complex consisting of A concentrated in degree n. Recall that $K(A, n) = \Gamma A[-n]$ defines the Eilenberg–Mac Lane sheaf associated to A, where $\Gamma : \mathbf{Ch}_+ \to s\mathbf{Ab}$ is the functor appearing in the Dold–Kan correspondence. Let $K(J, n) = \Gamma T(J[-n])$ where $T(J[-n])$ is the good truncation of $J[-n]$ in non-negative degrees. In particular, $T(J[-n])_0$ is the kernel of the boundary map $J_{-n} \to J_{-n-1}$.

Write $\mathbb{Z}X$ for the free simplicial abelian group on a simplicial set X, and write NA for the normalized chain complex of a simplicial abelian group A. Let $\pi_{ch}(C, D)$ denote the chain classes of maps between presheaves of $\mathbb{Z}$-graded chain complexes C and D.

Lemma 5. *Every local weak equivalence of simplicial presheaves $f : X \to Y$ induces an isomorphism*

$$\pi_{ch}(N\mathbb{Z}Y, J[-n]) \xrightarrow{\cong} \pi_{ch}(N\mathbb{Z}X, J[-n])$$

in chain homotopy classes for all $n \geq 0$.

Proof. Starting with the third quadrant bicomplex $\hom(\mathbb{Z}X_p, J^q)$ one constructs a spectral sequence

$$E_2^{p,q} = \operatorname{Ext}^q(\tilde{H}_p(X), A) \Rightarrow \pi_{ch}(N\mathbb{Z}X, J[-p-q])$$

(see [10]). The map f induces a homology sheaf isomorphism $N\mathbb{Z}X \to N\mathbb{Z}Y$, and then a comparison of spectral sequences gives the desired result. □

Recall [3] that the category of $\mathcal{C}^{op}$-diagrams in simplicial sets has a projective model structure for which the fibrations (respectively weak equivalences) are the maps $f : X \to Y$ which are defined sectionwise (aka. pointwise) in the sense that each map $f : X(U) \to Y(U)$, $U \in \mathrm{Ob}(\mathcal{C})$ is a fibration (respectively weak equivalence) of simplicial sets.

If two chain maps $f, g : N\mathbb{Z}X \to J[-n]$ are chain homotopic, then the corresponding maps $f_*, g_* : X \to K(J, n)$ are right homotopic in the projective model structure for $\mathcal{C}^{op}$-diagrams. Choose a sectionwise trivial fibration $\pi : W \to X$ such that W is projective cofibrant. Then $f_*\pi$ is left homotopic to $g_*\pi$ for some choice of cylinder object $W \otimes I$ for W, again in the projective structure. This means that there is a diagram

$$\begin{tikzcd}
& W \arrow[r, "\pi"] \arrow[dl, "1"'] \arrow[d, "i_0"] & X \arrow[dr, "f_*"] & \\
W & W \otimes I \arrow[l, "s"'] \arrow[rr, "h"] & & K(J,n) \\
& W \arrow[r, "\pi"'] \arrow[ul, "1"] \arrow[u, "i_1"'] & X \arrow[ur, "g_*"'] &
\end{tikzcd}$$

where the maps s, i_0, i_1 are all part of the cylinder object structure for $W \otimes I$, and are sectionwise weak equivalences. We therefore have relations

$$(1, f_*) \sim (\pi, f_*\pi) \sim (\pi s, h) \sim (\pi, g_*\pi) \sim (1, g_*)$$

in $\pi_0 H(X, K(J,n))$, where $H(X, K(J,n))$ is a cocycle category for the local model structure on simplicial presheaves. It follows that there is a well defined abelian group homomorphism

$$\phi : \pi_{ch}(N\mathbb{Z}X, J[-n]) \to \pi_0 H(X, K(J,n)).$$

This map is natural in simplicial presheaves X.

Lemma 6. *The map*

$$\phi : \pi_{ch}(N\mathbb{Z}X, J[-n]) \to \pi_0 H(X, K(J,n)).$$

is an isomorphism.

Proof. Suppose that $X \xleftarrow{f} Z \xrightarrow{g} K(J,n)$ is an object of $H(X, K(J,n))$. The map f is a local weak equivalence, so by Lemma 5 there is a unique chain homotopy class $[v] : N\mathbb{Z}X \to J[-n]$ such that $[v_* f] = [g]$. This chain homotopy class $[v]$ is independent of the choice of representative for the component of (f, g). We therefore have a well defined function

$$\psi : \pi_0 H(X, K(J,n)) \to \pi_{ch}(N\mathbb{Z}X, J[-n]).$$

The composites $\psi \cdot \phi$ and $\phi \cdot \psi$ are identity morphisms. □

Corollary 3. *Suppose that A is a sheaf of abelian groups on $\mathcal{C}$, and let $A \to J$ be an injective resolution of A in the category of abelian sheaves:*

1. Let X be a simplicial presheaf on $\mathcal{C}$. Then there is a natural isomorphism

$$\pi_{ch}(N\mathbb{Z}X, J[-n]) \cong [X, K(A,n)].$$

2. *There is a natural isomorphism*

$$H^n(\mathcal{C}, A) \cong [*, K(A,n)].$$

relating sheaf cohomology to morphisms in the homotopy category of simplicial presheaves (or sheaves).

Proof. This result is a consequence of Theorem 1 and Lemma 6. Observe that the map $K(A,n) \to K(J,n)$ is a local weak equivalence.

The second statement is a consequence of the first, and arises from the case where X is the terminal simplicial presheaf $*$. □

5 Group Extensions and 2-Groupoids

In this section we shall see that group extensions can be classified by path components of cocycles in 2-groupoids, by a very simple argument.

This is subject to knowing a few things about 2-groupoids and their homotopy theory. Recall that a 2-groupoid H is a groupoid enriched in groupoids. Equivalently, H is a groupoid enriched in simplicial sets such that all simplicial sets of morphisms $H(x, y)$ are nerves of groupoids. The object H is, in particular, a simplicial groupoid, and therefore has a bisimplicial nerve BH with associated diagonal simplicial set dBH.

One says that a map $G \to H$ of 2-groupoids is a weak equivalence if it induces a weak equivalence of simplicial sets $dBG \to dBH$. There is a natural weak equivalence of simplicial sets

$$dBH \simeq \overline{W}H$$

relating dBH to the space of universal cocycles $\overline{W}H$ [8, V.7], [20], so that $G \to H$ is a weak equivalence of 2-groupoids if and only if the induced map $\overline{W}G \to \overline{W}H$ is a weak equivalence of simplicial sets. One can also show that a map $G \to H$ is a weak equivalence if and only if it induces an isomorphism $\pi_0 G_0 \cong \pi_0 H_0$ of path components, and a weak equivalence of groupoids $G(x, y) \to H(f(x), f(y))$ for all objects x, y of G. Here, G_0 is the groupoid of 0-cells and 1-cells of G, or equivalently the groupoid in simplicial degree 0 for the corresponding groupoid enriched in simplicial sets.

The weak equivalences of 2-groupoids are part of a general picture: there is a model structure on groupoids enriched in simplicial sets, due to Dwyer and Kan (see [8, V.7.6, V.7.8]), for which a map $f : G \to H$ is a weak equivalence (respectively fibration) if and only if the induced map $f_* : \overline{W}G \to \overline{W}H$ is a weak equivalence (respectively fibration) of simplicial sets. This model structure is Quillen equivalent to the standard model structure for simplicial sets [8, V.7.11]. The model structure for groupoids enriched in simplicial sets restricts to a model structure for 2-groupoids, having the same definitions of fibration and weak equivalence [23], and it is easy to see that both model structures are right proper.

Here are some simple examples of 2-groupoids:

1. If K is a group, then there is a 2-groupoid $\mathbf{Aut}(K)$ with a single 0-cell, with 1-cells given by the automorphisms of K, and with 2-cells given by homotopies (aka. conjugacies) of automorphisms of K.
2. Suppose that $p : G \to G'$ is a surjective group homomorphism. Then there is a 2-groupoid $\tilde{p}$ with a single 0-cell, 1-cells given by the morphisms of G, and there is a 2-cell $g \to h$ if and only if $p(g) = p(h)$.

Suppose that K is the kernel of $p : G \to G'$. Then there are canonical morphisms of 2-groupoids

$$G' \xleftarrow{\pi} \tilde{p} \xrightarrow{F} \mathbf{Aut}(K), \tag{1}$$

The map π is p on 1-cells, and takes a 2-cell $g \to h$ to the identity on $p(g) = p(h)$, whereas the map F takes a 1-cell g to conjugation by g, and takes the 2-cell $g \to h$ to conjugation by $hg^{-1} \in K$. The map $\pi : \tilde{p} \to G'$ is a weak equivalence of 2-groupoids, since the groupoid of 1-cells and 2-cells of $\tilde{p}$ is the "Čech groupoid" associated to the underlying surjective function $G \to G'$. In general, if $f : X \to Y$ is a surjective function, then the associated Čech groupoid has objects given by the elements of X, and a unique morphism $x \to y$ if and only if $f(x) = f(y)$.

We have therefore produced a cocycle (1) in 2-groupoids from a short exact sequence

$$e \to K \xrightarrow{i} G \xrightarrow{p} G' \to e$$

Write $\mathbf{Ext}(G', K)$ for the usual groupoid of all such exact sequences. The cocycle construction is natural, and defines a functor

$$\phi : \mathbf{Ext}(G', K) \to H(G', \mathbf{Aut}(K))$$

taking values in the cocycle category $H(G', \mathbf{Aut}(K))$ in pointed 2-groupoids. All objects in the cocycle (1) have unique 0-cells, so the maps making up the cocycle are pointed in an obvious way.

Theorem 6. *The functor ϕ induces isomorphisms*

$$\pi_0\mathbf{Ext}(G', K) \cong \pi_0 H(G', \mathbf{Aut}(K)) \cong [BG', dB\mathbf{Aut}(K)]_*,$$

where $[\ ,\ ]_*$ denotes morphisms in the pointed homotopy category.

Proof. If the diagram

$$G' \xleftarrow{\pi} A \xrightarrow{F} \mathbf{Aut}(K)$$

is a pointed cocycle, then the base point $x \in A_0$ determines a 2-groupoid equivalence $A(x, x) \to G'$. The cocycle F can therefore be canonically replaced by its restriction to $A(x, x)$ at the base point x, and the 2-groupoid $A(x, x)$ can be identified with a 2-groupoid p_* arising from a surjective group homomorphism $p : L \to G'$ with 2-cells consisting of pairs (g, h) such that $p(g) = p(h)$.

Suppose given a cocycle

$$G' \xleftarrow{\pi} p_* \xrightarrow{F} \mathbf{Aut}(K)$$

where $\pi : p_* \to G'$ is determined a surjective group homomorphism $p : L \to G'$ as above. There is a group $E_F(p)$ which is the set of equivalence classes of pairs (k, x), $x \in L, k \in K$ such that $(k, x) \sim (k', x')$ if and only if $p(x) = p(x')$ and $k' = F(x, x')k$. Recall that $F(x, x')$ is a homotopy of the automorphisms $F(x), F(x')$ of K, and is therefore defined by conjugation by an element $F(x, x') \in K$. The product is defined by

$$[(k, x)] \cdot [(l, y)] = [(kF(x)(l), xy)]$$

and there is a short exact sequence

$$e \to K \to E_F(p) \to G' \to e$$

where $k \mapsto [(k, e)]$ and $[(k, x)] \mapsto p(x)$.

We have, with these constructions, described a functor

$$\psi : H(G', \mathbf{Aut}(K)) \to \mathbf{Ext}(G', K).$$

One shows that the associated function ψ_* on path components is the inverse of the function

$$\phi_* : \pi_0\mathbf{Ext}(G', K) \to \pi_0 H(G', \mathbf{Aut}(K)).$$

The homotopy category of groupoids enriched in simplicial sets is equivalent to the homotopy category of simplicial sets, and this equivalence is induced by the universal cocycles functor $\overline{W}$. It follows that there is a bijection

$$[G', \mathbf{Aut}(K)]_* \cong [BG', dB\mathbf{Aut}(K)]_*,$$

where the morphisms on the left are in the pointed homotopy category of groupoids enriched in simplicial sets. The set $[G', \mathbf{Aut}(K)]_*$ can be also identified with morphisms in the pointed homotopy category of 2-groupoids. One sees this by observing that for every cocycle

$$G' \xleftarrow{\simeq} B \to \mathbf{Aut}(K)$$

in groupoids enriched in simplicial sets, the object B is weakly equivalent to its fundamental groupoid, and so the cocycle can be canonically replaced by a cocycle in 2-groupoids. □

6 Classification of Gerbes

A gerbe is a stack G which is locally path connected in the sense that the sheaf of path components $\tilde{\pi}_0(G)$ is isomorphic to the terminal sheaf. Stacks are really just homotopy types of presheaves (or sheaves) of groupoids [9, 14, 22], so one may as well say that a gerbe is a locally connected presheaf of groupoids.

A morphism of gerbes is a morphism $G \to H$ of presheaves of groupoids which is a weak equivalence in the sense that the induced map $BG \to BH$ is a local weak equivalence of classifying simplicial sheaves. Write **gerbe** for the category of gerbes.

Write **Grp** for the "presheaf" of 2-groupoids whose objects are sheaves of groups, 1-cells are isomorphisms of sheaves of groups, and whose 2-cells are the homotopies of isomorphisms of sheaves of groups. The object **Grp** is not really a presheaf of 2-groupoids because it's too big in the sense that it does not take values in small 2-groupoids.

Write $H(*, \mathbf{Grp})$ for the category of cocycles

$$* \xleftarrow{\simeq} A \to \mathbf{Grp}$$

where A is a presheaf of 2-groupoids. One can sensibly discuss such a category, even though the object **Grp** is too big to be a presheaf. The category $H(*, \mathbf{Grp})$ is not small, and its path components do not form a set. Similarly, the path components of the category of gerbes do not form a set. It is, nevertheless, convenient to display the relationship between these objects in the following statement:

Theorem 7. *There is a bijection*

$$\pi_0 H(*, \mathbf{Grp}) \cong \pi_0(\mathbf{gerbe}).$$

The proof of Theorem 7 is a bit technical, and appears in [19]. It is relatively easy to say, however, how to get a cocycle from a gerbe G. Write $\tilde{G}$ for the 2-groupoid whose 0-cells and 1-cells are the objects and morphisms of G, respectively, and say that there is a unique 2-cell $\alpha \to \beta$ between any two arrows $\alpha, \beta{:}x \to y$. Then the canonical map $\tilde{G} \to *$ is an equivalence. There is a map $F(G) : \tilde{G} \to \mathbf{Grp}$ which associates to $x \in G(U)$ the sheaf $G(x, x)$ of automorphisms of x on $\mathcal{C}/U$, associates to $\alpha : x \to y$ the isomorphism $G(x, x) \to G(y, y)$ defined by conjugation by α, and associates to a 2-cell $\alpha \to \beta$ the homotopy defined by conjugation by $\beta\alpha^{-1}$. This cocycle construction effectively defines the function

$$\pi_0(\mathbf{gerbe}) \to \pi_0 H(*, \mathbf{Grp}).$$

A generalized Grothendieck construction is used to define its inverse – the construction of a group from a cocycle in the proof of Theorem 6 is a special case.

One can go further: the gerbes with band $L \in H^1(\mathcal{C}, \mathbf{Out})$ are classified by cocycles in the homotopy fibre over L of a morphism of fibrant presheaves of 2-groupoids approximating the map

$$\mathbf{Grp} \to \mathbf{Out}.$$

Here, **Out** is the groupoid of outer automorphisms, or automorphisms modulo the homotopy relation. One can also use the same techniques to classify gerbes locally equivalent to a fixed gerbe G. These results are also proved in [19].

Remark 2. Suppose that E is a sheaf. An E-gerbe is a morphism of presheaves of groupoids $G \to E$ such that the induced map $\tilde{\pi}_0 G \to E$ is an isomorphism of sheaves. Write $\mathbf{gerbe}/E$ for the category of E-gerbes. An E-gerbe is canonically a gerbe in the category of presheaves on the site $\mathcal{C}/E$ fibred over E [16]. It follows that Theorem 7 specializes to a homotopy classification statement

$$\pi_0(\mathbf{gerbe}/E) \cong \pi_0 H(E, \mathbf{Grp})$$

for E-gerbes.

7 The Parabolic Groupoid

Write $\mathcal{F}$ for the category of finite subsets of a countable set ω, and let $\mathrm{Mon}(\mathcal{F})_n$ be the set of all strings of subset inclusions

$$F : \ F_1 \subset F_2 \subset \cdots \subset F_n.$$

Say that such a string is a *formal flag* of length n, and write $\mathrm{Mon}(\mathcal{F})_n$ for the set of all such objects.

The set $\mathrm{Mon}(\mathcal{F})_n$ is the set of n-simplices of a simplicial set $\mathrm{Mon}(\mathcal{F})$. To see this, write $F_0 = \emptyset$ for each formal flag, and let $\theta : \mathbf{m} \to \mathbf{n}$ be an ordinal number morphism. Then the formal flag $\theta^*(F)$ (of length m) is the sequence of inclusions

$$\theta^* F : \ F_{\theta(1)} - F_{\theta(0)} \subset F_{\theta(2)} - F_{\theta(0)} \subset \cdots \subset F_{\theta(m)} - F_{\theta(0)}.$$

Suppose that T is an S-scheme, and write $\mathbf{Mod}(T)$ for the category of $\mathcal{O}_T$-modules on the T-scheme category $\mathrm{Sch}|_T$. Then the various categories of $\mathcal{O}_T$-modules $\mathbf{Mod}(T)$ assemble to form a presheaf of categories $\mathbf{Mod}$ on $\mathrm{Sch}|_S$.

Every finite set F determines a free $\mathcal{O}_S$-module $\mathcal{O}_S(F)$, and every function $F \to F'$ induces a morphism $\mathcal{O}_S(F) \to \mathcal{O}_S(F')$. It follows that there is a functor $\mathcal{O}_S : \mathcal{F} \to \mathbf{Mod}(S)$ taking values in $\mathcal{O}_S$-modules, and a corresponding morphism of presheaves of categories $\mathcal{O}_S : \Gamma^* \mathcal{F} \to \mathbf{Mod}$.

A morphism $\alpha : F \to F'$ of formal flags is a collection of $\mathcal{O}_S$-module homomorphisms $\alpha_k : \mathcal{O}_S(F_k) \to \mathcal{O}_S(F'_k)$, $1 \le k \le n$ such that the diagram

$$\begin{tikzcd}
\mathcal{O}_S(F_1) \arrow[r] \arrow[d, "\alpha_1"'] & \mathcal{O}_S(F_2) \arrow[r] \arrow[d, "\alpha_2"] & \cdots \arrow[r] & \mathcal{O}_S(F_n) \arrow[d, "\alpha_n"] \\
\mathcal{O}_S(F_1') \arrow[r] & \mathcal{O}_S(F_2') \arrow[r] & \cdots \arrow[r] & \mathcal{O}_S(F_n')
\end{tikzcd}$$

commutes. The formal flags of length n and their homomorphisms form an additive category, which will be denoted by $\mathbf{Fl}_n(S)$. Write $\mathbf{Par}_n(S)$ for the groupoid of the formal flags of length n and their isomorphisms.

Every formal flag morphism $\alpha : F \to F'$ uniquely induces a formal flag morphism $\theta^*(\alpha) : \theta^* F \to \theta^* F'$ for all ordinal number maps $\theta : \mathbf{m} \to \mathbf{n}$. It follows that the categories $\mathbf{Fl}_n(S)$ form a simplicial category $\mathbf{Fl}(S)$. Similarly, the groupoids $\mathbf{Par}_n(S)$ form a simplicial groupoid $\mathbf{Par}(S)$.

The simplicial category $\mathbf{Fl}(S)$ is the global sections object of a presheaf of simplicial categories defined on $Sch|_S$, which is denoted by $\mathbf{Fl}$. Similarly, the simplicial groupoid $\mathbf{Par}(S)$ is the global sections object of a simplicial presheaf of groupoids $\mathbf{Par}$ on the category of S-schemes. I say that $\mathbf{Par}$ is the *parabolic groupoid*.

Write $\mathrm{Ar}(\mathbf{n})$ for the category of arrows $i \leq j$ in the ordinal number $\mathbf{n}$. Let M be an exact category, and recall that Waldhausen's category $S_n M$ is defined to have objects consisting of all functors $P : \mathrm{Ar}(\mathbf{n}) \to M$, such that:

1. $P(i,i) = 0$ for all i.
2. All sequences
$$0 \to P(i,j) \to P(i,k) \to P(j,k) \to 0$$
are exact for $i \leq j \leq k$.

The morphisms of $S_n M$ are the natural transformations between diagrams in M. The categories $S_n(M)$ form a simplicial category $S_\bullet M$, and the simplicial set of objects $s_\bullet M = \mathrm{Ob}(S_\bullet M)$ is Waldhausen's $s_\bullet$-construction. Recall that there are natural weak equivalences:

1. $s_\bullet M \simeq BQ(M)$.
2. $s_\bullet M \simeq B\,\mathrm{Iso}\, S_\bullet M$, where $\mathrm{Iso}\, S_\bullet M$ is the simplicial groupoid of isomorphisms in $S_\bullet M$.

The assignment of the string of admissible monomorphisms

$$P(0,1) \to P(0,2) \to \cdots \to P(0,n)$$

to the object $P \in S_n M$ determines an equivalence of categories

$$m : S_n M \to \mathrm{Mon}_n(M),$$

where $\mathrm{Mon}_n(M)$ denotes the category of strings of admissible monomorphisms of length n.

Let $\mathcal{P}(S)$ denote the full subcategory of the category of $\mathcal{O}_S$-modules which consists of $\mathcal{O}_S$-modules which are locally free of finite rank. This is the category of "big site" vector bundles, defined on the category $\mathrm{Sch}\,|_S$.

Suppose that $F: F_1 \subset \cdots \subset F_n$ is a formal flag of length n. Then F determines an object $P(F) \in S_n\mathcal{P}(S)$ with

$$P(F)(i,j) = \mathcal{O}_S(F_j - F_i).$$

If $(i,j) \leq (k,l)$ is an arrow morphism, then the induced map

$$\bigoplus_{F_j - F_i} \mathcal{O}_S \to \bigoplus_{F_l - F_k} \mathcal{O}_S$$

is defined on summands corresponding to $x \in F_j - F_i$ to be 0 if $x \in F_k$ and is the inclusion

$$in_x : \mathcal{O}_S \to \bigoplus_{F_l - F_k} \mathcal{O}_S$$

if $x \notin F_k$.

The assignments $F \mapsto P(F)$ define a morphism of simplicial categories

$$P: \mathbf{Fl}(S) \to S_\bullet\mathcal{P}(S) \subset S_\bullet(\mathbf{Mod}(S)).$$

This morphism P restricts to a simplicial groupoid morphism

$$P: \mathbf{Par}(S) \to \mathrm{Iso}\, S_\bullet\mathcal{P}(S),$$

and this latter morphism is global sections of a morphism

$$P: \mathbf{Par} \to \mathrm{Iso}\, S_\bullet\mathcal{P},$$

of presheaves of simplicial groupoids on $Sch|_S$.

There is a functor $L: H(*, B\mathbf{Par}_n) \to \mathrm{Iso}\, S_n\mathcal{P}(S)$ which is defined by taking colimits.

In effect, every cocycle $Y \to B\mathbf{Par}_n$ can be identified with a functor $f: H \to \mathbf{Par}_n$ of presheaves of groupoids with $BH \simeq *$ (here, H is the fundamental groupoid πY of Y). The functor f determines a composite functor

$$f_*: H \xrightarrow{f} \mathbf{Par}_n \xrightarrow{P} \mathrm{Iso}\, S_n\mathcal{P}(S) \subset \mathrm{Iso}\, S_n(\mathbf{Mod}(S))$$

and hence determines functors

$$f_*(i,j): H \to \mathbf{Mod}(S),\ 0 \leq i \leq j \leq n,$$

with natural transformations between them determined by the morphisms

$$(i,j) \leq (k,l)$$

of the arrow category $\mathrm{Ar}(\mathbf{n})$. The abelian sheaf colimits

$$L(f)(i,j) = \varinjlim_{H} f_*(i,j)$$

define an object of $S_n(\mathbf{Mod}(S)))$ which is locally isomorphic to $P(F)$ for some formal flag of length n. To see this, note that if H has a global section $x \in \mathrm{Ob}(H)(S)$, then $f_*(x)$ is the object $P(F_x)$ for some formal flag $F_x : F_1 \subset \cdots \subset F_n$, and the local weak equivalence $x : * \to H$ determines sheaf isomorphisms

$$P(F_x)(i,j) \xrightarrow{\cong} \varinjlim_{H} f_*(i,j)$$

which are natural with respect to morphisms of $\mathrm{Ar}(\mathbf{n})$. More generally, H has sections locally, so that the diagram $L(f)$ is locally isomorphic to some $P(F)$.

The case $n = 1$ of the colimit construction L defines a functor

$$L : H(*, \sqcup_{n\geq 0} BGl_n) \to \mathrm{Iso}\,\mathcal{P}(S),$$

which can be viewed as a disjoint union of functors

$$L : H(*, BGl_n) \to \mathrm{Iso}\,\mathcal{P}_n(S),$$

where $\mathcal{P}_n(S)$ denotes the category of vector bundles of rank n.

The category $GL_n - \mathbf{tors}$ on S for the Zariski topology is the category of Gl_n-torsors, and the classical method of defining the well-known isomorphism

$$\pi_0(Gl_n - \mathbf{tors}) \xrightarrow{\cong} \pi_0\,\mathrm{Iso}\,\mathcal{P}_n(S) \tag{1}$$

starts by taking a representing cocycle $C(U) \to BGl_n$ for a torsor X on some trivializing open cover U of S, viewed as a functor $\pi C(U) \to Gl_n$, and then patching copies of $\mathcal{O}_U^n$ together in the $\mathcal{O}_S$-module category along the cocycle. The patched-together module is locally free of rank n, and is the colimit of the composite functor

$$\pi C(U) \to Gl_n = \mathrm{Aut}(\mathcal{O}_S^n) \subset \mathbf{Mod}(S).$$

The choice of representing cocycle $C(U) \to BGl_n$ does not matter up to isomorphism, and so one may as well make the canonical choice, which is the map $EGl_n \times_{Gl_n} X \to BGl_n$ which is associated to a Gl_n-torsor X, and is the output of the equivalence ψ given in (1) in the case of the group Gl_n. Thus, if we can show that the composite functor

$$Gl_n - \mathbf{tors} \underset{\simeq}{\xrightarrow{\psi}} H(*, BGl_n) \xrightarrow{L} \mathrm{Iso}\,\mathcal{P}_n(S) \tag{2}$$

is a weak equivalence of groupoids, we end up verifying that the function displayed in (1) is indeed an isomorphism. This is not hard: it suffices to show that the functor $L\psi$ induces an epimorphism in path components (one uses the observation that every vector bundle is the colimit of a trivializing cocycle to see this) and induces an isomorphism

$$\mathrm{Aut}(Gl_n) \xrightarrow{\cong} \mathrm{Aut}(\mathcal{O}_S^n),$$

where Gl_n is identified with the trivial Gl_n-torsor. All Gl_n-torsors are locally trivial, so a patching argument then implies that the composite functor is fully faithful.

The argument of the last paragraph is the classical argument for the existence of the bijection (1), and this same argument can be used for all of the standard parabolic groups. There is a similar composite functor

$$\mathrm{Aut}(F) - \mathbf{tors} \xrightarrow[\simeq]{\psi} H(*, B\,\mathrm{Aut}(F)) \xrightarrow{L} \mathrm{Iso}\,\mathrm{Mon}_n\,\mathcal{P}(S, F)$$

for each formal flag F of length n, where $\mathrm{Mon}_n\,\mathcal{P}(S, F)$ is the category of strings of locally split monomorphisms of vector bundles which are locally isomorphic to $\mathcal{O}_S(F)$, and one can use the same techniques as for the general linear group Gl_n to show that this composite is a weak equivalence of groupoids. This weak equivalence induces a bijection in path components, which is the classical identification of isomorphism classes of $\mathrm{Aut}(F)$-torsors with isomorphism classes of flags locally isomorphic to $\mathcal{O}_S(F)$.

One can summarize as follows:

Theorem 8. *1. The functor L induces a weak equivalence*

$$BH(*, B\mathbf{Par}_n) \simeq B\,\mathrm{Iso}\,S_n\mathcal{P}(S).$$

for all schemes S.

2. The morphism $P : \mathbf{Par}_n \to \mathrm{Iso}\,S_n\mathcal{P}$ of presheaves of groupoids is sectionwise weakly equivalent to the Zariski stack completion $\mathbf{Par}_n \to St\mathbf{Par}_n$.

Proof. To prove statement 1), we show that the composite functor

$$H(*, B\mathbf{Par}_n) \xrightarrow{L} \mathrm{Iso}\,S_n\mathcal{P}(S) \xrightarrow{m} \mathrm{Iso}\,\mathrm{Mon}_n(S)$$

is a weak equivalence of categories for all schemes S. Despite the abuse of notation, write $L = mL$ for this composite. The composite is defined by colimit in each index.

Suppose that

$$P_1 \to P_2 \to \cdots \to P_n$$

is a string of locally split monomorphisms of vector bundles. For a Zariski open covering $U \subset S$, this string of split monomorphisms is isomorphic to a string

$$\mathcal{O}_U(F_1) \to \cdots \to \mathcal{O}_U(F_n)$$

arising from a formal flag $F : F_1 \subset \cdots \subset F_n$. It follows that P is locally isomorphic to $L(F)$ for some formal flag F, and so the induced map

$$\pi_0 H(*, B\mathbf{Par}_n) \to \pi_0 \operatorname{Iso} S_n \mathcal{P}(S)$$

is surjective.

Suppose that $\underline{m} : m_1 \le m_2 \le \cdots \le m_n$ is a string of non-negative integers, and choose a fixed formal flag $F_{\underline{m}}$ such that $|F_i| = m_i$ for each such $\underline{m}$. Then there is a sectionwise weak equivalence

$$\bigsqcup_{\underline{m}} \operatorname{Aut}(F_{\underline{m}}) \to \mathbf{Par}_n \tag{3}$$

of presheaves of groupoids, and an induced weak equivalence

$$\bigsqcup_{\underline{m}} \operatorname{Aut}(F_{\underline{m}}) - \mathbf{tors} \to \mathbf{Par}_n - \mathbf{Tors}_d$$

The functor L restricts to a functor

$$L : H(*, B\operatorname{Aut}(F)) \to \operatorname{Iso} \operatorname{Mon}_n \mathcal{P}(S, F)$$

and the existence of the decomposition (3) implies that it suffices to show that this restricted functor is fully faithful for all F.

But we know from the "classical" argument given above that the composite

$$\operatorname{Aut}(F) - \mathbf{tors} \xrightarrow{\psi} H(*, B\operatorname{Aut}(F)) \xrightarrow{L} \operatorname{Iso} \operatorname{Mon}_n \mathcal{P}(S, F)$$

is a weak equivalence of groupoids, where ψ is the equivalence of (1). $\square$

The ordinary category of vector bundles $\operatorname{Vb}(S)$ is defined on the small Zariski site $Zar|_S$ for the scheme S. Restriction to the small site defines a morphism of simplicial groupoids

$$\operatorname{Iso} S_\bullet \mathcal{P}(S) \to \operatorname{Iso} S_\bullet \operatorname{Vb}(S), \tag{4}$$

and each groupoid morphism

$$\operatorname{Iso} S_n \mathcal{P}(S) \to \operatorname{Iso} S_n \operatorname{Vb}(S).$$

is a weak equivalence. To see this, note that there is an analogue of Theorem 8, with the same proof, which asserts that $\operatorname{Iso} S_n \operatorname{Vb}$ is the stack completion of the restriction of the groupoid $\mathbf{Par}_n$ to the small Zariski site $Zar|_S$, and then observe that restriction from the big site to the small site respects stack completion up to equivalence. It follows that the maps (4) are weak equivalences of simplicial groupoids, and so the standard K-theory space is weakly equivalent to the analogue defined with vector bundles on the big site.

Corollary 4. *The bisimplicial presheaves*

$$\mathbf{n} \mapsto B\mathbf{H}(*, \mathbf{Par}_n) \textit{ and } \mathbf{n} \mapsto B\,\mathrm{St}\,\mathbf{Par}_n$$

are sectionwise weakly equivalent, and are both models for the algebraic K-theory presheaf K^1 on the big Zariski site $(\mathrm{Sch}\,|_S)_{Zar}$.

Choose an open cover U of the scheme S, and write $U \to *$ for the corresponding sheaf epimorphism for the Zariski topology on S. Let F be a fixed formal flag, let $\mathrm{Aut}(F)$ be the corresponding parabolic group, and choose a homotopy inverse $\omega_U^{-1} : \mathrm{Aut}(F)^{C(U)} \to \mathrm{Aut}(F) - \mathbf{tors}_U$ of the weak equivalence of groupoids ω_U arising from Lemma 4. Let $\mathrm{Iso}\,\mathrm{Mon}_n\,\mathcal{P}(S, F)_U$ be the groupoid of isomorphisms of strings of locally split monomorphisms locally isomorphic to F and which trivialize over the cover U. Then the weak equivalence

$$L : H(*, B\,\mathrm{Aut}(F)) \to \mathrm{Iso}\,\mathrm{Mon}_n\,\mathcal{P}(S, F)$$

of the proof of Theorem 8 specializes to a weak equivalence

$$L_U : H(*, B\,\mathrm{Aut}(F))_U \to \mathrm{Iso}\,\mathrm{Mon}_n\,\mathcal{P}(S, F)_U,$$

by the same argument.

In the language of this paper, Schechtman shows in [26, Theorem 1.9] that the composite

$$\mathrm{Aut}(F)^{C(U)} \xrightarrow{\omega_U^{-1}} \mathrm{Aut}(F) - \mathbf{tors}_U \xrightarrow{\psi} H(*, B\,\mathrm{Aut}(F))_U \xrightarrow{L_U} \mathrm{Iso}\,\mathrm{Mon}_n\,\mathcal{P}(S, F)_U \tag{5}$$

is an equivalence of groupoids. Thus, the techniques displayed here give Schechtman's result. On the other hand, every torsor is trivialized along some open cover, so that the assertion that all maps L_U are weak equivalences implies Theorem 8.

The weak equivalence given by Schechtman's Theorem 1.9 is more general than that displayed in (5) – the full statement of his result is for the multiple $S_\bullet$-construction. Theorem 8 also has a multiple $S_\bullet$-version, but its statement is too fussy to reproduce here; in any case, its proof does not involve any new ideas.

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A Survey of Elliptic Cohomology

J. Lurie

Abstract This paper is an expository account of the relationship between elliptic cohomology and the emerging subject of *derived algebraic geometry*. We begin in Sect. 1 with an overview of the classical theory of elliptic cohomology. In Sect. 2 we review the theory of E_∞-ring spectra and introduce the language of derived algebraic geometry. We apply this theory in Sect. 3, where we introduce the notion of an *oriented group scheme* and describe connection between oriented group schemes and equivariant cohomology theories. In Sect. 4 we sketch a proof of our main result, which relates the classical theory of elliptic cohomology to the classification of oriented elliptic curves. In Sect. 5 we discuss various applications of these ideas, many of which rely upon a special feature of elliptic cohomology which we call 2-*equivariance*.

The theory that we are going to describe lies at the intersection of homotopy theory and algebraic geometry. We have tried to make our exposition accessible to those who are not specialists in algebraic topology; however, we do assume the reader is familiar with the language of algebraic geometry, particularly with the theory of elliptic curves. In order to keep our account readable, we will gloss over many details, particularly where the use of higher category theory is required. A more comprehensive account of the material described here, with complete definitions and proofs, will be given in [1].

1 Elliptic Cohomology

1.1 Cohomology Theories

To any topological space X one can associate the *singular cohomology groups* $A^n(X) = H^n(X; \mathbf{Z})$. These invariants have a number of good properties, which

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are neatly summarized by the following axioms (which, in a slightly modified form, are due to Eilenberg and Steenrod: see [2]):

1. For each $n \in \mathbf{Z}$, A^n is a contravariant functor from the category of pairs of topological spaces $(Y \subseteq X)$ to abelian groups (We recover the *absolute* cohomology groups $A^n(X)$ by taking $Y = \emptyset$).
2. If $f : X' \to X$ is a weak homotopy equivalence (in other words, if it induces a bijection $\pi_0(X') \to \pi_0(X)$ and an isomorphism $\pi_n(X', x) \to \pi_n(X, fx)$ for every $n > 0$ and every base point $x \in X'$), then the induced map $A^n(X) \to A^n(X')$ is an isomorphism.
3. To every triple $Z \subseteq Y \subseteq X$, there is an associated long exact sequence
$$\ldots \to A^n(X,Y) \to A^n(X,Z) \to A^n(Y,Z) \stackrel{\delta^n}{\to} A^{n+1}(X,Y) \to \ldots$$
Here the connecting morphism δ^n is a natural transformation of functors.
4. Let $(Y \subseteq X)$ be a pair of topological spaces, and $U \subseteq X$ an open subset whose closure is contained in the interior of Y. Then the induced map
$$A^n(X,Y) \to A^n(X-U, Y-U)$$
is an isomorphism.
5. If X is a disjoint union of a collection of spaces $\{X_\alpha\}$, then for every integer n the induced map $A^n(X) \to \prod A^n(X_\alpha)$ is an isomorphism.
6. If X is a point, then
$$A^n(X) = \begin{cases} 0 & \text{if } n \neq 0 \\ \mathbf{Z} & \text{if } n = 0. \end{cases}$$

Any collection of functors (and connecting maps δ^n) satisfying the above axioms is necessarily isomorphic to the integral cohomology functors $(X \subseteq Y) \mapsto H^n(X, Y; \mathbf{Z})$. More generally, we can give a similar characterization of cohomology with coefficients in any abelian group M: one simply replaces $\mathbf{Z}$ by M in the statement of the dimension axiom (6).

A *cohomology theory* is a collection of functors A^n (and connecting maps δ^n) that satisfy the first five of the above axioms. Every abelian group M gives rise to a cohomology theory, by setting $A^n(X,Y) = H^n(X,Y;M)$. But there are many other interesting examples; the study of these examples is the subject of *stable homotopy theory*.

If A is a cohomology theory, we will write $A^n(X)$ for $A^n(X, \emptyset)$, and $A(X)$ for $A^0(X)$. Generally speaking, we will be interested in *multiplicative* cohomology theories: that is, cohomology theories for which each of the groups $A^*(X)$ is equipped with the structure of a *graded* commutative ring (graded commutativity means that $uv = (-1)^{nm} vu$ for $u \in A^n(X)$, $v \in A^m(X)$; we will not take the trouble to spell out the complete definition).

Arguably the most interesting example of a cohomology theory, apart from ordinary cohomology, is complex K-theory. If X is a reasonably nice space (for example, a finite cell complex), then $K(X)$ coincides with the Grothendieck group of stable isomorphism classes of complex vector bundles on X. Complex K-theory is a multiplicative cohomology theory, with multiplication determined by tensor products of complex vector bundles.

We observe that the dimension axiom fails dramatically for complex K-theory. Instead of being concentrated in degree 0, the graded ring $K^*(*) \simeq \mathbf{Z}[\beta, \beta^{-1}]$ is a ring of Laurent polynomials in a single indeterminate $\beta \in K^{-2}(*)$. Here β is called the *Bott element*, because multiplication by β induces isomorphisms

$$K^n(X) \to K^{n-2}(X)$$

for every space X and every integer n: this is the content of the famous Bott periodicity theorem (see [3]).

The following definition abstracts some of the pleasant properties of complex K-theory.

Definition 1.1. Let A be a multiplicative cohomology theory. We will say that A is *even* if $A^i(*) = 0$ whenever i is odd. We will say that A is *periodic* if there exists an element $\beta \in A^{-2}(*)$ such that β is invertible in $A^*(*)$ (so that β has an inverse $\beta^{-1} \in A^2(*)$).

Complex K-theory is the prototypical example of an even periodic cohomology theory, but there are many other examples. Ordinary cohomology $H^*(X; R)$ with coefficients in a commutative ring R is obviously even, but not periodic: in fact, we have $H^n(*; R) = 0$ for $n \neq 0$. However, we can correct this problem by enforcing periodicity: namely, define

$$A^n(X) = \prod_{k \in \mathbf{Z}} H^{n+2k}(X; R).$$

Then A is an even periodic cohomology theory: we refer to this theory as *periodic* cohomology with coefficients in R. We will meet more exotic examples of even periodic cohomology theories in Sect. 1.3.

1.2 Formal Groups from Cohomology Theories

Let A be a cohomology theory, and X a topological space. The Atiyah–Hirzebruch spectral sequence allows one to compute the A-cohomology groups of the space X in terms of the A-cohomology groups of a point and the *ordinary* cohomology groups of the space X. More specifically, there is a spectral sequence with

$$E_2^{pq} = H^p(X; A^q(*)) \Rightarrow A^{p+q}(X).$$

In general, the Atiyah–Hirzebruch spectral sequence can be quite complicated. However, if A is an even periodic cohomology theory, and X is a space whose ordinary cohomology is concentrated in even degrees, then the situation simplifies drastically: the Atiyah–Hirzebruch spectral sequence degenerates at E_2, since there are no possible differentials for dimensional reasons. The most important example is the case in which X is the infinite dimensional complex projective space $\mathbf{C}\mathrm{P}^\infty$; in this case one can compute that

$$A(\mathbf{C}\mathrm{P}^\infty)$$

is (noncanonically) isomorphic to a formal power series ring $A(*)[[t]]$ in one variable over the commutative ring $A(*)$.

There is an analogous computation for the ordinary cohomology ring of $\mathbf{C}\mathrm{P}^\infty$: we have

$$H^*(\mathbf{C}\mathrm{P}^\infty; \mathbf{Z}) = \mathbf{Z}[t],$$

where $t \in H^2(\mathbf{C}\mathrm{P}^\infty, \mathbf{Z})$ is the first Chern class of the universal line bundle $\mathcal{O}(1)$ on $\mathbf{C}\mathrm{P}^\infty$. We have used the 2-periodicity of the cohomology theory A to shift the generator from degree 2 to degree 0.

Remark 1.1. The reader might object that the analogy is imperfect, since the ordinary cohomology ring $H^*(\mathbf{C}\mathrm{P}^\infty, \mathbf{Z})$ is a polynomial ring, rather than a power series ring. However, this is dependent on our procedure for extracting an ordinary ring from a positively graded ring $\{R^n\}_{n \geq 0}$. The usual convention is to define $R = \oplus_{n \geq 0} R^n$; however, one could instead consider the product $\prod_{n \geq 0} R^n$. The latter is more natural in the present context, because it can also be interpreted as the (degree 0) *periodic* cohomology of the space $\mathbf{C}\mathrm{P}^\infty$, which is a power series ring rather than a polynomial ring.

By analogy with the case of ordinary cohomology, we may view the parameter t in the isomorphism $A(\mathbf{C}\mathrm{P}^\infty) \simeq A(*)[[t]]$ as the first Chern class of the universal line bundle $\mathcal{O}(1)$. In fact, once we fix a choice of the parameter t, we can use t to *define* the first Chern class of any complex line bundle $\mathcal{L}$ on any space X. The space $\mathbf{C}\mathrm{P}^\infty$ is a *classifying space* for complex line bundles: that is, for any complex line bundle $\mathcal{L}$ on a (paracompact) space X, there is a classifying map $\phi : X \to \mathbf{C}\mathrm{P}^\infty$ and an isomorphism $\mathcal{L} \simeq \phi^* \mathcal{O}(1)$. The map ϕ is unique up to homotopy, so we may define

$$c_1(\mathcal{L}) = \phi^* t \in A(X).$$

This gives rise to a reasonably well-behaved theory of the first Chern class in A-cohomology. (By elaborating on this construction, one can construct a theory of higher Chern classes as well, but we will not need this).

In ordinary cohomology, there is a simple formula that describes the first Chern class of a tensor product of two line bundles:

$$c_1(\mathcal{L} \otimes \mathcal{L}') = c_1(\mathcal{L}) + c_1(\mathcal{L}').$$

There is no reason to expect this formula to hold in general, and in fact it does not.

Example 1.1. In complex K-theory, there is an even simpler theory of Chern classes for line bundles. Namely, if $\mathcal{L}$ is a complex line bundle on a space X, then we may regard $\mathcal{L}$ itself as representing an element of $K(X)$; we will denote this element by $[\mathcal{L}]$. We now define $c_1(\mathcal{L}) = [\mathcal{L}] - 1$. (The reason for the subtraction is that we wish to normalize, so that the trivial line bundle has first Chern class equal to zero). Now a simple computation shows that

$$c_1(\mathcal{L} \otimes \mathcal{L}') = c_1(\mathcal{L}) + c_1(\mathcal{L}') + c_1(\mathcal{L})c_1(\mathcal{L}').$$

Returning to the general case, what we can assert is that there is always *some* formula which expresses $c_1(\mathcal{L} \otimes \mathcal{L}')$ in terms of $c_1(\mathcal{L})$ and $c_1(\mathcal{L}')$. To see this, it suffices to consider the universal example: that is, let X be the classifying space $\mathbf{CP}^\infty \times \mathbf{CP}^\infty$ for *pairs* of complex line bundles. Like $\mathbf{CP}^\infty$, this is a space with only even-dimensional cohomology, so a relatively simple computation shows that

$$A(\mathbf{CP}^\infty \times \mathbf{CP}^\infty) \simeq A(*)[[t_1, t_2]].$$

Here we can take $t_1 = \pi_1^* t$ and $t_2 = \pi_2^* t$, where π_1 and π_2 are the projections from the product $\mathbf{CP}^\infty \times \mathbf{CP}^\infty$ onto the first and second factor, respectively. Phrased another way, the power series generators t_1 and t_2 are the first Chern classes of the universal line bundles $\pi_1^* \mathcal{O}(1)$ and $\pi_2^* \mathcal{O}(1)$ on $\mathbf{CP}^\infty \times \mathbf{CP}^\infty$. We now observe that

$$c_1(\pi_1^* \mathcal{O}(1) \otimes \pi_2^* \mathcal{O}(1)) = f(t_1, t_2) \in A(*)[[t_1, t_2]]$$

for some (uniquely determined) power series f. Roughly speaking, we can assert that, by universality, we have $c_1(\mathcal{L} \otimes \mathcal{L}') = f(c_1(\mathcal{L}), c_1(\mathcal{L}'))$ for *any* pair of complex line bundles $(\mathcal{L}, \mathcal{L}')$ on *any* space X.

Remark 1.2. The above assertion is somewhat vague, since it is not clear how to evaluate a formal power series on a pair of elements in the commutative ring $A(X)$. However, if X admits a finite cell decomposition, then for any line bundle $\mathcal{L}$ on X, the element $c_1(\mathcal{L}) \in A(X)$ is nilpotent (more specifically, $c_1(\mathcal{L})^n = 0$ as soon as $\mathcal{L}$ is generated by n sections). For such spaces X, it *does* make sense to evaluate the formal series $f(c_1(\mathcal{L}), c_1(\mathcal{L}'))$, and one obtains $c_1(\mathcal{L} \otimes \mathcal{L}')$.

The power series $f(u, v)$ is not arbitrary; it necessarily satisfies certain identities which reflect the properties of complex line bundles under multiplication.

$(L1)$ Because the first Chern class of the trivial bundle is zero, we have the identities

$$f(t, 0) = f(0, t) = t.$$

$(L2)$ Because the tensor product operation on complex line bundles is commutative up to isomorphism, we obtain the identity $f(u, v) = f(v, u)$.

($L3$) Because the tensor product operation on complex line bundles is associative up to isomorphism,

$$f(u, f(v, w)) = f(f(u, v), w).$$

A power series $f(u, v)$ with the properties enumerated above is called a *commutative, 1-dimensional formal group law* over the commutative ring $A(*)$. Such a formal group law determines a group structure on the formal scheme Spf $A(*)[[t]] =$ Spf $A(\mathbf{C}\mathrm{P}^\infty)$.

Remark 1.3. The formal group law f is not canonically associated to the cohomology theory A: it depends on the choice of a power series generator t for the ring $A(\mathbf{C}\mathrm{P}^\infty) \simeq A(*)[[t]]$. However, the underlying formal group is independent of the choice of t: namely, it is given by the formal spectrum Spf $A(\mathbf{C}\mathrm{P}^\infty)$. The group structure

$$\mathrm{Spf}\, A(\mathbf{C}\mathrm{P}^\infty) \times \mathrm{Spf}\, A(\mathbf{C}\mathrm{P}^\infty) \to \mathrm{Spf}\, A(\mathbf{C}\mathrm{P}^\infty)$$

is induced by a map $s : \mathbf{C}\mathrm{P}^\infty \times \mathbf{C}\mathrm{P}^\infty \to \mathbf{C}\mathrm{P}^\infty$ which classifies the operation of tensor product on complex line bundles. In more concrete terms, we can identify $\mathbf{C}\mathrm{P}^\infty$ with the space of lines in the field of rational functions $\mathbf{C}(x)$ (viewed as an infinite dimensional complex vector space); the map s is now induced by the multiplication map $\mathbf{C}(x) \otimes \mathbf{C}(x) \to \mathbf{C}(x)$.

Remark 1.4. Not every 1-dimensional formal group $\mathbb{G}$ over a commutative ring R arises from a formal group law: this requires the existence of a coordinate on the group $\mathbb{G}$, which in general exists only Zariski locally on Spec R. Consequently, it is convenient to introduce a slightly more general notion of periodicity. We will say that a multiplicative cohomology theory A is *weakly periodic* if the natural map

$$A^2(*) \otimes_{A(*)} A^n(*) \to A^{n+2}(*)$$

is an isomorphism for all $n \in \mathbf{Z}$. Note that this implies in particular that $A^2(*) \otimes_{A(*)} A^{-2}(*) \simeq A(*)$, so that $A^2(*)$ is a projective module of rank 1 over $A(*)$. We note that A is periodic if and only if it is weakly periodic *and* $A^2(*)$ is a free module over $A(*)$.

Example 1.2. The formal power series $f(u, v) = u + v$ determines a formal group over an arbitrary commutative ring R, which we call the *formal additive group* and denote by $\widehat{\mathbb{G}}_a$. As we have seen, this is the formal group which governs the behavior of Chern classes in (periodic) ordinary cohomology (with coefficients in R).

Example 1.3. The formal power series $f(u, v) = u + v + uv$ determines a formal group over an arbitrary commutative ring R, which we call the *formal multiplicative group* and denote by $\widehat{\mathbb{G}}_m$. In the case where $R = \mathbf{Z}$, this is the formal group which governs the behavior of Chern classes in complex K-theory. The role of the multiplicative group is easy to understand in this case: modulo a normalization, we have essentially defined the first Chern class of a line bundle $\mathcal{L}$ to be the class $[\mathcal{L}]$ represented by $\mathcal{L}$ in K-theory. Tensor product of line bundles then correspond to products in K-theory.

It is natural to ask whether there are any restrictions on the power series $f(u, v)$, other than the identities $(L1)$, $(L2)$, and $(L3)$. To address this question, it is convenient to introduce a cohomology theory MP, called *periodic complex cobordism*. This cohomology theory is in some sense a universal home for the first Chern class of complex line bundles; in particular, there is a *canonical* isomorphism $\mathrm{MP}(\mathbf{C}\mathrm{P}^\infty) \simeq \mathrm{MP}(*)[[t]]$. Quillen proved that the associated formal group law over $\mathrm{MP}(*)$ was also universal. In other words, the coefficient ring $\mathrm{MP}(*)$ is the *Lazard ring* which classifies formal group laws, so that for any commutative ring R there is a bijection between the set $\mathrm{Hom}(\mathrm{MP}(*), R)$ of commutative ring homomorphisms from $\mathrm{MP}(*)$ into R, and the set of power series $f(u, v)$ with coefficients in R that satisfy the three identities asserted above. We refer the reader to [4] for a proof of Quillen's theorem and further discussion.

The construction that associates the formal group $\mathbb{G} = \mathrm{Spf}\, A(\mathbf{C}\mathrm{P}^\infty)$ to an even periodic cohomology theory A has turned out to be a very powerful tool in stable homotopy theory. The reason is that the formal group $\mathbb{G}$ retains a remarkable amount of information about A. In many cases, it is possible to recover A from the formal group $\mathbb{G}$. Indeed, suppose that R is *any* commutative ring and $\mathbb{G}$ a formal group over R determined by a formal group law $f(u, v) \in R[[u, v]]$. According to Quillen's theorem, this data is classified by a (uniquely determined) ring homomorphism $\mathrm{MP}(*) \to R$. There is a natural candidate for a cohomology theory $A_{\mathbb{G}}$ which gives rise to the formal group $\mathbb{G}$. Namely, for every *finite* cell complex X, set

$$A^n_{\mathbb{G}}(X) = \mathrm{MP}^n(X) \otimes_{\mathrm{MP}(*)} R.$$

Remark 1.5. To give a definition which does not involve finiteness restrictions on X, one should work with homology rather than cohomology; we will not concern ourselves with this technical point.

In general, the above formula for $A_{\mathbb{G}}$ does not give a cohomology theory. The problem is that long exact sequences in MP-cohomology might not remain exact after tensoring with R. If R is *flat* over $\mathrm{MP}(*)$, then this problem disappears. However, a much weaker condition on $(R, \mathbb{G})$ will ensure the same conclusion. In order to formulate it, it is convenient to employ the language of algebraic stacks.

Let $\mathcal{M}_{\mathrm{FGL}}$ denote the moduli stack of *formal group laws*, so that for any commutative ring R the set $\mathrm{Hom}(\mathrm{Spec}\, R, \mathcal{M}_{\mathrm{FGL}})$ may be identified with the set of all power series $f(u, v) \in R[[u, v]]$ satisfying $(L1)$, $(L2)$, and $(L3)$. Then $\mathcal{M}_{\mathrm{FGL}}$ is an affine scheme: in fact, by Quillen's theorem we have $\mathcal{M}_{\mathrm{FGL}} = \mathrm{Spec}\, \mathrm{MP}(*)$. Let G denote the group scheme of all automorphisms of the formal affine line $\mathrm{Spf}\, \mathbf{Z}[[x]]$; in other words, $\mathrm{Hom}(\mathrm{Spec}\, R, G)$ is the set of all power series

$$g(x) = a_1 x + a_2 x^2 + \dots$$

with coefficients in R, such that a_1 is invertible; such power series form a group under composition. The group G acts on $\mathcal{M}_{\mathrm{FGL}}$: on the level of R-valued points, this action is given by the formula

$$f^g(u, v) = g^{-1} f(g(u), g(v)).$$

Let $\mathcal{M}_{\mathrm{FG}}$ denote the stack-theoretic quotient of $\mathcal{M}_{\mathrm{FGL}}$ by G: this is the moduli stack of *formal groups*. There is a natural map

$$\mathcal{M}_{\mathrm{FGL}} \to \mathcal{M}_{\mathrm{FG}}$$

which "forgets the coordinate"; it is a principal bundle with structure group G.

Remark 1.6. One can think of $\mathcal{M}_{\mathrm{FG}}$ as a kind of infinite-dimensional Artin stack.

Now, for any space X, the MP-cohomology groups $\mathrm{MP}^n(X)$ are modules over the commutative ring $\mathrm{MP}(*)$. By Quillen's theorem, these modules can be identified with quasi-coherent sheaves on $\mathcal{M}_{\mathrm{FGL}}$. However, one can say more. If A is a cohomology theory, a *stable cohomology operation* (of degree k) is a collection of natural transformations

$$A^n(X, Y) \to A^{n+k}(X, Y)$$

which are suitably compatible with the connecting homomorphism. For example, if A is a multiplicative cohomology theory, then every element of $u \in A^k(*)$ gives rise to a stable cohomology operation, given by multiplication by u. However, in the case where $A = \mathrm{MP}$, there are many other stable cohomology operations. For any space X, the cobordism group $\mathrm{MP}(X)$ is a module over the ring of *all* stable cohomology operations on MP. An elaboration of Quillen's theorem can be used to give an algebro-geometric interpretation of this additional structure: when X is finite cell complex, the quasi-coherent sheaves $\mathrm{MP}^n(X)$ on $\mathcal{M}_{\mathrm{FGL}}$ are endowed with an action of the group G (covering the action of G on $\mathcal{M}_{\mathrm{FGL}}$). In other words, we may interpret the complex cobordism groups $\mathrm{MP}^n(X)$ as quasi-coherent sheaves $\mathrm{MP}^n(X)$ on the moduli stack $\mathcal{M}_{\mathrm{FG}}$.

Remark 1.7. The finiteness restrictions on X in the preceding discussion can be dropped if we are willing to work with the MP-homology groups of X, rather than the MP-cohomology groups.

Now suppose that $\mathbb{G}$ is a (commutative, 1-dimensional) formal group over a commutative ring R. Then $\mathbb{G}$ is classified by a map

$$\operatorname{Spec} R \xrightarrow{\phi} \mathcal{M}_{\mathrm{FG}}$$

and we may define

$$A^n(X) = \phi^* \mathrm{MP}^n(X)$$

for every finite cell complex X; this is a quasi-coherent sheaf on Spec R which we may identify with an R-module. Again, this does not necessarily define a cohomology theory: however, it does give a cohomology theory whenever the map ϕ is flat. If ϕ is flat, then we say that the formal group $\mathbb{G}$ is *Landweber-exact*. Using the structure theory of formal groups, Landweber has given a criterion for a formal group $\mathbb{G}$ to be Landweber-exact (see [5]). We will not review Landweber's theorem here. However, we remark that Landweber's criterion is purely algebraic, easy to check, and is quite often satisfied.

Remark 1.8. In the case where $\mathbb{G}$ is the formal group given by a formal group law $f(u, v) \in R[[u, v]]$, the preceding definition of $A^n(X)$ coincides with the definition given earlier. In general, the formal group law f exists only (Zariski) locally on Spec R; namely, it exists whenever the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ is free as an R-module. The present definition of A has the advantage that it does not depend on a choice of coordinate on the formal group $\mathbb{G}$, and makes sense even when the Lie algebra $\mathfrak{g}$ is not free. Note that if $\mathfrak{g}$ is not free, then A cannot be a *periodic* cohomology theory in the sense of Definition 1.1: instead, we have $A^{2k}(*) \simeq \mathfrak{g}^{\otimes(-k)}$, so that A is weakly periodic in the sense of Remark 1.4.

Example 1.4. Let $R = \mathbf{Z}$, and let $\widehat{\mathbb{G}}_m$ be the formal multiplicative group (determined by the formal group law $f(u, v) = u + v + uv$). In this case, Landweber's criterion is satisfied, so that

$$A^n(X) = \mathrm{MP}^n(X) \otimes_{\mathrm{MP}(*)} \mathbf{Z}$$

defines a cohomology theory (for finite cell complexes X). Moreover, this cohomology theory is uniquely determined by the fact that it is even, periodic, and is associated to the formal multiplicative group $\widehat{\mathbb{G}}_m$ over $A(*) \simeq \mathbf{Z}$. We saw in Example 1.3 that complex K-theory has all of these properties. We deduce that A *is* complex K-theory: in other words, we can discover K-theory in a purely algebraic fashion, by thinking about the formal multiplicative group over the integers.

Example 1.5. Let $\widehat{\mathbb{G}}_a$ be the formal additive group over a commutative ring R, defined by the formal group law $f(u, v) = u + v$. Then $\widehat{\mathbb{G}}_a$ is Landweber-exact if and only if R is an algebra over the field $\mathbf{Q}$ of rational numbers. In this case, the associated cohomology theory is just periodic cohomology with coefficients in R.

Remark 1.9. Over the field $\mathbf{Q}$ of rational numbers, the formal additive group is isomorphic to the formal multiplicative group (in fact, *all* commutative, 1-dimensional formal groups over $\mathbf{Q}$ are isomorphic to one another). This reflects the fact that rationally, complex K-theory reduces to periodic cohomology with rational coefficients. More precisely, for every finite cell complex X, the Chern character gives an isomorphism

$$\mathrm{ch} : K(X) \otimes_{\mathbf{Z}} \mathbf{Q} \to \prod_k H^{2k}(X; \mathbf{Q}).$$

The Chern character should be thought of as a cohomological reflection of the isomorphism $\exp : \widehat{\mathbb{G}}_a \xrightarrow{\sim} \widehat{\mathbb{G}}_m$.

1.3 Elliptic Cohomology

In the last section, we reviewed the relationship between cohomology theories and formal groups. In particular, we saw that the most basic examples of formal groups (namely, the formal additive group $\widehat{\mathbb{G}}_a$ and the formal multiplicative group $\widehat{\mathbb{G}}_m$) were closely related to the most basic examples of cohomology theories (namely, ordinary cohomology and complex K-theory). It is natural to try to expand on these examples, by seeking out other cohomology theories that are related to interesting formal groups.

A key feature of the formal groups $\widehat{\mathbb{G}}_a$ and $\widehat{\mathbb{G}}_m$ is that they arise as the formal completions of the algebraic groups $\mathbb{G}_a$ and $\mathbb{G}_m$. (The homotopy-theoretic significance of this observation will become clear later, when we discuss equivariant cohomology theories). Since we are interested only in commutative, 1-dimensional formal groups, it is natural to consider algebraic groups which are also commutative and 1-dimensional. However, examples are in short supply: over an algebraically closed field, every 1-dimensional algebraic group is isomorphic to either the additive group $\mathbb{G}_a$, the multiplicative group $\mathbb{G}_m$, or an elliptic curve. The cohomology theories associated to the first two examples are classical, but the third case is more exotic: this is the subject of *elliptic cohomology*, to which this paper is devoted.

Definition 1.2. An *elliptic cohomology theory* consists of the following data:

1. A commutative ring R.
2. An elliptic curve E, defined over R.
3. A multiplicative cohomology theory A which is even and weakly periodic (see Remark 1.4).
4. Isomorphisms $A(*) \simeq R$ and $\widehat{E} \simeq \operatorname{Spf} A(\mathbf{CP}^{\infty})$ (of formal groups over $R \simeq A(*)$). Here $\widehat{E}$ denotes the formal completion of the elliptic curve E along its identity section.

Remark 1.10. In the situation of Definition 1.2, we will often abuse terminology and simply refer to A as an elliptic cohomology theory. In other words, we think of an elliptic cohomology theory as a cohomology theory A, together with some additional data relating A to an elliptic curve.

Remark 1.11. Let E be an elliptic curve over a commutative ring R. The formal completion $\widehat{E}$ of E is a commutative, 1-dimensional formal group over R. If $\widehat{E}$ is Landweber-exact, then the data of (3) and (4) is uniquely determined. Consequently, in the Landweber-exact case, an elliptic cohomology theory is essentially the same thing as an elliptic curve.

Remark 1.12. Our notion of elliptic cohomology theory is essentially the same as the notion of an *elliptic spectrum* as defined in defined in [6].

In view of Remark 1.11, there is a plentiful supply of elliptic cohomology theories: roughly speaking, there is an elliptic cohomology theory for every elliptic curve. Of course, we have a similar situation for other algebraic groups. The multiplicative group $\mathbb{G}_m$ can be defined over any commutative ring R. The formal completion of $\mathbb{G}_m$ is Landweber-exact if and only if R is a torsion-free $\mathbf{Z}$-module; in this case, we have an associated cohomology theory which is given by

$$A^n(X) = K^n(X) \otimes_{\mathbf{Z}} R$$

for every finite cell complex X. In other words, there are many cohomology theories associated to the multiplicative group, but they are all just variants of one universal example: complex K-theory, associated to the "universal" multiplicative group over the ring of integers $\mathbf{Z}$.

The analogous situation for elliptic cohomology is more complicated, because there is no universal example of an elliptic curve over a commutative ring. In other words, the moduli stack $\mathcal{M}_{1,1}$ of elliptic curves, defined by the equation

$$\mathrm{Hom}(\mathrm{Spec}\ R, \mathcal{M}_{1,1}) = \{\text{Elliptic curves } E \to \mathrm{Spec}\ R\},$$

is not an affine scheme. In fact, $\mathcal{M}_{1,1}$ is not even a scheme: the right hand side of the above equation really needs to be viewed as a groupoid, since elliptic curves can have nontrivial automorphisms. However, $\mathcal{M}_{1,1}$ is a Deligne–Mumford stack; that is, there is a sufficient supply of étale morphisms $\phi : \mathrm{Spec}\ R \to \mathcal{M}_{1,1}$. For every such morphism ϕ there is associated an elliptic curve E_ϕ. It turns out that, if ϕ is étale (or more generally, if ϕ is flat), then the formal group $\widehat{E}_\phi$ is Landweber-exact. Consequently, in this case we may associate to ϕ an elliptic cohomology theory A_ϕ.

The correspondence

$$\overline{\mathcal{O}} : [\phi : \mathrm{Spec}\ R \to \mathcal{M}_{1,1}] \mapsto A_\phi$$

determines a presheaf on $\mathcal{M}_{1,1}$, taking values in the category of cohomology theories (More precisely, it is a presheaf of cohomology theories on the category of *affine* schemes with an étale map to $\mathcal{M}_{1,1}$). This presheaf may, in some sense, be regarded as the "universal elliptic cohomology theory." To extract an actual cohomology theory from A, it is tempting to try to extract some sort of global sections $\Gamma(\mathcal{M}_{1,1}, \overline{\mathcal{O}})$. Unfortunately, the notion of a cohomology theory is not well-suited to this sort of construction: one cannot generally make sense of the global sections of a presheaf that takes values in the category of cohomology theories.

To remedy this difficulty, it is necessary to *represent* our cohomology theories. According to Brown's representability theorem (see [7]), any cohomology theory A has a *representing space* Z, so that there is a functorial identification

$$A(X) \simeq [X, Z]$$

of the A-cohomology of every cell complex X with the set $[X, Z]$ of homotopy classes of maps from X into Z. More generally, for each $n \in \mathbf{Z}$ there is a space $Z(n)$ and an identification

$$A^n(X) \simeq [X, Z(n)].$$

The connecting map in the A-cohomology long exact sequence endows the sequence of spaces $\{Z(n)\}$ with additional structure: namely, a sequence of homotopy equivalences

$$\delta(n) : Z(n) \to \Omega Z(n+1).$$

A sequence of (pointed) spaces $Z(n)$, together with homotopy equivalences $\delta(n)$ as above, is called a *spectrum*. Every spectrum determines a cohomology theory, and every cohomology theory arises in this way. However, the spectrum representing a cohomology theory A is not canonically determined by A: for example, the 0th space $Z(0)$ can only be recovered up to homotopy equivalence.

Consider the diagram

$$\begin{tikzcd} & \{ \text{Spectra} \} \arrow[d] \\ \{\phi : \operatorname{Spec} R \to \mathcal{M}_{1,1}\} \arrow[r] \arrow[ur, dashed] & \{\text{Cohomology Theories}\}. \end{tikzcd}$$

If we could supply the dotted arrow, then we could lift the presheaf $\overline{\mathcal{O}}$ to a presheaf of *spectra* on the moduli stack of elliptic curves. This would address the problem: there is a good theory of sheaves of spectra, which would allow us to form a spectrum of global sections. However, supplying the dotted arrow is no simple matter. It is an example of a lifting problem which, in principle, can be attacked using the methods of obstruction theory. However, the obstruction-theoretic calculations involved turn out to be very difficult, so that a direct approach is not feasible.

The fundamental insight, due to Hopkins and Miller, is that the requisite calculations are much more tractable if we try to prove a stronger result. By definition, an elliptic cohomology theory is required to have a multiplicative structure. If A is a multiplicative cohomology theory represented by a spectrum $\{Z(n)\}$, then by Yoneda's lemma, the multiplication map

$$A^k(X) \times A^l(X) \to A^{k+l}(X)$$

is represented by a map $m_{k,l} : Z(k) \times Z(l) \to Z(k+l)$, which is well-defined up to homotopy. Moreover, the identities that are satisfied by the multiplication operation may be rephrased in terms of certain diagrams involving the maps $m_{k,l}$, which are required to commute up to homotopy. In particular, the space $Z(0)$ has the structure of a commutative ring, when regarded as an object in the homotopy category of topological spaces. For sophisticated purposes, requiring the ring axioms to hold up to homotopy is not nearly good enough. However, one does not wish to require too much. We could ask that the ring axioms for $Z(0)$ hold not just up to homotopy, but on the nose, so that $Z(0)$ is a *topological commutative ring*. Any topological commutative ring represents a multiplicative cohomology theory. However, it turns out that the cohomology theories which arise in this way are not very interesting: they are all just variants of ordinary cohomology. In particular, the classifying space $\mathbf{Z} \times BU$ for complex K-theory is not homotopy equivalent to a topological commutative ring.

There is a notion that is intermediate between "commutative ring up to homotopy" and "topological commutative ring," which is suitable for describing the kind of multiplicative structure that exists on complex K-theory and in many other examples. Roughly speaking, one requires the representing space $Z(0)$ to have the structure of a commutative ring in the homotopy category of topological spaces, but *also* remembers the homotopies which make the relevant diagrams commute, which are required to satisfy further identities (also up to homotopies which are part of the structure and required to satisfy yet higher identites, and so on). A spectrum $\{Z(n)\}$ together with all of this data is called an E_∞*-ring spectrum*, or simply an E_∞*-ring*. We will not give a definition here, though we will give a brief outline of the theory in Sect. 2.1. For definitions and further details we refer the reader to the literature (for example [8] or [9]).

Returning to the subject of elliptic cohomology, we can now consider the diagram

$$\begin{tikzcd}
 & \{E_\infty - \text{ Rings }\} \arrow[d] \\
\{\phi : \operatorname{Spec} R \to \mathcal{M}_{1,1}\} \arrow[r] \arrow[ur, dashed] & \{\text{Multiplicative Cohomology Theories}\}.
\end{tikzcd}$$

Once again, our objective is to produce the dotted arrow in the diagram: as for spectra, there is a suitable sheaf theory for E_∞-rings. At first glance, this appears to be a much more difficult problem than the one considered earlier. Let $\phi : \operatorname{Spec} R \to \mathcal{M}_{1,1}$ be an étale map, and let A_ϕ be the associated elliptic cohomology theory. The Brown representability theorem guarantees that A_ϕ can be represented by a spectrum. However, the multiplicative structure on A_ϕ does not guarantee us an E_∞-structure on the representing spectrum. Thus, the problem of lifting $\overline{\mathcal{O}}$ to a sheaf of E_∞-rings is nontrivial, even when we restrict $\overline{\mathcal{O}}$ to a single object.

However, it turns out that problem of lifting $\overline{\mathcal{O}}$ to a presheaf of E_∞-rings is much more amenable to obstruction-theoretic attack. This is because E_∞-rings are very rigid objects. Although it is much harder to write down an E_∞-ring than a spectrum, it is also much harder to write down a map between E_∞-rings than a map between spectra. The practical effect of this, in our situation, is that it is much harder to write down the *wrong* maps between E_∞-rings and much easier to find the right ones. Using this idea, Goerss, Hopkins, and Miller were able to prove the following result

Theorem 1.1. *There exists a commutative diagram*

$$\begin{tikzcd}
 & \{E_\infty - \textit{Rings}\} \arrow[d] \\
\{\phi : \mathrm{Spec}\, R \to \mathcal{M}_{1,1}\} \arrow[ur, "\mathcal{O}_{\mathcal{M}^{\mathrm{Der}}}"] \arrow[r] & \{\textit{Multiplicative Cohomology Theories}\}.
\end{tikzcd}$$

Moreover, the functor $\mathcal{O}_{\mathcal{M}^{\mathrm{Der}}}$ is determined uniquely up to homotopy equivalence.

We are now in a position to extract a "universal" elliptic cohomology theory, by taking the global sections of the presheaf $\mathcal{O}_{\mathcal{M}^{\mathrm{Der}}}$. In the language of homotopy theory, this amounts to taking the *homotopy limit* of the functor $\mathcal{O}_{\mathcal{M}^{\mathrm{Der}}}$. We will denote this homotopy limit by tmf$[\Delta^{-1}]$. This is an E_∞-ring: in other words, a cohomology theory with a very sophisticated multiplicative structure. In particular, we can view tmf$[\Delta^{-1}]$ as a multiplicative cohomology theory. However, it is *not* an elliptic cohomology theory: in fact, it is neither even nor periodic (at least not of period 2). This is a reflection of the fact that the moduli stack $\mathcal{M}_{1,1}$ is not affine. In other words, tmf$[\Delta^{-1}]$ is not an elliptic cohomology theory because it does not correspond to any particular elliptic curve: rather, it corresponds to all elliptic curves at once.

For our purposes, it is the (pre)sheaf $\mathcal{O}_{\mathcal{M}^{\mathrm{Der}}}$ itself which is the principal object of interest, not the E_∞-ring tmf$[\Delta^{-1}]$ of global sections. Passage to global sections loses a great deal of interesting information, since the moduli stack $\mathcal{M}_{1,1}$ is not affine. In the next section, we will "rediscover" $\mathcal{O}_{\mathcal{M}^{\mathrm{Der}}}$ from a rather different point of view: namely, by thinking about equivariant cohomology theories.

Remark 1.13. The reason that we have written tmf$[\Delta^{-1}]$, rather than tmf, is that we considered above the moduli stack $\mathcal{M}_{1,1}$ of (smooth) elliptic curves. The cohomology theory tmf is associated to a similar picture, where we replace $\mathcal{M}_{1,1}$ by its Deligne–Mumford compactification $\overline{\mathcal{M}_{1,1}}$. We will discuss elliptic cohomology over $\overline{\mathcal{M}_{1,1}}$ in Sect. 4.3.

The notation tmf is an acronym for *topological modular forms*. It is so-named because there exists a ring homomorphism from tmf$^*(*)$ to the ring of integral modular forms. This homomorphism is an isomorphism after inverting the primes 2 and 3 (These primes are troublesome because of the existence of elliptic curves with automorphisms of orders 2 and 3).

2 Derived Algebraic Geometry

Many of the cohomology theories which appear "in nature" can be extended to *equivariant* cohomology theories. For example, if X is a reasonably nice (compact) space with an action of a compact Lie group G, one defines the G-equivariant K-theory of X to be the Grothendieck group $K_G(X)$ of G-equivariant vector bundles on X. We would like to consider the following general question: given a cohomology theory A, where should we look for G-equivariant versions of A-cohomology?

For any reasonable theory of equivariant cohomology, we expect that if X is a principal G-bundle over a space Y, then there is a natural isomorphism $A_G(X) \simeq A(Y)$. This gives a definition of $A_G(X)$ whenever the action of G on X is sufficiently "free." However, one can always replace X by a homotopy equivalent space with a free G-action. Namely, let $p : EG \to BG$ be a principal G-bundle whose total space EG is contractible. Such a principal bundle always exists, and is uniquely determined up to homotopy equivalence. Now any space X on which G acts can be replaced by the homotopy equivalent space $EG \times X$, which is a principal G-bundle. One can now define $A_G^{Bor}(X) = A((X \times EG)/G)$: this is called the Borel-equivariant cohomology theory associated to A.

There are some respects in which Borel-equivariant cohomology is not a satisfying answer to our question. For one thing, we note that $A_G^{Bor}(X)$ is entirely determined by the original cohomology theory A: in other words, it is nothing but a new notation for ordinary A-cohomology. More importantly, there are many cases in which it is not the answer that we want to get. For example, if A is complex K-theory, then the Borel-equivariant theory does *not* coincide with theory obtained by considering equivariant vector bundles. Instead, we have a natural map

$$K_G(X) \to K_G(X \times EG) \simeq K((X \times EG)/G) \simeq K_G^{Bor}(X)$$

where the first map is given by pullback along the projection $X \times EG \to X$. This map is generally not an isomorphism. When X is a point, we obtain the map

$$\phi : \mathrm{Rep}(G) \to K(BG)$$

from the representation ring of G to the K-theory of BG, which carries each representation of G to the associated bundle on the classifying space. This map is *never* an isomorphism unless G is trivial. However, it is not far from being an isomorphism: by the Atiyah–Segal completion theorem ([10]), ϕ identifies $K(BG)$ with the completion of $\mathrm{Rep}(G)$ with respect to a certain ideal (the ideal consisting of virtual representations of dimension zero).

Let us consider the situation in more detail for the circle group $G = S^1$. Every representation of G is a direct sum of irreducible representations. Since G is abelian, every irreducible representation is 1-dimensional, given by a character $G \to \mathbf{C}^*$. Furthermore, every character of G is simply an integral power of the "defining" character

$$\chi : G \simeq \{z \in \mathbf{C} : |z| = 1\} \hookrightarrow \mathbf{C}^*.$$

The representation ring of G is therefore isomorphic with the ring of Laurent polynomials $\mathbf{Z}[\chi, \chi^{-1}]$. We observe that this ring of Laurent polynomials may also be identified with the ring of functions on the multiplicative group $\mathbb{G}_m$. The classifying space of the circle group may be identified with $\mathbf{C}\mathrm{P}^{\infty}$; as we saw in Sect. 1.2, the K-theory of this space is a power series ring $\mathbf{Z}[[t]]$, which may be identified with the ring of functions on the *formal* multiplicative group $\widehat{\mathbb{G}}_m$. In this case, the map

$$K_G(*) \to K(BG)$$

is easily identified: it is given by restriction of functions from the *entire* multiplicative group to the formal multiplicative group. In other words, it is given by taking germs of functions near the identity. Concretely, this homomorphism is described by the formula $\chi \mapsto (t+1)$.

We saw in Sect. 1 that if A is an even periodic cohomology theory, then A determines a formal group $\widehat{\mathbb{G}} = \operatorname{Spf} A(\mathbf{C}\mathrm{P}^{\infty})$. The lesson to learn from the example of K-theory is that the problem of finding *equivariant* versions of the cohomology theory A may be related to the problem of realizing $\widehat{\mathbb{G}}$ as the formal completion of an algebraic group $\mathbb{G}$. In the case of K-theory, we can recover $\mathbb{G}$ as the spectrum of the equivariant K-group $K_{S^1}(*)$. In the case of elliptic cohomology, we do not expect $\mathbb{G}$ to be affine, and therefore we cannot expect to recover it from its ring of functions (though it is often possible to reconstruct $\mathbb{G}$ through a more elaborate procedure: we will discuss this problem in Sect. 5.5). Instead, we should view the correspondence as running in the other direction: given an algebraic group $\mathbb{G}$ having $\operatorname{Spf} A(\mathbf{C}\mathrm{P}^{\infty})$ as its formal completion, it is natural to *define* $A_{S^1}(*)$ to be global sections of the structure sheaf of $\mathbb{G}$. Passing from functions to their formal germs then gives a suitable "completion" map

$$A_{S^1}(*) \to A(\mathbf{C}\mathrm{P}^{\infty})$$

from equivariant A-cohomology to Borel-equivarient A-cohomology.

Of course, the above prescription does not solve the problem of defining equivariant versions of the cohomology theory A, even for the group $G = S^1$. Merely knowing the equivariant cohomology groups $A_G(*)$ of a point does not tell us how to define the equivariant cohomology groups $A_G(X)$ of a more general G-space X. This difficulty arises even when G is trivial: a cohomology theory A is not determined by its coefficient groups $A^n(*)$. To really exploit the ideas sketched above, we need to be able to extract from the algebraic group $\mathbb{G}$ not just cohomology *rings*, but cohomology *theories*. To accomplish this, we need $\mathbb{G}$ to be an algebraic group in a somewhat nonstandard setting: the sheaf of regular functions on $\mathbb{G}$ will be a sheaf not of ordinary commutative rings, but of E_∞-rings. Our goal in Sect. 2 is to introduce the ideas which are needed to make sense of these kinds of algebraic groups. We will use this theory to construct equivariant cohomology theories in Sect. 3, and relate it to the theory of elliptic cohomology in Sect. 4.

2.1 E_∞-Rings

In Sect. 1.3, we briefly mentioned the notion of an E_∞-ring spectrum, which reappear throughout this paper. However, the ideas involved are somewhat nonelementary, and to give a precise definition would take us too far afield. This section is intended as a nontechnical introduction to the subject of E_∞-ring theory; we will not give any definitions or prove any theorems, but will highlight some of the main features of the theory and explain how it can be used in practice.

Roughly speaking, an *E_∞-ring space* is a topological space A equipped with the structure of a commutative ring. As we explained in Sect. 1.3, there are several unsatisfactory ways of making this precise. Consider the diagram

$$\begin{tikzcd} A \times A \arrow[dr, "a"] \arrow[dd, "s"'] & \\ & A \\ A \times A \arrow[ur, "a"'] & \end{tikzcd}$$

where a denotes the addition map on A, and s the automorphism of $A \times A$ which permutes the factors. The commutativity of this diagram expresses the commutativity of the addition operation on A. However, this is a very strong condition which is often not satisfied in practice. What is much more common is that the diagram commutes *up to homotopy*. That means that there is a continuous map

$$h_t : A \times A \times [0, 1] \to A$$

with $h_0 = a$ and $h_1 = a \circ s$. For sophisticated applications, merely knowing the existence of h is usually not enough: one should really take h to be part of the ring structure on A. Moreover, h should not be arbitrary, but should itself satisfy certain identities (at least up to homotopy, which must again be specified). In other words, we do not want A merely to have the structure of a commutative ring up to homotopy, but up to *coherent* homotopy. Of course, deciding exactly what we mean by this is a highly nontrivial matter: exactly what homotopies should we take as part of the data, and what identities should they satisfy? There are several (equivalent) ways of answering these questions; we refer the reader to [8] for one possibility.

Let A be an E_∞-ring space; then, in particular, we can view A as a space and consider its homotopy groups $\pi_n A$ (here we use a canonical base point of A which is given by the "identity" with respect to addition). Since A is an abelian group in the homotopy category of topological spaces, each $\pi_n A$ is endowed with the structure of an abelian group; this agrees with the usual group structure on $\pi_n A$ for $n \geq 1$. In addition, the multiplication operation on A endows $\pi_* A = \oplus_{n \geq 0} \pi_n A$ with the structure of a graded ring. The ring $\pi_* A$ is commutative in the *graded* sense: that is, if $x \in \pi_n A$ and $y \in \pi_m B$, then $xy = (-1)^{nm} yx \in \pi_{n+m} A$. In particular, $\pi_0 A$ is a commutative ring, and each $\pi_n A$ has the structure of a module over $\pi_0 A$.

Any commutative ring can be regarded as an E_∞-ring space; we simply regard it as a topological commutative ring, given the discrete topology. For *any* E_∞-ring space A, there is a canonical map $A \to \pi_0 A$ which collapses each path component of A to a point.

A map of E_∞-ring spaces $A \to B$ is an *equivalence* if it gives rise to isomorphisms $\pi_n A \to \pi_n B$. We note that if A is an E_∞-ring space with $\pi_n A = 0$ for $n \geq 1$, then the projection $A \to \pi_0 A$ is an equivalence; in this case we say that A is *essentially discrete* and we may abuse terminology by identifying A with the ordinary commutative ring $\pi_0 A$. We can regard the higher homotopy groups $\pi_n A$ as a measure of the difference between A and the ordinary commutative ring $\pi_0 A$.

Remark 2.1. Recall that an ordinary commutative ring R is said to be *reduced* if it contains no nonzero nilpotent elements. If R is not reduced, then the nilpotent elements of R form an ideal I; then there is a projection $R \to R/I$, and R/I is a reduced ring. The relationship between E_∞-ring spaces and ordinary commutative rings should be regarded as analogous to the relationship between ordinary commutative rings and reduced commutative rings. If A is an E_∞-ring space, one should regard $\pi_0 A$ as the "underlying ordinary commutative ring" which is obtained from A by killing the higher homotopy groups, just as the reduced ring R/I is obtained by killing the nilpotent elements of R.

Any E_∞-ring space A determines a cohomology theory: for a (well-behaved) topological space X, one can define $A(X)$ to be the set of homotopy classes of maps from X into A. More generally, one can consider the space A^X of all maps from X into A, and endow it with the structure of an E_∞-ring space, computing all of the ring operations pointwise. One can then define $A^{-n}(X)$ to be the homotopy group $\pi_n A^X$ for $n \geq 0$. This definition has a natural extension to the case $n < 0$, and gives rise to a (multiplicative) cohomology theory which we will also denote by A. Of course, the functor

$$X \mapsto A(X)$$

determines A as a topological space, up to *weak* homotopy equivalence. The E_∞-ring structure on A can be regarded as determining a commutative ring structure on the functor $X \mapsto A(X)$, together with certain higher coherence conditions.

We therefore have two distinct (but related) points of view on what an E_∞-ring space A *is*. On the one hand, we may view A as a "commutative ring in homotopy theory"; from this point of view, the theory of E_∞-rings is a kind of generalized commutative algebra. On the other hand, we may view an E_∞-ring space A as a cohomology theory equipped with a good multiplicative structure, giving rise not only to multiplication maps on A-cohomology but also secondary operations such as Massey products and their higher analogues. Both of these points of view will be important for the applications we discuss in this paper.

The cohomology theory associated to an E_∞-ring space A is *connective*: that is, it has the property that $A^n(*) = 0$ for $n > 0$. Many cohomology theories which arise naturally do not have this property. For example, a nontrivial cohomology theory cannot be both connective and periodic (in the sense of Definition 1.1). Consequently, it is necessary to introduce a slightly more general notion than an E_∞-ring space: that of an *E_∞-ring spectrum*, or simply an *E_∞-ring*. Roughly speaking, an E_∞-ring is a cohomology theory A with all of the same good multiplicative properties that the cohomology theories associated to E_∞-ring spaces have, but without the requirement that A be connective. If A is an E_∞-ring, then one can associate to it a graded ring $\oplus_{n\in\mathbf{Z}}\pi_n A$ (by taking that A-cohomology groups of a point) which may be nonzero in both positive and negative degrees. Every E_∞-ring space may be regarded as an E_∞-ring; conversely, an E_∞-ring A with $\pi_n A = 0$ for $n < 0$ is equivalent to an E_∞-ring space. Finally, if A is an arbitrary E_∞-ring, then it has a *connective cover* $\tau_{\geq 0} A$, which satisfies

$$\pi_k(\tau_{\geq 0} A) = \begin{cases} \pi_k A & \text{if } k \geq 0 \\ 0 & \text{if } k < 0. \end{cases}$$

Remark 2.2. It is important to understand that the world of E_∞-rings is essentially *higher-categorical* in nature. In practice, this means that given two E_∞-rings A and B, one really has a *space* $\mathrm{Hom}(A, B)$ of maps from A to B, rather than simply a set.

If A is an E_∞-ring, then there is a good theory of modules over A, which are called *A-module spectra*. Every A-module spectrum M can be viewed as a spectrum, and therefore determines a cohomology theory $X \mapsto M^*(X)$. In particular, one can define homotopy groups $\pi_n M = M^{-n}(*)$, and these form a graded module $\pi_* M$ over the graded ring $\pi_* A$.

We will generally refer to A-module spectra simply as *A-modules*. However, there is a special case in which this could potentially lead to confusion. If A is an ordinary commutative ring, then we may regard A as an E_∞-ring. In this case, we can identify A-module spectra with objects of the *derived* category of A-modules: that is, chain complexes of A-modules, defined up to quasi-isomorphism.

The following definition will play an important role throughout this paper:

Definition 2.1. Let A be an E_∞-ring and let M be an A-module. We will say that M is *flat* if the following conditions are satisfied:

1. The module $\pi_0 M$ is flat over $\pi_0 A$, in the sense of classical commutative algebra.
2. For each n, the induced map

$$\pi_n A \otimes_{\pi_0 A} \pi_0 M \to \pi_n M$$

is an isomorphism.

We will say that a map $A \to B$ of E_∞-rings is *flat* if B is flat when regarded as an A-module.

2.2 Derived Schemes

In the last section, we reviewed the theory of E_∞-rings, and saw that it was a natural generalization of classical commutative algebra. In this section, we will explain how to incorporate these ideas into the foundations of algebraic geometry.

We begin by recalling the definition of a scheme. A *scheme* is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X, which is locally (on X) isomorphic to (Spec $A, \mathcal{O}_{\text{Spec } A}$) for some commutative ring A. We seek a modification of this definition in which we allow E_∞-rings to fill the role of ordinary commutative rings. The main challenge is to decide what we mean by Spec A, when A is an E_∞-ring. We will adopt the following rather simple-minded definition.

Definition 2.2. Let A be an E_∞-ring. Then the topological space Spec A, the *Zariski-spectrum* of A, is defined to be the spectrum of the ordinary commutative ring $\pi_0 A$: in other words, the set of prime ideals in $\pi_0 A$. We endow Spec A with the usual Zariski-topology, with a basis of open sets given by the loci $U_f = \{\mathfrak{p} | f \notin \mathfrak{p}\}$, $f \in \pi_0 A$.

To complete the definition of the Zariski-spectrum of an E_∞-ring A, we need to define the structure sheaf $\mathcal{O}_{\text{Spec } A}$. By general nonsense, it suffices to define $\mathcal{O}_{\text{Spec } A}$ on each of the basic open subsets $U_f \subseteq$ Spec A. If A were an ordinary commutative ring, we would define $\mathcal{O}_{\text{Spec } A}(U_f) = A[f^{-1}]$. This definition makes sense also in the E_∞-context. Namely, given an E_∞-ring A and an element $f \in \pi_0 A$, there exists a map of E_∞-rings $A \to A[f^{-1}]$, which is characterized by either of the following equivalent assertions:

1. The map $\pi_* A \to \pi_*(A[f^{-1}])$ identifies $\pi_*(A[f^{-1}])$ with $(\pi_* A)[f^{-1}]$.
2. For any E_∞-ring B, the induced map

$$\text{Hom}(A[f^{-1}], B) \to \text{Hom}(A, B)$$

   is a homotopy equivalence of the left hand side onto the subspace of the right hand side consisting of all maps $A \to B$ which carry f to an invertible element in $\pi_0 B$ (this is a union of path components of $\text{Hom}(A, B)$).

In virtue of the second characterization, the map $A \to A[f^{-1}]$ is well-defined up to canonical equivalence; moreover, it is sufficiently functorial to allow a definition of the structure sheaf $\mathcal{O}_{\text{Spec } A}$.

We are now prepared to offer our main definition.

Definition 2.3. A *derived scheme* is a topological space X equipped with a sheaf of E_∞-rings $\mathcal{O}_X$, which is locally equivalent to (Spec $A, \mathcal{O}_{\text{Spec } A}$) where A is an E_∞-ring.

Remark 2.3. As we mentioned earlier, the world of E_∞-rings is higher-categorical in nature. Consequently, one needs to be careful what one means by a *sheaf* of E_∞-rings. There are several approaches to making this idea precise. One is to

Quillen's theory of model categories. Namely, one can let $\mathcal{C}$ be a suitable model-category for E_∞-rings (for example, commutative monoids in symmetric spectra: see [9]). Now we can consider the category of *$\mathcal{C}$-valued presheaves* on a topological space X. This category of presheaves is itself endowed with a model structure, which simultaneously reflects the original model structure on $\mathcal{C}$ *and* the topology of X. Namely, we define a map $\mathcal{F} \to \mathcal{G}$ of presheaves to be a *cofibration* if it induces a cofibration $\mathcal{F}(U) \to \mathcal{G}(U)$ in $\mathcal{C}$ for every open subset $U \subseteq X$, and a *weak equivalence* if it induces a weak equivalence on stalks $\mathcal{F}_x \to \mathcal{G}_x$ for every point $x \in X$. One can then define a *sheaf* of E_∞-rings on X to be a fibrant and cofibrant object of this model category.

Remark 2.4. As with E_∞-rings themselves, derived schemes are higher-categorical objects by nature. That means that given derived schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, we can naturally associate a *space* of morphisms from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$. Namely, we define

$$\mathrm{Hom}((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)) = \coprod_{f: X \to Y} \mathrm{Hom}_0(\mathcal{O}_Y, f_* \mathcal{O}_X).$$

Here $\mathrm{Hom}_0(\mathcal{O}_Y, f_*\mathcal{O}_X)$ is the subspace of $\mathrm{Hom}(\mathcal{O}_Y, f_*\mathcal{O}_X)$ consisting of *local* maps of sheaves of E_∞-rings: that is, maps which induce local homomorphisms $\pi_0\mathcal{O}_{Y,f(x)} \to \pi_0\mathcal{O}_{X,x}$ of commutative rings for each $x \in X$. This is a union of path components of $\mathrm{Hom}(\mathcal{O}_Y, f_*\mathcal{O}_X)$, which may be defined following the description in Remark 2.3.

Example 2.1. Let A be an E_∞ ring. Then $(\mathrm{Spec}\ A, \mathcal{O}_{\mathrm{Spec}\ A})$ is a derived scheme. Derived schemes which arise via this construction we will call *affine*.

Remark 2.5. Let $(X, \mathcal{O}_X)$ be an ordinary scheme. Since every ordinary commutative ring can be regarded as an E_∞-ring, we may regard $\mathcal{O}_X$ as a presheaf of E_∞-rings on X. This presheaf is generally not a sheaf: this is because of the existence of nontrivial cohomology groups of the structure sheaf $\mathcal{O}_X$ over the open subsets of X. Let $\mathcal{O}'_X$ denote the sheafification of $\mathcal{O}_X$, in the setting of sheaves of E_∞-rings. Then $(X, \mathcal{O}'_X)$ is a derived scheme. We will abuse terminology by ignoring the distinction between $\mathcal{O}_X$ and $\mathcal{O}'_X$ (either one can be recovered from the other, via the functors of sheafification and π_0. We also note that the map $\mathcal{O}_X(U) \to \mathcal{O}'_X(U)$ is an equivalence whenever $U \subseteq X$ is affine). Thus, *every ordinary scheme can be regarded as a derived scheme*.

Conversely, let $(X, \mathcal{O}_X)$ be a derived scheme. Then the functor

$$U \mapsto \pi_0(\mathcal{O}_X(U))$$

is a presheaf of commutative rings on X. We let $\pi_0\mathcal{O}_X$ denote the sheafification of this presheaf. (A vanishing theorem of Grothendieck ensures that $(\pi_0\mathcal{O}_X)(U) \simeq \pi_0(\mathcal{O}_X(U))$ whenever U is affine). The pair $(X, \pi_0\mathcal{O}_X)$ is a scheme. We call it the *underlying ordinary scheme* of $(X, \mathcal{O}_X)$, and will occasionally denote it by $\overline{(X, \mathcal{O}_X)}$.

Remark 2.6. If we were to employ only E_∞-ring spaces, rather than arbitrary E_∞-rings, in our definition of derived schemes, then the functor

$$(X, \mathcal{O}_X) \to (X, \pi_0 \mathcal{O}_X)$$

would be a right adjoint to the inclusion functor from schemes to derived schemes. In other words, we would be able to regard $(X, \pi_0 \mathcal{O}_X)$ as the *maximal ordinary subscheme* of the derived scheme $(X, \mathcal{O}_X)$. There is an analogous construction in ordinary algebraic geometry: every scheme possesses a maximal reduced subscheme.

Without connectivity assumptions, no such interpretation is possible: there is no map which directly relates $(X, \mathcal{O}_X)$ and $(X, \pi_0 \mathcal{O}_X)$.

Much of the formalism of ordinary algebraic geometry can be carried over to derived algebraic geometry, without essential change. For example, if $(X, \mathcal{O}_X)$ is a derived scheme, then one can consider *quasi-coherent sheaves* on X: these are functors $\mathcal{F}$ which assign to every open subset $U \subseteq X$ a $\mathcal{O}_X(U)$-module spectrum $\mathcal{F}(U)$, which satisfy $\mathcal{F}(V) \simeq \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V)$ whenever $V \subseteq U$ are affine, and which satisfy an appropriate descent condition.

There is also a good theory of flatness in derived algebraic geometry.

Definition 2.4. Let $p : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a map of derived schemes. We will say that p is *flat* if, for every pair of open affine subsets $U \subseteq X$, $V \subseteq Y$ such that $p(U) \subseteq V$, the induced map of E_∞-rings

$$\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$$

is flat (in the sense of Definition 2.1).

Remark 2.7. As in ordinary algebraic geometry, one can give various equivalent formulations of Definition 2.4: for example, testing flatness only on particular affine covers of X and Y, or on stalks of the structure sheaves.

We note that if $p : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a flat map of derived schemes, then the underlying map of ordinary schemes $(X, \pi_0 \mathcal{O}_X) \to (Y, \pi_0 \mathcal{O}_Y)$ is flat in the sense of ordinary algebraic geometry. Conversely, if $(Y, \mathcal{O}_Y)$ is an ordinary scheme, then p is flat if and only if $(X, \mathcal{O}_X)$ is an ordinary scheme *and* p is flat when viewed as a map of ordinary schemes. In other words, the fibers of a flat morphism are classical schemes.

In many of the applications that we consider, we will need to deal with algebro-geometric objects of a more general nature than schemes. For example, the moduli stack $\mathcal{M}_{1,1}$ of elliptic curves cannot be represented by a scheme. However, $\mathcal{M}_{1,1}$ *is* an algebraic stack, in the sense of Deligne–Mumford. Algebraic stacks are usually *defined* to be a certain class of functors from commutative rings to groupoids; however, for Deligne–Mumford stacks one can also take a more geometric approach to the definition:

Definition 2.5. A *Deligne–Mumford stack* is a topos X with a sheaf $\mathcal{O}_X$ of commutative rings, such that the pair $(X, \mathcal{O}_X)$ is locally (on X) isomorphic with $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, where A is a commutative ring. Here $\operatorname{Spec} A$ denotes the étale topos of A, and $\mathcal{O}_{\operatorname{Spec} A}$ its canonical sheaf of rings.

As with schemes, it is possible to offer a *derived* version of Definition 2.5, and thereby obtain a notion of *derived Deligne–Mumford stack*. All of the above discussion carries over to this more general context, without essential change. The reason for introducing this definition is that we have already encountered a very interesting example:

Example 2.2. Let $\mathcal{M}_{1,1} = (\mathcal{M}, \mathcal{O}_{\mathcal{M}_{1,1}})$ denote the (ordinary) moduli stack of elliptic curves; here we let the symbol $\mathcal{M}$ denote the étale topos of the moduli stack. Let $\mathcal{O}_{\mathcal{M}^{\mathrm{Der}}}$ denote the sheaf of E_∞-rings on $\mathcal{M}$ constructed in Theorem 1.1. (To be more precise, Theorem 1.1 constructs a presheaf of E_∞-rings on a particular site for the topos $\mathcal{M}$, which extends in a canonical way to the sheaf $\mathcal{O}$). Then the pair $\mathcal{M}^{\mathrm{Der}} = (\mathcal{M}, \mathcal{O})$ is a derived Deligne–Mumford stack. We have $\pi_0 \mathcal{O}_{\mathcal{M}^{\mathrm{Der}}} \simeq \mathcal{O}_{\mathcal{M}_{1,1}}$, so that the underlying ordinary Deligne–Mumford stack of $\mathcal{M}^{\mathrm{Der}}$ is the classical moduli stack $\mathcal{M}_{1,1}$ of elliptic curves.

We can view Example 2.2 as offering a geometric interpretation of Theorem 1.1: namely, Theorem 1.1 asserts the existence of a certain "derived" version of the moduli stack of elliptic curves. This turns out to be a very useful perspective, because the derived stack $\mathcal{M}^{\mathrm{Der}}$ itself admits a moduli-theoretic interpretation. This observation leads both to a new construction of the sheaf $\mathcal{O}_{\mathcal{M}^{\mathrm{Der}}}$, and to a theory of equivariant elliptic cohomology. It also permits the study of elliptic cohomology using tools from derived algebraic geometry, which has many other applications.

3 Derived Group Schemes and Orientations

In Sect. 2, we argued that for a (multiplicative) cohomology theory A, there is a relationship between equivariant versions of A and group schemes over the commutative ring $A(*)$. To exploit this relationship, it is even better to have a group scheme $\mathbb{G}$ defined over A itself. The language of derived algebraic geometry enables us to make sense of this idea. Namely, suppose that A is an E_∞-ring; then Definition 2.3 allows us to speak of *derived A-schemes*; that is, maps $\mathbb{G} \to \operatorname{Spec} A$ in the setting of derived schemes. But how can we make sense of a group structure on $\mathbb{G}$? The question is somewhat subtle, since derived schemes are most naturally viewed as higher-categorical objects.

Let us first consider the case of ordinary schemes. Let $p : \mathbb{G} \to X$ be a map of schemes. What does it mean to say that $\mathbb{G}$ is a (commutative) group scheme over X? One possibility is to require that $\mathbb{G}$ be a commutative group object in the category of schemes over X: in other words, to require the existence of a multiplication map $\mathbb{G} \times_X \mathbb{G} \to \mathbb{G}$ satisfying various identities, which can be depicted as commutative diagrams. There is an alternative way of phrasing this, using Grothendieck's "functor of points" philosophy. Namely, given an X-scheme $q : S \to X$, we can define

$$\mathbb{G}(S) = \{r \in \operatorname{Hom}(S, \mathbb{G}) | p \circ r = q\}.$$

In other words, via the Yoneda embedding we may identify $\mathbb{G}$ with a functor from X-schemes to sets. To endow $\mathbb{G}$ with the structure of a commutative group object (over X) is to give a *lifting* of this functor to the category of abelian groups. Going still further, we can identify $\mathbb{G}$ with this lifting. From this point of view, a commutative group scheme over X is a functor from X-schemes to abelian groups, such that the underlying functor from X-schemes to sets happens to be representable by a scheme.

We can apply the same philosophy to derived algebraic geometry. However, there is one important difference: if X is a derived scheme, then derived schemes over X should not be viewed as an ordinary category. Rather, they behave in a higher-categorical fashion, so that for a pair of derived X-schemes S and S', the collection $\mathrm{Hom}_X(S, S')$ of commutative diagrams

$$\begin{tikzcd} S \arrow[rr] \arrow[dr] & & S' \arrow[dl] \\ & X & \end{tikzcd}$$

forms a space, rather than a set. However, this poses no major difficulties and we can make the following definition.

Definition 3.1. Let X be a derived scheme. A *commutative X-group* is a (topological) functor $\mathbb{G}$ from derived X-schemes to topological abelian groups, such that the composite functor

$$\{X\text{-schemes}\} \to \{\text{topological abelian groups}\} \to \{\text{ topological spaces }\}$$

is representable (up to weak homotopy equivalence) by a derived X-scheme that is *flat* over X. If $X = \mathrm{Spec}\ R$, we will also say that $\mathbb{G}$ is a *commutative R-group*.

We will often abuse terminology and not distinguish between a commutative X-group $\mathbb{G}$ over X and the derived X-scheme that represents its underlying space-valued functor.

Remark 3.1. Let X be an ordinary scheme. Then we may regard X as a derived scheme, and a commutative X-group in the sense of Definition 3.1 is the same thing as a commutative group scheme that is flat over X, in the sense of ordinary algebraic geometry. More generally, any commutative X-group gives rise to a flat commutative group scheme over $\overline{X}$, the underlying ordinary scheme of X.

If we were to remove the flatness hypothesis from Definition 3.1, neither of the above statements would be true. The problem is that the formation of fiber products $\mathbb{G} \times_X \mathbb{G}$ is not compatible with passage between schemes and derived schemes, unless we assume that $\mathbb{G} \to X$ is flat.

Let us now return to the study of equivariant cohomology. Suppose that we are given a cohomology theory which is represented by an E_∞-ring A, and that we are looking for a definition of "S^1-equivariant A-cohomology." An S^1-equivariant cohomology theory determines an ordinary cohomology theory, simply by restricting attention to spaces on which S^1 acts trivially. In the best of all possible worlds, this cohomology theory might itself be representable by an E_∞-ring A_{S^1}. In the case of complex K-theory, this E_∞-ring can be described as the ring of functions on a commutative K-group $\mathbb{G}_m$ (we will analyze this example in detail in Sect. 3.1). In the general case, we will take the commutative A-group $\mathbb{G}$ as our starting point, and try to recover a theory of equivariant A-cohomology from it.

Of course, the commutative A-group $\mathbb{G}$ is not arbitrary, because any reasonable theory of equivariant A-cohomology can be compared with Borel-equivariant A-cohomology. In particular, there should be an "completion" map

$$A_{S^1} \to A^{\mathbf{CP}^\infty}.$$

In the example of K-theory, this map can be interpreted as a restriction map, from regular functions defined on all of $\mathbb{G}_m$ to formal functions defined only near the identity section of $\mathbb{G}_m$. Morally speaking, this is induced by a map

$$\operatorname{Spf} A^{\mathbf{CP}^\infty} \to \mathbb{G}$$

where the left hand side is a *formal* commutative A-group. In the case where A is even and periodic, it is possible to make sense of the formal spectrum of $A^{\mathbf{CP}^\infty}$ using a formal version of derived algebraic geometry. Fortunately, this turns out to be unnecessary: it is possible to formulate the relevant structure in simpler terms. In order to do this, we first note that $\mathbf{CP}^\infty$ has the structure of a (topological) abelian group. Namely, we may view $\mathbf{CP}^\infty$ as the space of 1-dimensional complex subspaces in the ring of rational functions $\mathbf{C}(x)$; multiplication on $\mathbf{C}(x)$ determines a group structure on $\mathbf{CP}^\infty$.

Definition 3.2. Let X be a derived scheme, and let $\mathbb{G}$ be a commutative X-group. A *preorientation* of $\mathbb{G}$ is a map of topological abelian groups

$$\mathbf{CP}^\infty \to \mathbb{G}(X).$$

A *preoriented X-group* is a commutative X-group $\mathbb{G}$ together with a preorientation of $\mathbb{G}$.

Suppose that X is the spectrum of an E_∞-ring A and let $\mathbb{G}$ be a commutative X-group, with A_{S^1} the E_∞-ring of global functions on $\mathbb{G}$. Given an X-valued point of $\mathbb{G}$, restriction of functions induces a map $A_{S^1} \to A$. If $\mathbb{G}$ is equipped with a preorientation, then we get a collection of maps $A_{S^1} \to A$ indexed by $\mathbf{CP}^\infty$, which we may identify with a map $A_{S^1} \to A^{\mathbf{CP}^\infty}$; this is the "completion" map we are looking for. The requirement that $\mathbf{CP}^\infty \to \mathbb{G}(X)$ be a map of topological abelian groups corresponds to the condition that the map

$$\operatorname{Spf} A^{\mathbf{CP}^\infty} \to \mathbb{G}$$

should be compatible with the group structures.

It is possible to formulate the notion of a preorientation in even simpler terms. We regard $\mathbf{C}\mathrm{P}^{\infty}$ as the set of nonzero elements in the field of rational functions $\mathbf{C}(x)$, modulo scaling. Let $\mathbf{C}\mathrm{P}^{n}$ denote the subspace corresponding to nonzero polynomials of degree $\leq n$. Then $\mathbf{C}\mathrm{P}^{0}$ is the identity element of $\mathbf{C}\mathrm{P}^{\infty}$. By the fundamental theorem of algebra, every nonzero element of $\mathbf{C}(x)$ factors as a product of (powers of) linear factors, which are unique up to scaling. Consequently, $\mathbf{C}\mathrm{P}^{\infty}$ is generated, as an abelian group, by the subspace $\mathbf{C}\mathrm{P}^{1}$. Moreover, it is *almost* freely generated: the only relation is that the point $\mathbf{C}\mathrm{P}^{0} \subseteq \mathbf{C}\mathrm{P}^{1}$ be the identity element. It follows that giving a preorientation $\mathbf{C}\mathrm{P}^{\infty} \to \mathbb{G}(X)$ of a derived commutative group scheme over X is equivalent to giving a map from the 2-sphere $\mathbf{C}\mathrm{P}^{1}$ into $\mathbb{G}(X)$, which carries the base point to the identity element of $\mathbb{G}(X)$. Thus, up to homotopy, preorientations of $\mathbb{G}$ are classified by elements of the homotopy group $\pi_2\mathbb{G}(X)$.

Of course, given any commutative A-group $\mathbb{G}$, one can always find a preorientation of $\mathbb{G}$: namely, the zero map. In order to rule out this degenerate example, we need to impose a further condition. Let us return to the example of complex K-theory, and let $\mathbb{G}_m = \mathrm{Spec} K_{S^1}$ the multiplicative group over K. In this case, $K^{\mathbf{C}\mathrm{P}^{\infty}}$ is precisely the E_∞-ring of formal functions on $\mathbb{G}_m$ near the identity section. In other words, our preorientation of $\mathbb{G}_m$ gives rise to a map

$$s : \mathrm{Spf} K^{\mathbf{C}\mathrm{P}^{\infty}} \to \mathbb{G}_m$$

which is as nontrivial as possible: it realizes the left hand side as the formal completion $\widehat{\mathbb{G}}_m$ of $\mathbb{G}_m$. We wish to axiomatize this condition, without making reference to formal derived geometry. To do so, we note that in the example above, both $\mathrm{Spf} K^{\mathbf{C}\mathrm{P}^{\infty}}$ and $\widehat{\mathbb{G}}_m$ are 1-dimensional formal groups. Consequently, to test whether or not s is an isomorphism of formal groups, it suffices to check that the derivative of s is invertible. In order to discover the object that plays the role of this derivative, we need to introduce a few more definitions.

Let A be an E_∞-ring and $\mathbb{G}$ a preoriented A-group. Let $\mathbb{G}_0$ denote the underlying ordinary scheme of $\mathbb{G}$. Let $\Omega_{\mathbb{G}_0/\pi_0 A}$ denote the sheaf of differentials of $\mathbb{G}_0$ over $\pi_0 A$, and let ω denote the pullback of this sheaf along the identity section

$$\mathrm{Spec}\pi_0 A \to \mathbb{G}_0.$$

We will identify ω with the $\pi_0 A$-module of global sections of ω. In the case where $\mathbb{G}_0$ is *smooth*, we can identify ω with the dual of the (abelian) Lie algebra of $\mathbb{G}_0$.

Let $\sigma : S^2 \to \mathbb{G}(A)$ be the preorientation of $\mathbb{G}$. Let $\mathrm{Spec}\, B = U \subseteq \mathbb{G}$ be an affine open subscheme of $\mathbb{G}$ containing the identity section; then we may identify σ with a map $S^2 \to U(A)$. Since U is affine, we may identify this with a map of E_∞-rings $B \to A^{S^2}$. This in turn induces a map of ordinary $\pi_0 A$-algebras $\pi_0 B \to A(S^2) \simeq \pi_0 A \oplus \pi_2 A$. The first component is a ring homomorphism $\pi_0 B \to \pi_0 A$ corresponding to the identity section of $\mathbb{G}_0$, while the second is a $\pi_0 A$-algebra derivation of $\pi_0 B$ into $\pi_2 A$. This derivation is classified by a map $\beta : \omega \to \pi_2 A$ of $\pi_0 A$-modules.

Definition 3.3. Let A be an E_∞-ring and $\mathbb{G}$ a commutative A-group equipped with a preorientation $\sigma : S^2 \to \mathbb{G}(A)$. We will say that σ is an *orientation* if the following conditions are satisfied:

1. The underlying map of ordinary schemes $\mathbb{G}_0 \to \mathrm{Spec}\pi_0 A$ is smooth of relative dimension 1.
2. The map $\beta : \omega \to \pi_2 A$ induces *isomorphisms*

$$\pi_n A \otimes_{\pi_0 A} \omega \to \pi_{n+2} A$$

for every integer $n \in \mathbf{Z}$.

An *oriented A-group* is a commutative A-group equipped with an orientation. More generally, if X is a derived scheme, an *oriented X-group* is a preoriented X-group whose restriction to every open affine Spec $A \subseteq X$ is an oriented A-group.

Remark 3.2. Let A be an E_∞-ring and $\mathbb{G}$ an oriented A-group. Then condition (2) of Definition 3.3 forces A to be weakly periodic. Conversely, if A is weakly periodic, then condition (2) is equivalent to the assertion that β is an isomorphism.

3.1 Orientations of the Multiplicative Group

Let A be an E_∞-ring. In the last section, we introduced the notion of a (pre)orientation on a commutative A-group. In this section, we specialize to the case where $\mathbb{G}$ is the multiplicative group $\mathbb{G}_m$.

As in ordinary algebraic geometry, we can make sense of the multiplicative group $\mathbb{G}_m$ over *any* E_∞ ring A. In order to do so, we begin with a few general remarks about group algebras. Let R be a commutative ring, and M an abelian group. Then the *group ring* $R[M]$ can be characterized by the following universal property: to give a commutative ring homomorphism $R[M] \to S$, one must give a commutative ring homomorphism $R \to S$, together with a homomorphism from M to the multiplicative group of S. We note that $R[M]$ is not merely an R-algebra, but an R-Hopf algebra. More precisely, Spec $R[M]$ is a commutative group scheme over R, the group structure coming from the fact that the collection of homomorphisms from M to $S^\times$ forms a group under pointwise multiplication, for every R-algebra S.

The above discussion generalizes without essential change to derived algebraic geometry. Given an E_∞-ring A and a topological abelian group M, we can form a group algebra $A[M]$. In the special case where M is the group $\mathbf{Z}$ of integers, we may informally write $A[\mathbf{Z}]$ as $A[t, t^{-1}]$ and we define the multiplicative group $\mathbb{G}_m$ to be Spec $A[\mathbf{Z}]$. We note that $\pi_*(A[\mathbf{Z}]) = (\pi_* A)[t, t^{-1}]$. In particular, the multiplicative group $\mathbb{G}_m$ is flat over Spec A, so we may regard it as a commutative A-group. The underlying ordinary scheme of $\mathbb{G}_m$ is just the usual multiplicative group over $\mathrm{Spec}\pi_0 A$; in particular, it is smooth of relative dimension 1.

Let us consider the problem of constructing a *preorientation* of the multiplicative group $\mathbb{G}_m$. By definition, this is given by a homomorphism of topological abelian groups, from $\mathbf{CP}^\infty$ into $\mathbb{G}_m(A)$. This also can be rewritten in terms of group algebras: it is the same thing as a map of A-algebras from $A[\mathbf{CP}^\infty]$ into A.

Let S denote the *sphere spectrum*: this is the initial object in the world of E_∞-rings. To give an A-algebra map from $A[M]$ into B is equivalent to giving a map of E_∞-rings from $S[M]$ into B. In particular, to give a preorientation of the multiplicative group $\mathbb{G}_m$ over A is equivalent to giving a map of E_∞-rings from $S[\mathbf{CP}^\infty]$ into A. In other words, the E_∞-ring $S[\mathbf{CP}^\infty]$ *classifies* preorientations of $\mathbb{G}_m$.

Remark 3.3. The group algebra $S[\mathbf{CP}^\infty]$ is more typically denoted by $\Sigma^\infty \mathbf{CP}^\infty_+$, and is called the (unreduced) *suspension spectrum* of the space $\mathbf{CP}^\infty$.

Let us now suppose that we have a preorientation σ of $\mathbb{G}_m$ over an E_∞-ring A, classified by a map $S[\mathbf{CP}^\infty] \to A$. The underlying ordinary scheme of $\mathbb{G}_m$ is the ordinary multiplicative group $\mathrm{Spec}(\pi_0 A)[t, t^{-1}]$, and the sheaf of differentials on this scheme has a canonical generator $\frac{dt}{t}$. The restriction ω of this sheaf along the identity section is again canonically trivial, and the map

$$\omega \to \pi_2 A$$

induced by the preorientation σ can be identified with an element $\beta_\sigma \in \pi_2 A$. Examining Definition 3.3, we see that σ is an *orientation* if and only if β_σ is *invertible*.

We note that β_σ is functorial: if we are given an E_∞-map $f : A \to B$, then we get an induced preorientation $f^*\sigma$ of the multiplicative group over B, and $\beta_{f^*\sigma}$ is the image of β_σ under the induced map

$$\pi_2 A \to \pi_2 B.$$

In particular, β_σ itself is the image of a universal element $\beta \in \pi_2 S[\mathbf{CP}^\infty]$ under the map $S[\mathbf{CP}^\infty] \to A$.

The identification of $\beta \in \pi_2 S[\mathbf{CP}^\infty]$ is a matter of simple calculation. The group algebra $S[\mathbf{CP}^\infty]$ can be identified with an E_∞-ring *space*, and this space admits a canonical (multiplicative) map from $\mathbf{CP}^\infty$. The class β can be identified with the composite map

$$S^2 \simeq \mathbf{CP}^1 \subseteq \mathbf{CP}^\infty \to S[\mathbf{CP}^\infty].$$

(At least up to translation: this composite map carries the base point of S^2 to the multiplicative identity of $S[\mathbf{CP}^\infty]$, rather than the additive identity).

To classify *orientations* on $\mathbb{G}_m$, we need to "invert" the element β in $S[\mathbf{CP}^\infty]$. In order words, we want to construct a map of E_∞-rings $f : S[\mathbf{CP}^\infty] \to S[\mathbf{CP}^\infty][\beta^{-1}]$ such that, for every E_∞-ring A, composition with f induces a homotopy equivalence of $Map(S[\mathbf{CP}^\infty][\beta^{-1}], A)$ with the subspace of $Map(S[\mathbf{CP}^\infty], A)$ consisting of maps which carry β to an invertible element in $\pi_* A$. This is a bit more subtle. It is easy to show that an E_∞-ring with the desired universal property exists.

With some additional work, one can show that it has the expected structure: that is, the natural map

$$(\pi_* S[\mathbf{CP}^\infty])[\beta^{-1}] \to \pi_*(S[\mathbf{CP}^\infty][\beta^{-1}])$$

is an isomorphism. The structure of this E_∞-ring is the subject of the following theorem of Snaith.

Theorem 3.1. *The E_∞-ring $S[\mathbf{CP}^\infty][\beta^{-1}]$ is equivalent to (periodic) complex K-theory.*

Remark 3.4. The original formulation of Theorem 3.1 does not make use of the theory of E_∞-rings. However, it is easy to construct an E_∞ map $S[\mathbf{CP}^\infty][\beta^{-1}] \to K$ (in our language, this map classifies the orientation of $\mathbb{G}_m$ over K-theory), and the real content of Snaith's theorem is that this map is an equivalence.

Remark 3.5. In Example 1.4, we saw that Landweber's theorem could be used to produce complex K-theory, *as a cohomology theory*, starting with purely algebraic data. We can view Theorem 3.1 as a much more sophisticated version of the same idea: we now recover complex K-theory *as an E_∞-ring*, as the solution to a moduli problem. We will see later that the moduli-theoretic interpretation of Theorem 3.1 leads to purely algebraic constructions of equivariant complex K-theory as well.

Remark 3.6. The topological $\mathbf{CP}^\infty$ is a classifying space for complex line bundles; it is therefore natural to imagine that the points of $\mathbf{CP}^\infty$ *are* complex lines. Following this line of thought, we can imagine a similar description of the E_∞-ring space $S[\mathbf{CP}^\infty]$: points of $S[\mathbf{CP}^\infty]$ are given by formal sums of complex lines. Of course, this space is very different from the classifying space $\mathbf{Z} \times BU$ for complex K-theory, whose points are given by (virtual) vector spaces. The content of Theorem 3.1 is that this difference disappears when the Bott element is inverted. A very puzzling feature of Theorem 3.1 is the apparent absence of any direct connection of the theory of vector bundles with the problem of orienting the multiplicative group.

Remark 3.7. According to Theorem 3.1, the E_∞-ring K classifies orientations of *the* multiplicative group $\mathbb{G}_m$. However, one could consider a more general problem of orienting a commutative A-group $q : \mathbb{G} \to \operatorname{Spec} A$ which happened to *look like* the multiplicative group $\mathbb{G}_m$, in the sense that the underlying ordinary group scheme $\mathbb{G}_0$ is a 1-dimensional torus over $\operatorname{Spec} \pi_0 A$. It turns out that this problem is not really more general: as in ordinary algebraic geometry, tori are *rigid*, so that $\mathbb{G}$ is isomorphic to the usual multiplicative group $\mathbb{G}_m$ after passing to a double cover of $\operatorname{Spec} A$. In other words, we can understand all of the relevant structure by thinking about the usual multiplicative group $\mathbb{G}_m$ together with its automorphism group, which is cyclic of order 2. The automorphism group also acts on the classifying E_∞-ring K, and this action corresponds to the operation of complex conjugation (on complex vector bundles).

In other words, by thinking not only about *the* multiplicative group but *all* multiplicative groups, we can recover not only complex K-theory but also real K-theory.

3.2 Orientations of the Additive Group

In Sect. 3.1 we studied the problem of *orienting* the multiplicative group $\mathbb{G}_m$. In this section, we wish to discuss the analogous problem for the additive group $\mathbb{G}_a$. Our first task is to *define* what we mean by the additive group $\mathbb{G}_a$.

One choice would be to define $\mathbb{G}_a$ so as to represent the functor which carries an E_∞-ring A to its underlying "additive group." However, we immediately encounter two problems. First, the addition on A is generally not commutative enough: we can regard A has having an "underlying space" which is an infinite loop space, but this underlying space is generally not homotopy equivalent to a topological abelian group. We can construct a derived group scheme which represents this functor: let us denote it by $\mathbb{A}^1$. But the group structure on $\mathbb{A}^1$ is not sufficiently commutative to carry out the constructions we will need in Sect. 3.3.

A second problem is that the derived scheme $\mathbb{A}^1$ is generally not *flat* over A. The A-scheme $\mathbb{A}^1$ may be written as Spec $A\{X\}$, where $A\{X\}$ denotes the free E_∞-algebra generated over A by one indeterminate X. However, the homotopy groups of $A\{X\}$ are perhaps not what one would naively expect: one has

$$\pi_k A\{X\} = \bigoplus_{n \geq 0} A^{-k}(B\Sigma_n).$$

Here Σ_n denotes the symmetric group on n letters. This calculation coincides with the naive expectation $(\pi_k A)[X]$ if and only if, for every $n \geq 0$, the classifying space $B\Sigma_n$ is acyclic with respect to A-cohomology.

Remark 3.8. We would encounter the same difficulties if we used the "naive" procedure above to define the multiplicative group. Namely, there is a derived scheme GL_1, which associates to every E_∞-ring A the underlying "multiplicative group" of A. This derived scheme GL_1 is defined over the sphere spectrum S. However, it is *not* a commutative S-group in the sense of Definition 3.1, because it is neither flat over S nor can it be made to take values in topological abelian groups. In particular, it does not coincide with the commutative S-group $\mathbb{G}_m$ defined in Sect. 3.1. Instead, there is a natural map

$$\mathbb{G}_m \to GL_1$$

which is an equivalence over the rational numbers $\mathbf{Q}$. In general, one may regard $\mathbb{G}_m$ as universal among commutative S-groups admitting a homomorphism to GL_1.

Since defining the additive group $\mathbb{G}_a$ over a general E_∞-ring A seems to be troublesome, we will choose a less ambitious starting point. We can certainly make sense of the *ordinary* additive group $\mathbb{G}_a = \mathrm{Spec}\mathbf{Z}[X]$ over the ring of integers $\mathbf{Z}$. This is a commutative group scheme over the ordinary scheme $\mathrm{Spec}\mathbf{Z}$. Since it is flat over $\mathbf{Z}$, we may also regard it as a commutative $\mathbf{Z}$-group in the sense of Definition 3.1. (We note that the ordinary commutative ring $\mathbf{Z}[X]$, when regarded as an E_∞-algebra over $\mathbf{Z}$, is not *freely* generated by X: this is the difference between $\mathbb{G}_a$ and the derived scheme $\mathbb{A}^1$ considered above).

Let us now suppose we are given a map of E_∞-rings $\mathbf{Z} \to R$, and consider the problem of finding a preorientation of $\mathbb{G}_a$ over R: in other words, the problem of finding a homomorphism of topological abelian groups $\mathbf{CP}^\infty \to \mathbb{G}_a(R)$. As in the case of the multiplicative group $\mathbb{G}_m$, this is equivalent to giving a map of E_∞-rings $A \to R$, for a certain $\mathbf{Z}$-algebra A. A calculation similar to the one given in Sect. 3.1 allows us to identify A with the group algebra $\mathbf{Z}[\mathbf{CP}^\infty]$; this is an E_∞-ring with homotopy groups given by

$$\pi_* \mathbf{Z}[\mathbf{CP}^\infty] = H_*(\mathbf{CP}^\infty; \mathbf{Z}),$$

where multiplication is given by the Pontryagin product. As a ring, $\pi_*\mathbf{Z}[\mathbf{CP}^\infty]$ is a free divided power series algebra over $\mathbf{Z}$, on a single generator $\beta \in \pi_2\mathbf{Z}[\mathbf{CP}^\infty]$.

A preorientation $\sigma : \mathbf{Z}[\mathbf{CP}^\infty] \to R$ of the additive group over R is an orientation if and only if $\sigma(\beta) \in \pi_2 R$ is invertible. Consequently, the universal E_∞-ring over which we have an orientation of $\mathbb{G}_a$ is the localization

$$\mathbf{Z}[\mathbf{CP}^\infty][\beta^{-1}] \simeq \mathbf{Q}[\mathbf{CP}^\infty][\beta^{-1}] \simeq K \otimes \mathbf{Q}.$$

This is the E_∞-ring which represents *periodic* rational cohomology.

Remark 3.9. Let us say that A is a *rational* E_∞-ring if there is a map of E_∞-rings from the field $\mathbf{Q}$ to A (such a map is automatically unique, up to a contractible space of choices). Equivalently, A is rational if the ring $\pi_0 A$ is a vector space over $\mathbf{Q}$.

If A is rational, then the difficulties in defining the additive group over A dissolve: classifying spaces BG for finite groups are acyclic with respect to A-cohomology, so the free E_∞-algebra $A\{X\}$ has the expected homotopy groups and is flat over A; moreover, the underlying "space" of any A-algebra is (naturally) homotopy equivalent to a topological abelian group. Consequently, we get an equivalence $\mathbb{A}^1 \simeq \mathbb{G}_a$ over A, so $\mathbb{A}^1$ has the structure of a commutative A-group.

The point of the above discussion is that one can make sense of the additive group $\mathbb{G}_a$ in a bit more generality. For our purposes, this turns out to be irrelevant: although $\mathbb{G}_a$ can be defined over $\mathbf{Z}$, it can only be oriented over $\mathbf{Q}$.

Remark 3.10. Like the multiplicative group $\mathbb{G}_m$, the additive group $\mathbb{G}_a$ has nontrivial automorphisms. Provided that we work over $\mathbf{Q}$, these are parametrized by the multiplicative group $\mathbb{G}_m$. Consequently, $\mathbb{G}_m$ acts also on the E_∞-ring $\mathbf{Q}[\beta, \beta^{-1}]$ which classifies orientations of $\mathbb{G}_a$. Passing to invariants with respect to this action, we recover ordinary (nonperiodic) cohomology with coefficients in $\mathbf{Q}$.

3.3 *The Geometry of Preorientations*

Throughout this section, we fix an E_∞-ring A and a commutative A-group $\mathbb{G}$. Supposing that there exists a good equivariant version of A-cohomology, we would expect that for any compact Lie group G and any G-space X, the equivariant cohomology $A_G(X)$ is a module over $A_G(*)$. If $G = S^1$, then $A_G(*)$ can be identified with the ring of functions on $\mathbb{G}$; it is therefore natural to expect $A_G(X)$ to be obtained as the global sections of a quasi-coherent sheaf on $\mathbb{G}$. More generally, for every compact Lie group G one can construct a derived scheme M_G and obtain $A_G(*)$ as the E_∞-algebra of functions on M_G, and $A_G(X)$ as the global sections of a certain quasi-coherent sheaf on M_G. In this section, we address the first step: constructing the derived scheme M_T in the case where T is a compact *abelian* Lie group.

Let $X^*(T)$ denote the character group $\mathrm{Hom}(T, S^1)$; then $X^*(T)$ is a finitely generated abelian group (isomorphic to $\mathbf{Z}^n$ if T is connected). We can recover T from its character group via the isomorphism $T \simeq \mathrm{Hom}(X^*(T), S^1)$ (a special case of the Pontryagin duality theorem).

We let M_T denote the derived A-scheme which classifies maps of abelian groups from $X^*(T)$ into $\mathbb{G}$. In other words, for every commutative A-algebra B, the space of B-valued points $M_T(B)$ is homotopy equivalent to a space of homomorphisms from $X^*(T)$ into $\mathbb{G}(B)$ (if T is not connected, the abelian group $X^*(T)$ is not a free abelian group and one must be careful to use the *derived* mapping space between the topological abelian groups).

Example 3.1. If $T = S^1$, then $M_T = \mathbb{G}$. More generally, if T is an n-dimensional torus, then M_T is the n-fold fiber power of $\mathbb{G}$ over Spec A. In particular, $M_{\{e\}} =$ Spec A.

Example 3.2. Let T be a cyclic group $\mathbf{Z}/n\mathbf{Z}$ of order n. Then M_T is equivalent to the *kernel* $\mathbb{G}[n]$ of the multiplication-by-n-map

$$\mathbb{G} \xrightarrow{n} \mathbb{G}.$$

Remark 3.11. If $\mathbb{G}$ is affine, then each M_T is also affine: in this case, we can dispense with derived schemes entirely and work at the level of E_∞-rings. However, we will be primary interested in the case of elliptic cohomology, where $\mathbb{G}$ is *not* affine and the geometric language is indispensable.

The geometric object M_T is meant to encode the T-equivariant A-cohomology of a point. Let $*/T$ denote the *orbifold* quotient of a point by the group T. Then the T-equivariant A-cohomology of a point ought to be identified with the *orbifold* A-cohomology of $*/T$: that is, it ought to depend only on $*/T$, and not on T itself. In other words, it ought to be independent of the chosen basepoint of $*/T$.

We may rephrase the situation as follows. Given a commutative A-group $\mathbb{G}$, we have constructed a functor

$$T \mapsto M_T$$

from compact abelian Lie groups to derived A-schemes. We wish to factor this through the classifying space functor. In other words, we want a functor $\widetilde{M}$, defined on the (topological) category of spaces of the form BT (T a compact abelian Lie group), such that

$$\widetilde{M}(BT) \simeq M_T.$$

The above formula determines the behavior of $\widetilde{M}$ on objects. To finish the job, we note that the space of maps from BT to BT' is homotopy equivalent to a product $BT' \times \mathrm{Hom}(T, T')$: the first factor is given by the image of the basepoint of BT in BT', and the second factor is a model for the space of base-point-preserving maps from BT to BT'. The functor M is already defined on the second factor. To complete the definition of $\widetilde{M}$, we need to produce a map

$$BT' \to \mathrm{Hom}(M_T, M_{T'}).$$

Moreover, this should be suitably functorial in T and T'. Functoriality in T implies that we need only define this map in the universal case $T = \{e\}$; that is, we need to produce a map

$$\mathrm{Hom}(X^*(T'), \mathbf{C}\mathrm{P}^\infty) \simeq BT' \to M_{T'}(A) = \mathrm{Hom}(X^*(T'), \mathbb{G}(A)).$$

This map is required to be functorial in the character group $X^*(T')$; it is therefore determined by its behavior in the universal case where $X^*(T') = \mathbf{Z}$. Consequently, we have sketched the proof of the following:

Proposition 3.1. *Let A be an E_∞-ring, let $\mathbb{G}$ a commutative A-group and let M_T be defined as above. Let $\mathcal{C}$ be the (topological) category of spaces having the homotopy type of BT, where T is a compact abelian Lie group.*

The following data are equivalent:

1. *Covariant (topological) functors $\widetilde{M}$ from $\mathcal{C}$ to derived A-schemes, together with functorial identifications $\widetilde{M}(BT) \simeq M_T$.*
2. *Preorientations $\sigma : \mathbf{C}\mathrm{P}^\infty \to \mathbb{G}(A)$ of $\mathbb{G}$.*

Remark 3.12. It is possible to sharpen Proposition 3.1 further. The functor $T \mapsto M_T$ is one way of encoding the commutative group structure on $\mathbb{G}$. Consequently, specifying a preoriented A-group is equivalent to specifying a functor $\widetilde{M}$ from $\mathcal{C}$ to A-schemes, such that $\widetilde{M}$ preserves certain Cartesian diagrams. This point of view is relevant when it comes to studying elliptic cohomology over the *compactified* moduli stack of elliptic curves, where there is a similar functor $\widetilde{M}$ which does *not* commute with products.

3.4 Equivariant A-Cohomology for Abelian Groups

Let A be an E_∞-ring and $\mathbb{G}$ a preoriented A-group. In the last section, we constructed a derived A-scheme M_T, for every compact abelian Lie group T. Moreover, we showed that M_T really depends only on the classifying space BT, and not on a choice of basepoint on BT.

Let T be a compact abelian Lie group, and X a space on which T acts. We wish to define the T-equivariant cohomology group $A_T(X)$. In the case where X is a point, we have already described the appropriate definition: we should take the global sections of the structure sheaf of the space M_T. More generally, we will obtain $A_T(X)$ as the global sections of a certain sheaf $\mathcal{F}_T(X)$ on M_T.

We will say that a T-space X is *finite* if it admits a filtration

$$\emptyset = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n = X$$

where $X_{i+1} = X_i \coprod_{T/T_0 \times S^k} (T/T_0 \times D^{k+1})$ is obtained from X_i by attaching a T-equivariant cell $(T/T_0 \times D^{k+1})$.

Theorem 3.2. *There exists a collection of functors $\{\mathcal{F}_T\}$, defined for every compact abelian Lie group T and essentially uniquely prescribed by the following properties:*

1. *For every compact abelian Lie group T, the functor $\mathcal{F}_T$ is a contravariant functor from finite T-spaces to quasi-coherent sheaves on M_T, which carries T-equivariant homotopy equivalences to equivalences of quasi-coherent sheaves.*
2. *For fixed T, the functor $\mathcal{F}_T$ carries finite homotopy colimits of T-spaces to homotopy limits of quasi-coherent sheaves.*

(3) *If X is a point, then $\mathcal{F}_T = \mathcal{O}_{M_T}$.*
(4) *Let $T \subseteq T'$, let X be a finite T-space, and $X' = (X \times T')/T$ the induced finite T'-space. Let $f : M_{T'} \to M_T$ be the induced morphism of derived schemes. Then $\mathcal{F}_{T'}(X') = f_* \mathcal{F}_T(X)$.*

Here is a sketch of the proof: suppose we wish to define $\mathcal{F}_T(X)$, where X is a finite T-space. Using (2), we can reduce to the case where X is an individual cell: in fact, to the case where X is a T-orbit T/T_0, where $T_0 \subseteq T$ is a closed subgroup. By condition (4), we have $\mathcal{F}_T(T/T_0) = f_* \mathcal{F}_{T_0}(*)$, where $f : M_{T_0} \to M_T$ is the induced map. Finally, $\mathcal{F}_{T_0}(*)$ is determined by condition (3).

Remark 3.13. We could extend the functor $\mathcal{F}_T$ formally to *all* T-spaces, but we have refrained from doing so because inverse limit constructions behave poorly in the setting of quasi-coherent sheaves on M_T. For general T-spaces, it is T-equivariant *homology* which has better formal properties at the level of sheaves on M_T.

Remark 3.14. Suppose that X is a T-space on which T acts transitively. Then X is abstractly isomorphic to T/T_0, but the isomorphism is not canonical unless we specify a base point on X. Consequently, the identification $\mathcal{F}_T(X) \simeq f_* \mathcal{O}_{M_{T_0}}$ is not quite canonical either; however, the ambiguity that results from the failure to specify a base point on X is *precisely* accounted for by the fact that M_T is depends only on the classifying space BT. In other words, a preorientation of $\mathbb{G}$ is precisely the datum needed to make the above prescription work.

Remark 3.15. To flesh out our sketch of the proof of Theorem 3.2, we would need to sharpen requirements (1) through (4) somewhat. For example, the isomorphisms

$$f_* \mathcal{F}_T(X) \simeq \mathcal{F}_{T'}(X')$$

should be suitably compatible with the formations of *chains* of subgroups $T \subseteq T' \subseteq T''$. We will not spell out the precise axiomatics of the situation here.

We are now prepared to define equivariant A-cohomology. Namely, for any compact abelian Lie group T and any finite T-space X, let $A_T(X) = \Gamma(M_T, \mathcal{F}_T(X))$ be the global sections of the sheaf $\mathcal{F}_T(X)$. Then $A_T(X)$ is a cohomology theory defined on finite T-spaces. We may, if we wish, go further to extract cohomology *groups* via the formula

$$A_T^n(X) = \pi_{-n} A_T(X).$$

The reader who is familiar with equivariant homotopy theory might, at this point, raise an objection. We have defined a cohomology theory on the category of T-spaces, which assigns to each finite T-space X the abelian group $A_T^0(X)$. By general nonsense, this functor is represented by a G-space $Z(0)$. Moreover, the functors $A_T^n(X)$ are represented by spaces $Z(n)$, which are deloopings of $Z(0)$. In other words, $Z(0)$ is a T-equivariant infinite loop space. However, in equivariant stable homotopy theory one demands more: namely, one wishes to be able to deloop not only with respect to ordinary spheres, but also spheres with a nontrivial (linear) action of T. To extract the necessary deloopings, we need to introduce a few more definitions.

By construction, the sheaves $\mathcal{F}_T(X)$ are not merely sheaves of *modules* on M_T, but actually sheaves of E_∞-algebras. By functoriality, there are natural maps

$$\mathcal{F}_T(X) \to \mathcal{F}_T(X \times Y) \leftarrow \mathcal{F}_T(Y)$$

and therefore a map $\mathcal{F}_T(X) \otimes \mathcal{F}_T(Y) \to \mathcal{F}_T(X \times Y)$.

Let T be a compact abelian Lie group, and let $X_0 \subseteq X$ be finite T-spaces. We define $\mathcal{F}_T(X, X_0)$ to be the fiber of the map

$$\mathcal{F}_T(X) \to \mathcal{F}_T(X_0).$$

The multiplication maps defined above extend to give maps $\mathcal{F}_T(X, X_0) \otimes \mathcal{F}_T(Y) \to \mathcal{F}_T(X \times Y, X_0 \times Y)$.

Proposition 3.2. *Suppose that A is an E_∞ ring and let $\mathbb{G}$ be an oriented A-group. Let T be a compact abelian Lie group, let V be a finite dimensional unitary representation of T, and let $SV \subseteq BV$ be the unit sphere and the unit ball in V, respectively. Then:*

1. *The quasi-coherent sheaf* $\mathcal{L}_V = \mathcal{F}_T(BV, SV)$ *is a line bundle on* M_T.
2. *For every finite* T*-space* X*, the natural map*

$$\mathcal{L}_V \otimes \mathcal{F}_T(X) \to \mathcal{F}_T(X \times BV, X \times SV)$$

is an equivalence.

To give the flavor of the proof of Proposition 3.2, let us consider the case where T is the circle group, and V its defining 1-dimensional representation. The sheaf $\mathcal{L}_V$ is defined to be the fiber of the map

$$\mathcal{F}_T(BV) \to \mathcal{F}_T(SV).$$

The T-space BV is equivariantly contractible, and the T-space SV is T-equivariantly homotopy equivalent to T itself. Consequently, the sheaf on the left hand side is the structure sheaf $\mathcal{O}_{\mathbb{G}}$, and the sheaf on the right hand side is the structure sheaf of the identity section of $\mathbb{G}$. Thus, $\mathcal{F}_T(BV, SV)$ can be viewed as the ideal sheaf for the identity section of $\mathbb{G}$. Assertion (1) of Proposition 3.2 asserts that $\mathcal{L}_V$ is invertible: this follows from the assumption that $\mathbb{G}$ is oriented, which implies that the underlying ordinary scheme $\mathbb{G}_0$ is smooth of relative dimension 1 over $\operatorname{Spec}\pi_0 A$.

Now let us suppose that $\mathbb{G}$ is an oriented derived commutative group scheme over an E_∞-ring A. For every finite dimensional complex representation V of a compact abelian Lie group T, we let $\mathcal{L}_V = \mathcal{F}_T(BV, SV)$ be the line bundle on M_T whose existence is asserted by Proposition 3.2. There are natural maps

$$\mathcal{L}_V \otimes \mathcal{L}_{V'} \to \mathcal{L}_{V \oplus V'}$$

which are equivalences in view of assertion (2) of Proposition 3.2. Consequently, the definition $\mathcal{L}_V$ extends to the case where V is a *virtual* representation of T.

For every finite T-space X and every virtual representation V of T, we define

$$A_T^V(X) = \pi_0 \Gamma(M_T, \mathcal{F}_T(X) \otimes \mathcal{L}_V^{-1}).$$

Each functor $X \mapsto A_T^V(X)$ is represented by a T-space $Z(V)$. If V is an actual representation of T, then $Z(V)$ is, up to T-equivariant homotopy equivalence, a delooping of $Z(0)$ with respect to the 1-point compactification of V. Consequently, when $\mathbb{G}$ is *oriented*, then the above construction yields an actual T-equivariant cohomology theory, defined in degrees indexed by the virtual representations of T.

3.5 The Nonabelian Case

Let A be an E_∞-ring, and $\mathbb{G}$ an oriented A-group. In Sect. 3.4, we constructed an equivariant cohomology theory A_G for every compact *abelian* Lie group G. We now wish to treat the case where G is nonabelian. We will do so by formally extrapolating from the abelian case. Namely, we claim the following:

Proposition 3.3. *There exists a family of functors $X \mapsto A_G(X)$, defined for all compact Lie groups G, and essentially characterized by the following properties:*

1. *For every compact Lie group G, A_G is a contravariant functor from G-spaces to spectra, which preserves equivalences.*
2. *For every inclusion $H \subseteq G$ of compact Lie groups, there are natural equivalences*
$$A_H(X) \simeq A_G((X \times G)/H).$$
3. *The functor A_G carries homotopy colimits of G-spaces to homotopy limits of spectra.*
4. *In the case where G is* abelian *and X is a finite G-space, the functor A_G coincides with the functor defined in Sect. 3.4.*
5. *Let $E^{ab}G$ be a G-space with the following property: for any closed subgroup $H \subseteq G$, the set $(E^{ab}G)^H$ of H-fixed points of Y is empty if H is nonabelian, and weakly contractible if H is abelian. Then, for any G-space X, the natural map*
$$A_G(X) \to A_G(X \times E^{ab}G)$$
is an equivalence.

We sketch the proof. By (5), we can reduce to defining $A_G(X)$ in the case where the action of G on X has only abelian stabilizer groups. Replacing X by a G-cell complex if necessary, we can assume that X is composed of cells modelled on G/H, where $H \subseteq G$ is abelian. Using property (3), we can reduce to the case where $X = G/H$. Property (2) then forces $A_G(X) = A_H(*)$, which is determined by property (4).

We remark that conditions (1) through (3) are obvious and natural demands to place on any good theory of equivariant cohomology. Condition (4) is an equally natural demand, given that we want to build on the definition that we have already given in the abelian case. Condition (5), on the other hand, is more mysterious. We could define a *different* version of equivariant A-cohomology if we were to replace (4) and (5) by the following conditions:

(4′) In the case where G is *trivial*, the functor $A_G(X)$ coincides with the function spectrum A^X.

(5′) For any G-space X, the natural map
$$A_G(X) \to A_G(X \times EG)$$
is an equivalence.

Properties (1) through (5′) characterize Borel-equivariant cohomology. Condition (5) is considerably weaker than condition (5′), but it is still a rather severe assumption. It asserts that the equivariant cohomology theory associated to a nonabelian group G is formally determined by equivariant cohomology theories associated to abelian subgroups of G. In other words, $A_G(X)$ should be given by a Borel construction *relative to abelian subgroups of G*. Why should we expect a good equivariant version of A-cohomology to satisfy (5), when the analogous condition ($5'$) is unreasonable? We offer several arguments:

- In the case where A is complex K-theory, and we take $\mathbb{G}$ to be the multiplicative group $\mathbb{G}_m$ with its natural orientation, the equivariant cohomology theories described by Proposition 3.3 coincide with ordinary equivariant K-theory, even for nonabelian groups. In other words, assumption (5) above *is satisfied* in the case of complex K-theory.
- In the case where G is abelian and X is a finite G-space, the constructions of Sect. 3.4 give much more than the G-equivariant cohomology theory A_G. Namely, we had also a geometric object M_G, and an interpretation of $A_G(X)$ as the global sections of a certain sheaf $\mathcal{F}_G$ on M_G. There is similar geometry associated to the case where G is nonabelian, at least provided that G is connected.
- Suppose that one can construct a derived scheme M_G and a functor $\mathcal{F}_G$ as in Sect. 3.3 and Sect. 3.4, where G is a nonabelian compact Lie group. Suppose further that Proposition 3.2 remains valid in this case. Then it is possible to prove that assumption (5) holds, using the method of *complex-oriented descent* (as explained, for example, in [11]).
- When $\mathbb{G}$ is an oriented elliptic curve and G is a *connected* compact Lie group, the theory of G-equivariant A-cohomology described by Proposition 3.3 is closely related to interesting geometry, such as the theory of regular $G_{\mathbf{C}}$-bundles on elliptic curves and nonabelian theta functions.

4 Oriented Elliptic Curves

In Sect. 3, we described the theory of oriented A-groups $\mathbb{G}$, where A is an E_∞-ring. In this section, we consider the most interesting case: where $\mathbb{G}$ is an elliptic curve.

Definition 4.1. Let A be an E_∞-ring. An *elliptic curve* over A is a commutative A-group $E \to \operatorname{Spec} A$, having the property that the underlying map $\overline{E} \to \operatorname{Spec} \pi_0 A$ is an elliptic curve (in the sense of classical algebraic geometry).

Remark 4.1. If A is an ordinary commutative ring, regarded as an E_∞-ring, then Definition 4.1 is equivalent to the usual definition of an elliptic curve over A.

Remark 4.2. In ordinary algebraic geometry, one need not take the group structure on an elliptic curve $E \to \operatorname{Spec} A$ as part of the data. The group structure on E is *uniquely* determined, as soon as one specifies a base point $\operatorname{Spec} A \to E$. In derived algebraic geometry, this is generally not true: the group structure on E is not determined by the underlying derived scheme, even after a base point has been specified.

We now come to the main result of this survey:

Theorem 4.1. *Let $\mathcal{M}^{\mathrm{Der}} = (\mathcal{M}, \mathcal{O}^{\mathrm{Der}})$ denote the derived Deligne–Mumford stack of Example 2.2. For every E_∞-ring A, let $E(A)$ denote the classifying space for the (topological) category of* oriented *elliptic curves over A. Then there is a natural homotopy equivalence*

$$\mathrm{Hom}(\mathrm{Spec}\, A, \mathcal{M}^{\mathrm{Der}}) \simeq E(A).$$

In other words, $\mathcal{M}^{\mathrm{Der}}$ may be viewed as a moduli stack for *oriented* elliptic curves in derived algebraic geometry, just as its underlying ordinary stack classifies elliptic curves in classical algebraic geometry.

Remark 4.3. We have phrased Theorem 4.1 in reference to the work of Goerss–Hopkins–Miller, which establishes the existence and uniqueness of the structure sheaf $\mathcal{O}^{\mathrm{Der}}$. However, our proof of Theorem 4.1 does not require that we know existence of $\mathcal{O}^{\mathrm{Der}}$ in advance. Instead, we could begin by considering the functor

$$A \mapsto E(A)$$

and prove that it is representable by a derived Deligne–Mumford stack $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. The hard part is to show that $\mathcal{X}$ is equivalent to the étale topos of the ordinary moduli stack of elliptic curves, and that $\mathcal{O}_{\mathcal{X}}$ is a sheaf of E_∞-rings which satisfies the conclusions of Theorem 1.1. Consequently, our method yields a *new* proof of the existence of $\mathcal{O}^{\mathrm{Der}}$.

The rest of this section is devoted to sketching the proof of Theorem 4.1.

4.1 Construction of the Moduli Stack

Let A be an E_∞-ring. As in Theorem 4.1, we let $E(A)$ denote the classifying space of the (topological) category of *oriented* elliptic curves over Spec A. The first step is to prove that the functor

$$A \mapsto E(A)$$

is representable by a derived Deligne–Mumford stack. As a first approximation, we let $E'(A)$ denote the classifying space of the (topological) category of *preoriented* elliptic curves over Spec A. Let $\mathcal{M}_{1,1} = (\mathcal{M}, \mathcal{O}_{\mathcal{M}_{1,1}})$ denote the *classical* moduli stack of elliptic curves. We observe that every elliptic curve over an ordinary commutative ring R admits a *unique* preorientation (namely, zero). Consequently, the restriction of E' to ordinary commutative rings is represented by $\mathcal{M}_{1,1}$. We now apply the following general representability result:

Proposition 4.1. *Let $\mathcal{F}$ be a functor from connective E_∞-rings to spaces. Suppose that $\mathcal{F}$ satisfies the following conditions:*

1. *There exists a Deligne–Mumford stack $(\mathcal{X}, \mathcal{O})$ which represents the restriction of $\mathcal{F}$ to ordinary commutative rings. In other words, for any commutative ring R,*

the space $\mathcal{F}(R)$ is homotopy equivalent to the classifying space of the groupoid $\mathrm{Hom}(\mathrm{Spec}\ R, (\mathcal{X}, \mathcal{O}))$.

2. *The functor $\mathcal{F}$ satisfies étale descent.*
3. *The functor $\mathcal{F}$ has a well-behaved deformation theory.*

Then there exists a derived Deligne–Mumford stack $(\mathcal{X}, \widetilde{\mathcal{O}})$ which represents the functor $\mathcal{F}$. Moreover, $\widetilde{\mathcal{O}}(U)$ is a connective E_∞-ring whenever U is affine.

Proof (Proof sketch). In virtue of condition (2), the assertion is local on $\mathcal{X}$. We may therefore reduce to the case where $(\mathcal{X}, \mathcal{O})$ is isomorphic to Spec R, where R is a commutative ring. The idea is to obtain $(\mathcal{X}, \widetilde{\mathcal{O}}) = \mathrm{Spec}\widetilde{R}$, where $\widetilde{R}$ is a connective E_∞-ring with $\pi_0 \widetilde{R} = R$. One builds $\widetilde{R}$ as the inverse limit of a convergent tower of approximations, which are constructed using condition (3). For details, and a discussion of the meaning of (3), we refer the reader to [12].

We wish to apply Proposition 4.1 to the functor $A \mapsto E'(A)$. Condition (1) is clear: on ordinary commutative rings, E' is represented by the classical moduli stack $\mathcal{M}_{1,1}$. The remaining conditions are also not difficult to verify, using general tools provided by derived algebraic geometry. We conclude that there exists a derived Deligne–Mumford stack $(\mathcal{M}, \mathcal{O}')$ which represents the functor $A \mapsto E'(A)$, at least when the E_∞-ring A is connective. However, using the fact that elliptic curves are *flat* over A, one can show that $E'(A)$ is equivalent to $E'(\tau_{\geq 0} A)$, where $\tau_{\geq 0} A$ is the connective cover of A. It follows that $(\mathcal{M}, \mathcal{O}')$ represents the functor E' for *all* E_∞-rings A. Moreover, $\mathcal{M}$ is the étale topos of the ordinary moduli stack $\mathcal{M}_{1,1}$ of elliptic curves, $\pi_0 \mathcal{O}' \simeq \mathcal{O}_{\mathcal{M}_{1,1}}$, and $\pi_i \mathcal{O}'$ is a quasi-coherent sheaf on $\mathcal{M}_{1,1}$ for $i > 0$. With a bit more effort, one can show that each $\pi_i \mathcal{O}'$ is a coherent sheaf on $\mathcal{M}_{1,1}$.

Let ω denote the line bundle on $\mathcal{M}_{1,1}$ which associates to each elliptic curve $E \to \mathrm{Spec}\ R$ the R-module of invariant differentials on E. The preorientation of the universal elliptic curve over $(\mathcal{M}, \mathcal{O}')$ gives rise to a map $\beta : \omega \to \pi_2 \mathcal{O}'$ of coherent sheaves on $\mathcal{M}$. We can now define a new sheaf of E_∞-rings $\mathcal{O}$ on $\mathcal{M}$ by "inverting" β. This sheaf has the property that

$$\pi_n \mathcal{O} \simeq \varinjlim_k \{\pi_{n+2k} \mathcal{O}' \otimes_{\mathcal{O}_{\mathcal{M}_{1,1}}} \omega^{-k}\}$$

in the category of quasi-coherent sheaves on $\mathcal{M}_{1,1}$.

Remark 4.4. Let $U \to \mathcal{M}$ be affine, and suppose that the restriction of ω to U is free. Then $\mathcal{O}'(U) = A$ is a connective E_∞-ring, and we may identify β with an element of $\pi_2 A$. By definition, $\mathcal{O}(U)$ is the A-algebra $A[\beta^{-1}]$ obtained by inverting β. We note that $\mathcal{O}$ is *not* a connective sheaf of E_∞-rings; rather it is (locally) 2-periodic, by construction.

To complete the proof of Theorem 4.1, it will suffice to prove the following:

1. For $n = 2k$, the natural map $\omega^k \to \pi_n \mathcal{O}$ is an isomorphism of quasi-coherent sheaves on $\mathcal{M}_{1,1}$.

2. For $n = 2k + 1$, the quasi-coherent sheaf $\pi_n \mathcal{O}$ is trivial.

Indeed, suppose that (1) and (2) are satisfied. We first observe that $(\mathcal{M}, \mathcal{O})$ is a derived Deligne–Mumford stack. The assertion is local on $\mathcal{M}$, so we may restrict ourselves to an étale $U \to \mathcal{M}$ such that $(U, \mathcal{O}_{\mathcal{M}_{1,1}}|U)$ is affine and $\omega|U$ is trivial. In this case, $(U, \mathcal{O}_{\mathcal{M}_{1,1}}|U) \simeq \operatorname{Spec} R$, where R is a commutative ring; $A = \mathcal{O}'(U)$ is a connective E_∞-ring with $\pi_0 A \simeq R$, and we may identify the map β with an element in $\pi_2 A$. Then $\mathcal{O}(U) = A[\beta^{-1}]$. Condition (1) asserts that $R \simeq \pi_0 A[\beta^{-1}]$, so that $(U, \mathcal{O}|U)$ is equivalent to Spec $A[\beta^{-1}]$.

By construction, the derived Deligne–Mumford stack $(\mathcal{M}, \mathcal{O})$ represents the functor $A \mapsto E(A)$. To complete the proof, it suffices to show that $\mathcal{O}$ coincides with the sheaf $\mathcal{O}^{\mathrm{Der}}$ of E_∞-rings constructed by Goerss, Hopkins and Miller. For this, it suffices to show that over each affine $U = \operatorname{Spec} R$ of $\mathcal{M}$, the E_∞-ring $\mathcal{O}(U)$ gives rise to the elliptic cohomology theory associated to the universal (classical) elliptic curve $\mathcal{E}_U$ over U. In other words, we need to produce an isomorphism of the formal spectrum of $\mathcal{O}(U)(\mathbf{CP}^\infty)$ with the formal completion of the underlying ordinary elliptic curve of $\mathcal{E}_U$. By construction, we have such an isomorphism not only at the level of classical formal groups, but at the level of derived formal groups.

It remains to prove that (1) and (2) are satisfied. To simplify the discussion, we will consider only condition (2): the first condition can be handled by a similar but slightly more complicated argument.

Let n be an odd integer. We wish to show that colimit $\pi_n \mathcal{O}$ of the directed system

$$\{\pi_{n+2k} \mathcal{O}' \otimes_{\mathcal{O}_{\mathcal{M}_{1,1}}} \omega^{-k}\}$$

is zero. Since $\pi_n \mathcal{O}$ is generated by the images of $\pi_{n+2k} \mathcal{O}' \otimes_{\mathcal{O}_{\mathcal{M}_{1,1}}} \omega^{-k}$, it suffices to show that each of these images is zero. Replacing n by $n+2k$ if necessary, it suffices to show that $f : \pi_n \mathcal{O}' \to \pi_n \mathcal{O}$ is the zero map. Let $\mathcal{F}$ be the image of f. Then $\mathcal{F}$ is a quotient of the coherent sheaf $\pi_n \mathcal{O}'$, and therefore coherent. If $\mathcal{F} \neq 0$, then $\mathcal{F}$ has support at some closed point $\kappa : \operatorname{Spec} \mathbf{F}_q \to \mathcal{M}_{1,1}$. Consequently, to prove that condition (2) holds, it suffices to prove that (2) holds in a formal neighborhood of the point κ. In other words, we need not consider the entire moduli stack $\mathcal{M}_{1,1}$ of elliptic curves: it is sufficient to consider (a formal neighborhood of) a *single* elliptic curve defined over a finite field $\mathbf{F}_q$.

Remark 4.5. The above argument reduces the proof of Theorem 4.1 in characteristic zero to a statement in characteristic p. This reduction is not necessary: in characteristic zero, it is fairly easy to verify (1) and (2) directly. We sketch how this is done. Let A be an E_∞-algebra over the field $\mathbf{Q}$ of rational numbers. The theory of elliptic curves over A then reduces to the classical theory of elliptic curves, in the sense that they are classified by maps Spec $A \to \mathcal{M}_{1,1}$.

It follows that, rationally, $\mathcal{O}'$ is an algebra over $\pi_0 \mathcal{O}' \simeq \mathcal{O}_{\mathcal{M}_{1,1}}$. Moreover, it is easy to work out the structure of this algebra: namely, $\pi_* \mathcal{O}'$ is the symmetric algebra on ω over $\mathcal{O}_{\mathcal{M}_{1,1}}$. Consequently, we observe that the direct system $\{\omega^{-k} \otimes_{\mathcal{O}_{\mathcal{M}_{1,1}}} \pi_{n+2k} \mathcal{O}'\}$ is actually *constant* for $n + 2k > 0$, isomorphic to ω^m if $n = 2m$ is even and 0 otherwise.

If we do not work rationally, the directed system $\{\omega^{-k} \otimes_{\mathcal{O}_{\mathcal{M}_{1,1}}} \pi_{n+2k}\mathcal{O}'\}$ is not constant. The sheaves $\pi_n\mathcal{O}'$ are complicated and we do not know how to compute them individually; only in the limit do we obtain a clean statement.

4.2 The Proof of Theorem 4.1: The Local Case

In Sect. 4.1, we reduced the proof of Theorem 4.1 to a local calculation, which makes reference only to a formal neighborhood of a closed point $\kappa : \mathrm{Spec}\mathbf{F}_q \to \mathcal{M}_{1,1}$ of the moduli stack of elliptic curves. In this section, we will explain how to perform this calculation, by adapting the theory of *p-divisible groups* to the setting of derived algebraic geometry. Let p be a prime number, fixed throughout this section.

Definition 4.2. Let A be an E_∞-ring. Let $\mathbb{G}$ be a functor from commutative A-algebras to topological abelian groups. We will say that $\mathbb{G}$ is a *p-divisible group of height h* over A if the following conditions hold:

1. The functor $B \mapsto \mathbb{G}(B)$ is a sheaf (in the ∞-categorical sense) with respect to the flat topology on A-algebras.
2. For each n, the multiplication map $p^n : \mathbb{G} \to \mathbb{G}$ is surjective (in the flat topology) with kernel $\mathbb{G}[p^n]$.
3. The colimit of the system $\{\mathbb{G}[p^n]\}$ is equivalent to $\mathbb{G}$ (as sheaves of topological abelian groups with respect to the flat topology).
4. For each $n \geq 0$, the functor $\mathbb{G}[p^n]$ is a commutative A-group, that is finite and flat over A of rank p^{nh}.

Remark 4.6. If A is an ordinary commutative ring, then Definition 4.2 recovers the usual notion of a p-divisible group over A. Since a p-divisible group $\mathbb{G}$ is determined by the flat derived A-schemes $\mathbb{G}[p^n]$, the theory of p-divisible groups over an arbitrary E_∞-ring A is equivalent to the theory of p-divisible groups over the connective cover $\tau_{\geq 0}A$.

Example 4.1. Let E be an elliptic curve over an E_∞-ring A. Let $E[p^n]$ denote the fiber of the map $p^n : E \to E$, and let $\mathbb{G}$ be the direct limit of the system $\{E[p^n]\}_{n\geq 0}$. Then $\mathbb{G}$ is a p-divisible group over A of height 2, which we will denote by $E[p^\infty]$.

Our approach to the proof of Theorem 4.1 rests on a derived analogue of the Serre–Tate theorem, which asserts that the deformation theory of an elliptic curve is equivalent to the deformation theory of its p-divisible group, provided that we work p-adically

Theorem 4.2. *Let A be an E_∞-ring. Suppose that $\pi_0 A$ is a complete, local, Noetherian ring whose residue field k has characteristic p, and that each of the homotopy groups $\pi_n A$ is a finitely generated module over $\pi_0 A$. Then there is a Cartesian diagram (of ∞-categories)*

$$\begin{tikzcd}
\{\textit{Elliptic curves } E \to \operatorname{Spec} A\} \arrow[r] \arrow[d] & \{\textit{p-divisible groups } E[p^\infty] \textit{over } A\} \arrow[d] \\
\{\textit{Elliptic curves } E_0 \to \operatorname{Spec} k\} \arrow[r] & \{\textit{p-divisible groups } E_0[p^\infty] \textit{over } k\}.
\end{tikzcd}$$

According to Theorem 4.2, giving an elliptic curve over A is equivalent to giving an elliptic curve E_0 over the residue field k of $\pi_0 A$, together with a lifting of the p-divisible group $E_0[p^\infty]$ to a p-divisible group over A. In other words, the deformation theory of elliptic curves is the same as the deformation theory of their p-divisible groups (in characteristic p).

Let A be an E_∞-ring satisfying the hypotheses of Theorem 4.2. It is possible to analyze the theory of p-divisible groups over A along the same lines as one analyzes the theory of p-divisible groups over $\pi_0 A$. Namely, every p-divisible group $\mathbb{G}$ over A has a *unique* filtration

$$\mathbb{G}_{\text{inf}} \to \mathbb{G} \to \mathbb{G}_{et}.$$

Here $\mathbb{G}_{\text{inf}}$ is a "purely infinitesimal" p-divisible group obtained from the p-power torsion points of a (uniquely determined) commutative *formal* A-group, and $\mathbb{G}_{et}$ is an étale p-divisible group associated to a p-adic local system on Spec A $=$ Spec$\pi_0 A$. As with commutative A-groups, we can speak of *preorientations* and *orientations* on a p-divisible group $\mathbb{G}$. A preorientation of $\mathbb{G}$ is equivalent to a preorientation of its infinitesimal part $\mathbb{G}_{\text{inf}}$. An orientation of $\mathbb{G}$ is an identification of $\mathbb{G}_{\text{inf}}$ with the p-power torsion on Spf $A^{\mathbf{CP}^\infty}$; in particular, an orientation of $\mathbb{G}$ can exist only if $\mathbb{G}$ is 1-dimensional.

Now suppose that k is a perfect field of characteristic p, and that we are given a point $\kappa : \operatorname{Spec} k \to \mathcal{M}_{1,1}$ corresponding to an elliptic curve E_0 over k. Let $\mathcal{O}'_\kappa$ denote the formal completion of the sheaf $\mathcal{O}'$ at the point κ. Then $\mathcal{O}'_\kappa$ is a connective E_∞-ring such that $\pi_0\mathcal{O}'_\kappa$ is the formal completion of the sheaf of functions on the classical moduli stack $\mathcal{M}_{1,1}$ at the point κ. There is a preoriented elliptic curve $E \to \operatorname{Spec}\mathcal{O}'_\kappa$, which we may regard as the universal deformation of E_0 (as a preoriented elliptic curve). Let $\overline{E}$ denote the underlying ordinary scheme of E; the $\pi_0\mathcal{O}'_\kappa$-module ω of invariant differentials on $\overline{E}$ is a free $\pi_0\mathcal{O}'_\kappa$-module of rank 1. Fixing a generator of ω, the preorientation of E gives an element $\beta \in \pi_2\mathcal{O}'_\kappa$. Let $\mathcal{O}_\kappa = \mathcal{O}'_\kappa[\beta^{-1}]$. As we saw in Sect. 4.1, Theorem 4.1 amounts to proving that $\mathcal{O}_\kappa$ is even and $\pi_0\mathcal{O}_\kappa \simeq \pi_0\mathcal{O}'_\kappa$.

In view of Theorem 4.2, we may reinterpret the formal spectrum Spf$\mathcal{O}'_\kappa$ in the language of p-divisible groups. Namely, let $\mathbb{G}_0$ denote the p-divisible group of the elliptic curve $E_0 \to \operatorname{Spec} k$. Since k is an ordinary commutative ring, $\mathbb{G}_0$ has a unique preorientation. The formal spectrum Spf$\mathcal{O}'_\kappa$ is the parameter space for the universal deformation of E_0 as a preoriented elliptic curve. By Theorem 4.2, this is also the parameter space for the universal deformation of $\mathbb{G}_0$ as a preoriented p-divisible group. More informally, we might also say that the E_∞-ring $\mathcal{O}_\kappa$ classifies *oriented* p-divisible groups which are deformations of $\mathbb{G}_0$.

There are now two cases to consider, according to whether or not the elliptic curve $E_0 \to \operatorname{Spec} k$ is supersingular. If E_0 *is* supersingular, then its p-divisible group $\mathbb{G}_0$ is entirely infinitesimal. It follows that any *oriented* deformation E of E_0 over an E_∞-ring R is uniquely determined: E is determined by its p-divisible group $E[p^\infty] \simeq (\mathrm{Spf} R^{\mathbf{CP}^\infty})[p^\infty]$. In concrete terms, this translates into a certain mapping property of the deformation ring $\mathcal{O}_\kappa$. One can show that this mapping property characterizes the *Lubin–Tate spectrum* E_2 associated to the formal group $\widehat{E_0}$, which is known to have all of the desired properties. (We refer the reader to [13] for a proof of the Hopkins-Miller theorem on Lubin–Tate spectra, which is very close to establishing the universal property that we need here).

If the elliptic curve $E_0 \to \mathrm{Spec}\ k$ is not supersingular, then we need to work a bit harder. In this case, any p-divisible group $\mathbb{G}$ deforming $\mathbb{G}_0 = E_0[p^\infty]$ admits a filtration

$$\mathbb{G}_{\mathrm{inf}} \to \mathbb{G} \to \mathbb{G}_{et}$$

where both $\mathbb{G}_{\mathrm{inf}}$ and $\mathbb{G}_{et}$ have height 1. Consequently, to understand the deformation theory of $\mathbb{G}_0$ we must understand three things: the deformation theory of $(\mathbb{G}_0)_{\mathrm{inf}}$, the deformation theory of $(\mathbb{G}_0)_{et}$, and the deformation theory of the space of extensions $Ext(\mathbb{G}_{et}, \mathbb{G}_{\mathrm{inf}})$.

The deformation theory of the étale p-divisible group $(\mathbb{G}_0)_{et}$ is easy. An étale p-divisible group over A is given by a p-adic local system over Spec A, which is the same thing as a p-adic local system on $\mathrm{Spec}\pi_0 A$. If A is a complete, local Noetherian ring with residue field k, these may be identified with p-adic local systems on Spec k: in other words, finite free $\mathbf{Z}_p$-modules with a continuous action of the absolute Galois group of k. In particular, this description is independent of A: $(\mathbb{G}_0)_{et}$ has a *unique* deformation to every formal thickening of k, even in derived algebraic geometry. To simplify the discussion which follows, we will suppose that k is algebraically closed. Then we may identify $(\mathbb{G}_0)_{et}$ with the constant (derived) group scheme $\mathbf{Q}_p/\mathbf{Z}_p$.

It is possible to analyze the infinitesimal part $(\mathbb{G}_0)_{\mathrm{inf}}$ in the same way, using the fact that it is the *dual* of an étale p-divisible group and therefore also has a trivial deformation theory. However, this is not necessary: remember that we are really interested in deformations of $\mathbb{G}_0$ which are *oriented*. As we saw in the supersingular case, the orientation of $\mathbb{G}$ determines the infinitesimal part $\mathbb{G}_{\mathrm{inf}}$, and the E_∞-ring which controls the universal deformation of $\mathbb{G}_{\mathrm{inf}}$ as an *oriented* p-divisible group is a Lubin–Tate spectrum E_1 (in this case, an unramified extension of the p-adic completion of complex K-theory).

Finally, we need to study the deformation theory $Ext(\mathbf{Q}_p/\mathbf{Z}_p, \mathbb{G}_{\mathrm{inf}})$. As in classical algebraic geometry, one shows that in a suitable setting one has isomorphisms

$$\mathbb{G}_{\mathrm{inf}} \simeq \mathrm{Hom}(\mathbf{Z}_p, \mathbb{G}_{\mathrm{inf}}) \simeq Ext(\mathbf{Q}_p/\mathbf{Z}_p, \mathbb{G}_{\mathrm{inf}}).$$

Consequently, to obtain the universal deformation E_∞-ring of $\mathbb{G}_0$ as an oriented p-divisible group, one should take the E_∞-ring of functions on the universal deformation of $(\mathbb{G}_0)_{\mathrm{inf}}$ as an oriented p-divisible group. Since this deformation is oriented, we may identify this E_∞-ring with $E_1^{\mathbf{CP}^\infty}$. A simple computation now shows that $\mathcal{O}_\kappa \simeq E_1^{\mathbf{CP}^\infty}$ has the desired properties.

Remark 4.7. The arguments employed above are quite general, and can be applied to p-divisible groups which do not necessarily arise from elliptic curves. For example, one can produce "derived versions" of certain Shimura varieties, at least p-adically, using the same methods. For more details, we refer the reader to [14].

4.3 Elliptic Cohomology near ∞

Classically, an elliptic curve $E \to \operatorname{Spec} \mathbf{C}$ is determined, up to noncanonical isomorphism, by its j-invariant $j(E) \in \mathbf{C}$. Consequently, the moduli *space* of elliptic curves is isomorphic to $\mathbf{C}$, which is not compact. One can compactify the moduli space by allowing elliptic curves to develop a nodal singularity.

We wish to extend the theory of elliptic cohomology to this compactification. To carry out this extension, it suffices to work locally in a formal neighborhood of $j = \infty$. Complex analytically, we can construct a family of elliptic curves over the punctured disk $\{q \in \mathbf{C} : 0 < |q| < 1\}$, which assigns to a complex parameter q the elliptic curve $E_q = \mathbf{C}^*/q^{\mathbf{Z}}$. This family has a natural extension over the disk $\{q \in \mathbf{C} : |q| < 1\}$, where E_q specializes to a nodal rational curve at $q = 0$. Provided that one is willing to work in a *formal* neighborhood of $q = 0$, one can even give an algebraic construction of the elliptic curve E_q. This algebraic construction gives a generalized elliptic curve T, the *Tate curve*, which is defined over $\mathbf{Z}[[q]]$. The fiber of T over $q = 0$ is isomorphic to a nodal rational curve, and its general fiber "is" $\mathbb{G}_m/q^{\mathbf{Z}}$, in a suitable sense.

We will sketch a construction of the Tate curve which makes sense in derived algebraic geometry. For this, we will assume that the reader is familiar with the language of *toric varieties* (for a very readable account of the theory of toric varieties, we refer the reader to [15]).

Fix an E_∞-ring R. Let Λ be a lattice (that is, a free $\mathbf{Z}$-module of finite rank) and let $F = \{\sigma_\alpha\}_{\alpha \in A}$ be a rational polyhedral fan in $\mathbf{Z}$. For each $\alpha \in A$, let $\sigma_\alpha^\vee \subseteq \Lambda^\vee$ denote the dual cone to σ_α, regarded as a commutative monoid. Then the monoid algebra $R[\sigma_\alpha^\vee]$ is an E_∞-ring, and we may define $U_\alpha = \operatorname{Spec} R[\sigma_\alpha^\vee]$. The correspondence $\alpha \mapsto U_\alpha$ is functorial: it carries inclusions of cones to open immersions of affine derived schemes. We may therefore construct a derived scheme $X_F = \varinjlim \{U_\alpha\}_{\alpha \in A}$ by gluing these affine charts together, using the pattern provided by the fan F. We call X_F the *toric variety over R defined by F*. When R is an ordinary commutative ring, this is a well-known classical construction; it makes perfect sense in derived algebraic geometry as well.

Let $F_0 = \{\{0\}, \mathbf{Z}_{\geq 0}\}$ be the fan in $\mathbf{Z}$ giving rise to the toric variety $X_{F_0} = \operatorname{Spec} R[\mathbf{Z}_{\geq 0}]$. We will write the E_∞-ring $R[\mathbf{Z}_{\geq 0}]$ as $R[q]$, though we should note that it is *not* the free E_∞-ring on one generator over R (the generator q satisfies relations that force it to *strictly* commute with itself). For each $n \in \mathbf{Z}$, let σ_n denote the cone

$$\{(a,b) \in \mathbf{Z} \times \mathbf{Z} | na \leq b \leq (n+1)a\},$$

and let F denote the fan in $\mathbf{Z} \times \mathbf{Z}$ consisting of the cones σ_n, together with all their faces. Projection onto the first factor gives a map $F \to F_0$ of fans, and therefore a map of toric varieties $f : X_F \to \operatorname{Spec} R[q]$. The fiber of f over any point $q \neq 0$ can be identified with the multiplicative group $\mathbb{G}_m$, while the fiber of f over the point $q = 0$ is an infinite chain of rational curves, each intersecting the next in a node.

Consider the automorphism $\tau : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z} \times \mathbf{Z}$ defined by $\tau(a,b) = (a, b+a)$. It is clear that $\tau(\sigma_n) = \sigma_{n+1}$, so that τ preserves the fan F and therefore defines an automorphism of X_F, which we shall also denote by τ. The action of τ is not free: for $q_0 \neq 0$, τ acts on the fiber $f^{-1}\{q_0\} \simeq \mathbb{G}_m$ by multiplication by q_0. However, the action of τ *is* free when it is restricted to the fiber $f^{-1}(0)$.

Let $\widehat{X}_F$ denote the formal completion of X_F along the fiber $f^{-1}\{0\}$, and let $R[[q]]$ denote the formal completion of $R[q]$ along the closed subset defined by the equation $q = 0$. The group $\tau^{\mathbf{Z}}$ acts freely on $\widehat{X}_F$, and thus we may define a formal derived scheme $\widehat{T}$ by taking the quotient of $\widehat{X}_F$ by the action of $\tau^{\mathbf{Z}}$. We note that the fiber of $\widehat{T}$ over $q = 0$ can be identified with a nodal rational curve; in particular, it is proper over Spec R. Using a generalization of the Grothendieck existence theorem to derived algebraic geometry, one can show that $\widehat{T}$ is the formal completion of a (uniquely determined) derived scheme $T \to \operatorname{Spec} R[[q]]$. We call T the *Tate curve*; its restriction to the punctured formal disk $\operatorname{Spec} R((q)) = \operatorname{Spec} R[[q]][q^{-1}]$ is an elliptic curve over $R((q))$, which we may think of as the quotient of the multiplicative group $\mathbb{G}_m$ by the subgroup $q^{\mathbf{Z}}$.

Of course, we are primarily interested in *oriented* elliptic curves. The Tate curve is essentially given as a quotient of the multiplicative group $\mathbb{G}_m$. In particular, its formal completion is equivalent to the formal completion of the multiplicative group. Consequently, giving an orientation of the Tate curve is *equivalent* to giving an orientation of the multiplicative group $\mathbb{G}_m$. By Theorem 3.1, this is equivalent to working over the E_∞-ring K (the complex K-theory spectrum). In other words, the Tate curve T over $K((q))$ is an *oriented* elliptic curve, which is therefore classified by a map

$$\operatorname{Spec} K((q)) \to \mathcal{M}^{\mathrm{Der}}.$$

Of course, there are additional symmetries which we should take into account. The involution $(a,b) \mapsto (a,-b)$ preserves the fan F and intertwines with the action of τ, and therefore induces an involution on T (the inverse map with respect to the group structure). This involution preserves the orientation on T, provided that we allow it to act also on the ground ring K. We therefore actually obtain a map Spec $K((q))/\{\pm 1\} \to \mathcal{M}^{\mathrm{Der}}$, where the group $\{\pm 1\}$ operates on K by complex conjugation.

One can define a new derived Deligne–Mumford stack by forming a pushout square

$$\begin{tikzcd}
\operatorname{Spec} K((q))/\{\pm 1\} \arrow[r] \arrow[d] & \mathcal{M}^{\mathrm{Der}} \arrow[d] \\
\operatorname{Spec} K[[q]]/\{\pm 1\} \arrow[r] & \overline{\mathcal{M}^{\mathrm{Der}}}.
\end{tikzcd}$$

Here $\overline{\mathcal{M}^{\mathrm{Der}}}$ is a compactification of the derived moduli stack $\mathcal{M}^{\mathrm{Der}}$. The underlying ordinary Deligne–Mumford stack of $\overline{\mathcal{M}^{\mathrm{Der}}}$ is the classical Deligne–Mumford compactification of $\mathcal{M}_{1,1}$.

Remark 4.8. Much of the theory of elliptic cohomology carries over to the compactified moduli stack $\overline{\mathcal{M}^{\mathrm{Der}}}$. However, there are often subtleties at ∞. For example, we can glue the universal oriented elliptic curve $\mathcal{E}$ over $\mathcal{M}^{\mathrm{Der}}$ with the Tate curve T, to obtain a universal oriented *generalized* elliptic curve $\overline{\mathcal{E}}$ over $\overline{\mathcal{M}^{\mathrm{Der}}}$. However, $\overline{\mathcal{E}}$ is not a derived group scheme over $\overline{\mathcal{M}^{\mathrm{Der}}}$: it has a group structure only on the smooth locus of the map $\overline{\mathcal{E}} \to \overline{\mathcal{M}^{\mathrm{Der}}}$.

Consequently, the construction of geometric objects that we described in Sect. 3.3 needs to be modified. Since $\overline{\mathcal{E}}$ is not a derived group scheme, we cannot define the geometric object M_G associated to a compact abelian Lie group G to be the derived scheme $\mathrm{Hom}(G^{\vee}, \overline{\mathcal{E}})$. It is possible to make the necessary modifications, giving an explicit construction of M_G near ∞ using the theory of toric varieties. However, the construction is somewhat complicated and we will not describe it here.

5 Applications

5.1 2-*Equivariant Elliptic Cohomology*

Let A be an E_∞-ring and let $E \to \mathrm{Spec}\, A$ be an oriented elliptic curve. In Sect. 3.3 we used this data to construct a derived A-scheme M_G, for every compact *abelian* Lie group G, which is the natural "home" for the G-equivariant version of A-cohomology described in Sect. 3.4. It is possible to construct a derived scheme M_G with similar properties even when G is nonabelian. If G is connected and T is a maximal torus of G, the derived scheme M_G looks *roughly* like a quotient of M_T by the action of the Weyl group of G. In particular, when G is connected, M_G is a derived scheme whose underlying classical scheme can be identified with the moduli *space* for regular G_{alg}-bundles on E; here G_{alg} is the reductive algebraic group associated to G (see [16] for a discussion of regular G_{alg}-bundles on elliptic curves over the complex numbers).

Remark 5.1. It is possible to describe $M_{SU(n)}$ and $M_{U(n)}$ more explicitly, in the language of derived algebraic geometry. However, we do not know of any moduli-theoretic interpretation of the derived scheme M_G in general. This seems to be a difficult question, primarily because the underlying classical object is a moduli space rather than a moduli stack.

One way of thinking about equivariant elliptic cohomology is that it is a correspondence which associates derived schemes to certain topological spaces. In particular, to the classifying space BG it assigns the derived scheme M_G. It is possible to extend this process to a more general class of topological spaces, to obtain what we call 2-*equivariant* elliptic cohomology. For an explanation of this terminology we refer the reader to Sect. 5.4. The correspondence associated to an oriented elliptic curve $E \to \operatorname{Spec} A$ is summarized in the following table.

Topological space	Associated geometric object
$*$	$\operatorname{Spec} A$
$\mathbf{C}\mathrm{P}^\infty$	E
$B\mathbf{Z}/n\mathbf{Z}$	$E[n]$
BG	M_G
$K(\mathbf{Z}/n\mathbf{Z}, 2)$	μ_n
$K(\mathbf{Z}, 3)$	$\mathbb{G}_m$
$K(\mathbf{Z}, 4)$	$B\mathbb{G}_m$

Every level $l \in H^4(BG, \mathbf{Z})$ classifies a fibration

$$K(\mathbf{Z}, 3) \to X \to BG,$$

and 2-equivariant elliptic cohomology associates to this a fibration of geometric objects: in other words, a principal $\mathbb{G}_m$ over M_G, which we may identify with a line bundle $\mathcal{L}_l$ over M_G.

Remark 5.2. The line bundle $\mathcal{L}_l$ gives rise to a line bundle on the classical moduli space $\overline{M}_G$ of regular G-bundles on elliptic curves. This is the "theta" bundle whose global sections are nonabelian θ-functions: in other words, the spaces of conformal blocks for the modular functor underlying the Wess–Zumino–Witten model.

Example 5.1. Let us say that a commutative A-group $X \to \operatorname{Spec} A$ is an *abelian scheme* over $\operatorname{Spec} A$ if the underlying map of ordinary schemes $\overline{X} \to \operatorname{Spec} \pi_0 A$ is an abelian scheme, in the sense of classical algebraic geometry. There is a good duality theory for abelian schemes: namely, given an abelian scheme $X \to \operatorname{Spec} A$, one can define a dual abelian scheme $X^\vee$ which classifies extensions of X by the multiplicative group $\mathbb{G}_m$.

In classical algebraic geometry, every elliptic curve is canonically isomorphic to its dual. The analogous result is not true in derived algebraic geometry. However, it is true for *oriented* elliptic curves $E \to \operatorname{Spec} A$. Let $l : \mathbf{C}\mathrm{P}^\infty \times \mathbf{C}\mathrm{P}^\infty \to K(\mathbf{Z}, 4)$ classify the cup-product operation

$$H^2(X; \mathbf{Z}) \times H^2(X; \mathbf{Z}) \to H^4(X; \mathbf{Z}).$$

Then $\mathcal{L}_l$ is a line bundle on the product $E \times_{\operatorname{Spec} A} E$. The symmetry and bi-additivity of the cup product operation translate into the assertion that $\mathcal{L}_l$ is a *symmetric biextension* of E by $\mathbb{G}_m$ (see [17] for a discussion in the classical setting). This gives an identification of E with the dual elliptic curve $E^\vee$.

We will not describe the construction of 2-equivariant elliptic cohomology here. However, we should point out that it is essentially uniquely determined by the properties we have asserted above: namely, that it is a functorial process which assigns to each level $l : BG \to K(\mathbf{Z}, 4)$ a line bundle $\mathcal{L}_l$ on M_G, and that it determines an identification of the elliptic curve E with its dual.

5.2 Loop Group Representations

Let us consider a punctured formal disk around ∞ on the moduli stack of elliptic curves. As we saw in Sect. 4.3, over this disk elliptic cohomology is given by the E_∞-ring $K((q))$, where K is complex K-theory. In other words, near ∞, elliptic cohomology reduces to K-theory. However, equivariant elliptic cohomology does *not* reduce to equivariant K-theory. Both of these equivariant theories are given by the constructions of Sect. 3. However, the inputs to these constructions are different: to get equivariant K-theory, we use the multiplicative group $\mathbb{G}_m$; for elliptic cohomology, we use the Tate curve $\mathbb{G}_m/q^{\mathbf{Z}}$. Thus, we should expect G-equivariant K-theory to be somewhat less complicated than G-equivariant elliptic cohomology. The former is related to the representation theory of the group G, but the latter is related to the representation theory of the *loop group* LG.

We begin with a quick review of the theory of loop groups: for a more extensive discussion, we refer the reader to [18]. Fix a connected compact Lie group G, which for simplicity we will assume to be simple. Then the group $H^4(BG; \mathbf{Z})$ is *canonically* isomorphic to $\mathbf{Z}$. Let us fix a nonnegative integer l, which we identify with an element of $H^4(BG; \mathbf{Z})$ and therefore with a map $BG \to K(\mathbf{Z}, 4)$. Let LG denote the loop group of (smooth) loops $S^1 \to G$. The level l classifies a central extension

$$S^1 \to \widetilde{LG} \to LG.$$

Moreover, there is an action of S^1 on the group $\widetilde{LG}$, which descends to the natural S^1-action on LG given by rotation of loops. We let $\widetilde{LG}^+$ denote the semidirect product of $\widetilde{LG}$ by S^1, via this action. We will refer to the circle of this semidirect product decomposition as the *energy circle*, and the circle $S^1 \subseteq \widetilde{LG} \to \widetilde{LG}^+$ as the *central circle*.

Definition 5.1. Let V be a (Hilbert) representation of $\widetilde{LG}^+$. We will say that V is a *positive energy representation of level* l if it satisfies the following conditions:

1. The central circle $S^1 \subseteq \widetilde{LG}^+$ acts on V via the defining character $S^1 \hookrightarrow \mathbf{C}^*$.
2. The Hilbert space V decomposes as a direct sum of finite-dimensional eigenspaces $V_{(n)}$ with respect to the action of the energy circle.
3. The energy eigenspaces $V_{(n)}$ are zero for $n \ll 0$.

Remark 5.3. If we ignore the action of the energy circle, then there are finitely many irreducible positive energy representations of $\widetilde{LG}$, up to isomorphism. However, each of these extends to a representation of $\widetilde{LG}^+$ in infinitely many ways: given any positive energy representation V of $\widetilde{LG}^+$, we can obtain a new positive energy representation $V(1)$ by tensoring with the defining representation of the energy circle (in other words, by *shifting* the energy).

Remark 5.4. Every positive energy representation V of $\widetilde{LG}^+$ can be decomposed as a direct sum of irreducible representations V_α. The irreducible constituents V_α may be infinite in number, so long as the lowest energy of V_α tends to ∞ with α.

Consider the Tate curve E over $K((q))$, constructed in Sect. 4.3. Using the constructions described in Sect. 5.1 we saw that there is a derived scheme $M_G \to \operatorname{Spec} K((q))$ and a line bundle $\mathcal{L}_l$ on M_G, naturally associated to the level $l : BG \to K(\mathbf{Z}, 4)$. The global sections $\Gamma(M_G, \mathcal{L}_l)$ can be identified with an $K((q))$-module, which we may informally refer to as the *G-equivariant elliptic cohomology of a point, at level l.*

We can now state the relationship between elliptic cohomology and the theory of loop group representations:

Theorem 5.1. *Let G be a compact simple Lie group and l a nonnegative level on G. There is a natural identification of $\Gamma(M_G, \mathcal{L}_l)$ with the K-theory of the (topological) category of positive energy representations of $\widetilde{LG}^+$ at level l. Under this correspondence, multiplication by $q \in \pi_0 K((q))$ corresponds to the energy shift $V \mapsto V(1)$.*

The classical analogue of this result, which identifies the K-*group* of positive energy representations of $\widetilde{LG}^+$ with the global sections of the theta bundle over the underlying ordinary scheme of M_G, is proven in [19].

Remark 5.5. Theorem 5.1 is related to the work of Freed, Hopkins, and Telemann (see [20]), which identifies the K-theory of loop group representations with the twisted G-equivariant K-theory of the group G itself.

5.3 The String Orientation

Let V be a finite dimensional real vector space of dimension $d \geq 5$, equipped with a positive definite inner product. The orthogonal group $\mathrm{O}(V)$ of automorphisms of V is not connected. Consequently, we define $\mathrm{SO}(V) \subseteq \mathrm{O}(V)$ to be the connected component of the identity; the map $\mathrm{SO}(V) \subseteq \mathrm{O}(V)$ may be regarded as "killing" $\pi_0 \mathrm{O}(V)$, in the sense that $\mathrm{SO}(V)$ is a connected space with $\pi_n \mathrm{SO}(V) \simeq \pi_n \mathrm{O}(V)$ an isomorphism for $n > 0$.

The group $SO(V)$ is not simply connected, but it has a simply connected double cover $\mathrm{Spin}(V)$. The map $\mathrm{Spin}(V) \to \mathrm{SO}(V)$ has the effect of "killing" $\pi_1 \mathrm{SO}(V)$, in the sense that $\pi_n \mathrm{Spin}(V) \to \pi_n \mathrm{SO}(V)$ is an isomorphism for $n \neq 1$, but $\mathrm{Spin}(V)$ is simply connected.

An algebraic topologist would see no reason to stop there. The group $\pi_2 \mathrm{Spin}(V) \simeq \pi_2 \mathrm{SO}(V) \simeq \pi_2 \mathrm{O}(V)$ is trivial, but the homotopy group $\pi_3 \mathrm{Spin}(V) \simeq \pi_3 \mathrm{SO}(V)$

$\simeq \pi_3\mathrm{O}(V)$ is (canonically) isomorphic to $\mathbf{Z}$. We may therefore construct a map $\mathrm{String}(V) \to \mathrm{Spin}(V)$ which induces an isomorphism $\pi_n\mathrm{String}(V) \to \pi_n\mathrm{Spin}(V)$ for $n \neq 3$, but such that $\pi_3\mathrm{String}(V) = 0$.

Remark 5.6. One might wonder what sort of an object $\mathrm{String}(V)$ *is*. It is certainly not a finite dimensional Lie group. One can construct a topological space with the desired properties using standard methods of homotopy theory (attaching cells to kill homotopy groups). Using more sophisticated methods, one can realize $\mathrm{String}(V)$ as a topological group. An explicit realization of $\mathrm{String}(V)$ as an (infinite dimensional) topological group is described in [21].

An alternative point of view is to consider $\mathrm{String}(V)$ as the total space over a certain S^1-gerbe over $\mathrm{Spin}(V)$. This has the advantage of being a "finite dimensional" object that can be studied using ideas from differential geometry (see for example [22]).

Let M be a smooth manifold of dimension n. Choosing a Riemannian metric on M, we can reduce the structure group of the tangent bundle of M to $\mathrm{O}(V)$. An *orientation* (*spin structure*, *string structure*) on M is a further reduction of the structure group of the tangent bundle T_M to $\mathrm{SO}(V)$ ($\mathrm{Spin}(V)$, $\mathrm{String}(V)$). The manifold M admits an orientation if and only if the first Stiefel–Whitney class $w_1(M)$ vanishes. An orientation of M can be lifted to a spin structure if and only if the second Stiefel–Whitney class $w_2(M)$ vanishes. Likewise, a spin structure on M extends to a string structure if and only if a certain characteristic class $p \in H^4(M;\mathbf{Z})$ vanishes. The characteristic class p has the property that $2p$ coincides with the first Pontryagin class $p_1(M)$, though p itself is well-defined only after a spin structure on M has been chosen.

Let A be a multiplicative cohomology theory. Then A determines a (dual) homology theory, which we will also denote by A. A class $\eta \in A_n(M)$ is an *A-orientation* of M provided that cap product by η induces an isomorphism

$$A^*(*) \to A_{n-*}(M, M - \{m\})$$

for every point $m \in M$. In this case, we will say that M is *A-oriented* and that η is the *A-fundamental class of M*.

In the case where A is ordinary integral homology, giving an A-orientation of M is equivalent to reducing the structure group of the tangent bundle of M to $\mathrm{SO}(V)$. Similarly, giving a spin structure on M allows one to define an orientation of M with respect to complex K-theory (even with respect to *real* K-theory). The idea is that K-homology classes can be represented by elliptic differential operators; the appropriate candidate for the fundamental class of M is the Dirac operator, which is well-defined once M has been endowed with a spin structure. In this section, we would like to discuss the "elliptic" analogue of this result: if M is a *string* manifold, then M has a canonical orientation with respect to elliptic cohomology.

The relationship between elliptic cohomology orientations of a manifold and string structures goes back to the work of Witten (see [23]). Heuristically, the elliptic cohomology of a space M can be thought of as the S^1-equivariant K-theory of the loop space LM. To obtain an elliptic cohomology orientation of M, one wants to write down the Dirac equation on LM. Witten computed the index of this hypothetical Dirac operator using a localization formula, and thereby defined the *Witten genus* $w(M) \in \mathbf{Z}((q))$ of the manifold M, having the property that the coefficient of q^n in $w(M)$ is the χ^n-isotypic part of the index of the Dirac operator, where $\chi : S^1 \to \mathbf{C}^*$ is the identity character. Moreover, he made the following very suggestive observation: if $p_1(M) = 0$, then $w(M)$ is the q-expansion of a modular form.

Remark 5.7. The statement that the elliptic cohomology of M is given by the S^1-equivariant K-theory of the loop space LM is only heuristically true. However, it is true in the "limiting" case where we consider elliptic cohomology near ∞, and we consider only "small" loops in M. As we saw in Sect. 4.3, in a formal disk around ∞, elliptic cohomology may be identified with $K((q))$. Moreover, $K((q))(M)$ is a completion of $K_{S^1}(M)$, where we identify M with the subset of LM consisting of *constant* loops (so that S^1 acts trivially on this space). The results described in Sect. 5.2 may be considered as a less trivial illustration of the same principle.

In order to study the problem of orienting various classes of manifolds more systematically, it is convenient to pass to the limit by defining

$$\mathrm{O} = \varinjlim \{\mathrm{O}(\mathbb{R}^d)\}_{d \geq 0}$$

$$\mathrm{SO} = \varinjlim \{\mathrm{SO}(\mathbb{R}^d)\}_{d \geq 0}$$

$$\mathrm{Spin} = \varinjlim \{\mathrm{Spin}(\mathbb{R}^d)\}_{d \geq 0}$$

$$\mathrm{String} = \varinjlim \{\mathrm{String}(\mathbb{R}^d)\}_{d \geq 0}$$

Let G denote any of these groups (though much of what we say below applies to other structure groups as well). If M is a smooth manifold of *any* dimension, we will say that a G*-structure* on M is a reduction of the structure group of the *stabilized* tangent bundle of M from O to G. In this case, we will say that M is a G*-manifold*.

Fix a topological group G with a map $G \to \mathrm{O}$, and consider the class of G*-manifolds*: that is, smooth manifolds M whose structure group has been reduced to G. To this data, one can associate a *Thom spectrum* MG. Roughly speaking, one defines $MG_n(X)$ to be the set of bordism classes of n-dimensional G-manifolds M equipped with a map $M \to X$. In particular, if M is a G-manifold of dimension n, there is a canonical element $\eta \in MG_n(M)$; this is an orientation of M with respect to the cohomology theory MG. One can view MG as the universal cohomology theory for which every G-manifold has an orientation. More generally, if A is an arbitrary cohomology theory, then equipping every G-manifold M with an A-orientation is more or less equivalent to giving a map $s : MG \to A$. In good cases, the cohomology theory A will be represented by an E_∞-ring, and s will be a map of E_∞-rings: roughly speaking, this may be thought of as asserting that the orientations on G-manifolds determined by s are compatible with the formation of products of G-manifolds.

The statements we made earlier, regarding orientations of manifolds with respect to ordinary homology and K-theory, can be reinterpreted as asserting the existence of certain natural maps

$$\mathrm{MSO} \to \mathbf{Z}$$
$$\mathrm{MSpin} \to \mathrm{KO}$$

between E_∞-rings. Similarly, the string orientability of elliptic cohomology can be formulated as the existence of an E_∞-map

$$\sigma : \mathrm{MString} \to \mathrm{tmf}.$$

The map σ is sometimes called the *topological Witten genus*; when composed with the map $\mathrm{tmf} \to K((q))$ and evaluated on a string manifold M, it reduces to the ordinary Witten genus described above.

The existence of the map σ is known, thanks to the work of Ando, Hopkins, Rezk, Strickland, and others. We refer the reader to [6] for a discussion of the problem in a somewhat simpler setting. We will sketch here an alternative construction of the map σ, using the theory of 2-equivariant elliptic cohomology sketched in Sect. 5.1. For simplicity, we let $E \to \operatorname{Spec} A$ be an oriented elliptic curve over an E_∞-ring A; we will construct a natural map $\sigma_A : \mathrm{MString} \to A$. The map σ itself is obtained by passing to the (homotopy) inverse limit.

We first rephrase the notion of an orientation from a point of view which is more readily applicable to our situation. Given a group homomorphism $s : G \to \mathrm{O}$, we obtain a map of classifying spaces $BG \to B\mathrm{O}$, which we may think of as a (stable) vector bundle over BG. This vector bundle determines a spherical fibration over BG, which we will view as a bundle of invertible S-modules over BG. Given an E_∞-ring A, we may tensor with A to obtain a bundle of A-modules over BG; let us denote this bundle by $\mathcal{A}_s$. To give a map of E_∞-rings $MG \to A$ is equivalent to giving a trivialization of the local system $\mathcal{A}_s$, which is suitably compatible with the group structure on BG. To simplify the discussion, we will focus only on trivializing $\mathcal{A}_s$; the compatibility with the group structure is established by a more careful application of the same ideas.

Let us now specialize to the case $G = \mathrm{Spin}$, and suppose that we are given an oriented elliptic curve over A. The theory of equivariant elliptic cohomology associates to the compact Lie group $\mathrm{Spin}(n)$ a derived A-scheme M_{Spin}. Moreover, the functoriality of the construction gives a map of topological spaces

$$\phi : B\mathrm{Spin}(n) \simeq \mathrm{Hom}(*, B\mathrm{Spin}(n)) \to \mathrm{Hom}(\operatorname{Spec} A, M_{\mathrm{Spin}}).$$

Let $l : B\mathrm{Spin}(n) \to K(\mathbf{Z}, 4)$ be the canonical generator; then the theory of 2-equivariant elliptic cohomology associates to l a line bundle $\mathcal{L}_l$ over M_{Spin}. Thus, for every A-valued point of M_{Spin}, we get an invertible A-module. The morphism ϕ therefore gives a local system of invertible A-modules on $B\mathrm{Spin}(n)$; let us denote it by $\mathcal{A}_\phi$.

The existence of the string orientation on elliptic cohomology rests on the following comparison result, which we will not prove here

Theorem 5.2. *Let* $s : \mathrm{Spin}(n) \to \mathrm{O}$ *be the natural homomorphism and let* $E \to \operatorname{Spec} A$ *be an oriented elliptic curve over an* E_∞*-ring* A*. There is a canonical isomorphism*

$$\mathcal{A}_\phi \simeq \mathcal{A}_s$$

of local systems of invertible A*-modules over* $B\mathrm{Spin}(n)$*.*

To provide A with a string orientation, we need to prove that $\mathcal{A}_s$ becomes trivial after pulling back along the map $B\mathrm{String}(n) \to B\mathrm{Spin}(n)$. By Theorem 5.2, it will suffice to prove that $\mathcal{A}_\phi$ becomes trivial after pullback to $B\mathrm{String}(n)$. The theory of 2-equivariant elliptic cohomology provides a commutative diagram

$$\begin{tikzcd} B\mathrm{String}(n) \arrow[r] \arrow[d] & \mathrm{Hom}(\operatorname{Spec} A, M_{\mathrm{String}(n)}) \arrow[d] \\ B\mathrm{Spin}(n) \arrow[r] & \mathrm{Hom}(\operatorname{Spec} A, M_{\mathrm{Spin}(n)}) \end{tikzcd}$$

Consequently, to prove that $\mathcal{A}_\phi$ becomes trivial after pullback to $B\mathrm{String}(n)$, it suffices to prove that $\mathcal{L}_l$ becomes trivial after pullback along the map of derived schemes $M_{\mathrm{String}(n)} \to M_{\mathrm{Spin}(n)}$. But $M_{\mathrm{String}(n)}$ is precisely the total space of the principal $\mathbb{G}_m$-bundle underlying $\mathcal{L}_l$: in other words, it is universal among derived schemes over $M_{\mathrm{Spin}(n)}$ over which $\mathcal{L}_l$ has a trivialization.

Remark 5.8. The argument given above not only establishes the existence of the string orientation $\mathrm{MString} \to A$; it also explains why the covering $\mathrm{String}(n) \to \mathrm{Spin}(n)$ arises naturally when one considers the problem of finding orientations with respect to elliptic cohomology.

5.4 Higher Equivariance

The purpose of this section is to place the theory of 2-equivariant elliptic cohomology, which we discussed in Sect. 5.1, into a larger context.

Let A be an even, periodic cohomology theory, and suppose that $A(*) = k$ is a *field*. In this case, there are not many possibilities for the formal group $\widehat{\mathbb{G}} = \operatorname{Spf} A(\mathbf{CP}^\infty)$. If k is of characteristic zero, any 1-dimensional formal group over k is isomorphic to the formal additive group; correspondingly, A is necessarily equivalent to periodic ordinary cohomology with coefficients in k. If k is of characteristic p, then the formal group $\mathbb{G}$ is classified up to isomorphism (over the algebraic closure $\overline{k}$) by a single invariant $1 \leq n \leq \infty$. The invariant n is called the *height* of $\mathbb{G}$, and may be thought of as a measure of the size of the p-torsion subgroup $\mathbb{G}[p] \subseteq \mathbb{G}$.

By convention, we say that the additive group in characteristic zero has height zero (since it has no nontrivial p-torsion).

To a formal group $\mathbb{G}$ of height n over an algebraically closed field k of characteristic p, one can associate an (essentially unique) even, periodic cohomology theory. This cohomology theory is called *Morava K-theory* and denoted by $K(n)$.

The Morava K-theory of Eilenberg–Mac Lane spaces has been computed by Ravenel and Wilson (see [24]). In particular, they show that for $m > n$, the natural map

$$K(n)^*(*) \to K(n)^*(K(\mathbf{Z}/p\mathbf{Z}, m))$$

is an isomorphism. In other words, the Morava K-theory $K(n)$ cannot tell the difference between the space $K(\mathbf{Z}/p\mathbf{Z}, m)$ and a point. We may informally summarize the situation by saying that $K(n)$ does not "see" the homotopy groups of a space X in dimensions larger than n. Of course, this is not literally true: however, it is true provided that the homotopy groups of X satisfy certain finiteness properties.

If $\mathbb{G}$ is a formal group over a commutative ring R, then it need not have a well-defined height; however, it has a height when restricted to each residue field of R. We will say that an cohomology theory A has *height* $\leq n$ if it is even, weakly periodic, and the associated formal group has height $\leq n$ at every point. Our discussion of Morava K-theory can be generalized as follows: if A is a cohomology theory of height $\leq n$, then A only "sees" the first n homotopy groups of a space X. As we remarked above, this is not true in general, but it is true provided that we make suitable finiteness assumptions on X.

Let us first consider the case of a cohomology theory A of height ≤ 0. Then $R = A(*)$ is an algebra over the field $\mathbf{Q}$ of rational numbers, and A is automatically a periodic variant of ordinary cohomology with coefficients in R. It follows that $A(X)$ is insensitive to the homotopy groups of X, provided that those homotopy groups are finite. For spaces with finite homotopy groups, $A(X)$ simply measures the number of connected components of X.

We can get more information by considering cohomology theories of height ≤ 1. The prototypical example is complex K-theory. The above discussion indicates that, under suitable finiteness hypotheses, $K(X)$ should be sensitive only to the fundamental groupoid of X. In other words, the only really interesting K-group to compute is $K(BG)$, where G is a finite group. However, this example turns out to be quite interesting: there is a natural map $\eta : \mathrm{Rep}(G) \to K(BG)$, defined for any finite group G. The Atiyah–Segal completion theorem asserts that η is not far from being an isomorphism: it realizes $K(BG)$ as the completion of $\mathrm{Rep}(G)$ with respect to the augmentation ideal consisting of virtual representations having virtual dimension zero.

There are many respects in which $\mathrm{Rep}(G)$ is a better behaved object than $K(BG)$: for example, it is a finitely generated abelian group, while $K(BG)$ is not. Moreover, there is a moral sense in which $K(BG)$ "ought to be" $\mathrm{Rep}(G)$: the fact that η is not an isomorphism is somehow a technicality. We can regard equivariant K-theory as a way of formally "defining away" the technicality. Namely, the equivariant K-group $K_G(*)$ is a refined version of $K(BG)$, which coincides with $\mathrm{Rep}(G)$ by definition.

One might ask if there are other examples of spaces X for which $K(X)$ is in need of refinement. According to the discussion above, the answer is essentially no: $K(X)$ should only be sensitive to the fundamental groupoid of X, so that the general "expected answer" for $K(X)$ is the representation ring $\mathrm{Rep}(\pi_1 X)$ (provided that X is connected).

When we consider cohomology theories of higher height, the above argument breaks down. Let Ell denote an elliptic cohomology theory, necessarily of height ≤ 2. Following the above discussion, we should imagine that $Ell(X)$ is sensitive to the first two homotopy groups of a space X. Consequently, it is most interesting to compute $Ell(X)$ when X is a connected space satisfying $\pi_1 X = G$, $\pi_2 X = A$, and $\pi_n X = 0$ for $n > 2$. In the above discussion, we assumed that the groups G and A were finite. However, by analogy with K-theory, we should also allow the case where G and A are compact Lie groups (so that X is allowed to be a space like BG, or the classifying space of a circle gerbe over G). By analogy with K-theory, one might guess that there is an "expected" answer for $Ell(X)$, and that $Ell(X)$ is related to this expected answer by some sort of completion result in the spirit of the Atiyah–Segal theorem. This prediction turns out to be correct: there is an expected answer, which is dictated by the geometry of elliptic curves. Moreover, as with equivariant K-theory, one can build a coherent and useful theory out of the expected answers. This is the theory of 2-equivariant elliptic cohomology which we discussed in Sect. 5.1.

Remark 5.9. There is no reason to stop at height 2. Given an E_∞-ring A and an *oriented* p-divisible group $\mathbb{G}$ over A of height n, one can construct a theory of *n-equivariant A-cohomology*. Namely, let X be a space whose homotopy groups are all finite p-groups, vanishing in dimension $> n$. Then there is a natural procedure for using $\mathbb{G}$ to construct a spectrum A_X, which can be thought of as the "expected answer" for $A(X)$. Moreover, there is a map $A_X \to A(X)$ which can be described by an Atiyah–Segal completion theorem. The significance of these "expected answers" is not yet clear, but they are closely related to the generalized character theory of [11].

5.5 Elliptic Cohomology and Geometry

In this paper, we have discussed elliptic cohomology from an entirely algebraic point of view. In doing so, we have ignored what is perhaps the most interesting question of all: what *is* elliptic cohomology? In other words, given a topological space X, what does it mean to give an elliptic cohomology class $\eta \in \mathrm{tmf}(X)$? No satisfactory answer to this question is known, as of this writing. However, a number of very interesting ideas have been put forth. We saw in Sect. 5.2 that equivariant elliptic cohomology is related to the theory of loop group representations. Graeme Segal has suggested that elliptic cohomology should bear some relationship to Euclidean field theories (see [25]). Building on his ideas, Stolz and Teichner have proposed that the classifying space for elliptic cohomology might be interpreted as a moduli space for supersymmetric quantum field theories. To support their view, they show in [21] that supersymmetry predicts that the coefficients in the q-expansion of the partition function of such a theory should be *integral*. Alternative speculations on the problem can be found in [26] and [27], among other places.

Our moduli-theoretic interpretation of elliptic cohomology has the advantage of being well-suited to proving comparison results with other theories. Suppose that we are given some candidate cohomology theory A, which we suspect is equivalent to elliptic cohomology. For simplicity, we will assume that A is a candidate for the elliptic cohomology theory associated to an elliptic curve over an *affine* base Spec $\pi_0 A$ (the non-affine case can be treated as well, but requires a more elaborate discussion). We will suppose that A enjoys all of the good formal properties of elliptic cohomology: namely, that it is representable by an E_∞-ring, and that it has well-behaved equivariant and 2-equivariant analogues.

To relate A to elliptic cohomology, we would like to produce a map $f : \mathrm{tmf} \to A$. Better yet, we would like to have a map Spec $A \to \mathcal{M}^{\mathrm{Der}}$: we can then obtain f by considering the induced map on global sections of the structure sheaves. To provide such a map, we need to give an oriented elliptic curve $E \to$ Spec A. We will proceed under the assumption that E exists, and explain how to reconstruct it from the cohomology theory A.

We wish to produce E not just as a derived scheme, but as an oriented A-group. Equivalently, for every lattice Λ, we want to construct the abelian variety $E^\Lambda = \mathrm{Hom}(\Lambda, E)$, and we want the construction to be functorial in Λ. Let $T = \mathrm{Hom}(\Lambda, S^1)$ denote the Pontryagin dual of Λ. If E^Λ were affine, we would expect to recover it as Spec $A_T(*)$. Unfortunately, abelian varieties are not affine, so we must work a bit harder. To this end, select a map $q : BT \to K(\mathbf{Z}, 4)$ which classifies a *positive definite* quadratic form on $\Lambda^\vee$. The level q determines an ample line bundle $\mathcal{L}_q$ on E^Λ. Moreover, the global sections of the kth power of $\mathcal{L}_q$ should be given by T-equivariant A-cohomology *at level* kq: we can make sense of this A-module in virtue of the assumption that we have a good 2-equivariant theory. We can now assemble these modules of sections, for varying $k \geq 0$, into a graded E_∞-ring $R_\bullet$, and attempt to recover $\mathcal{E}^\Lambda$ as the projective spectrum of $R_\bullet$.

Of course, the above construction will only work to produce an abelian variety over Spec A if certain conditions are met. These conditions can be reduced to certain computations: for example, one must show that if T is a torus and $q : BT \to K(\mathbf{Z}, 4)$ classifies a positive definite quadratic form of discriminant d, then the T-equivariant A-cohomology of a point, at level q, is a locally free A-module of rank d. Supposing that this and other algebraic conditions are satisfied, the above construction will give us an elliptic curve $E \to$ Spec A. Moreover, by comparing equivariant A-cohomology with Borel-equivariant A-cohomology, we can supply E with a preorientation $\sigma : \mathbf{CP}^\infty \to E(A)$. The condition that σ be an orientation is again a matter of computation: essentially, we would need to show that an appropriate version of the Atiyah–Segal completion theorem holds, at least locally on the elliptic curve E. Provided that all of the necessary computations yield favorable results, we obtain an oriented elliptic curve $E \to \operatorname{Spec} A$, which is classified by the desired map $\operatorname{Spec} A \to \mathcal{M}^{\mathrm{Der}}$.

Unfortunately, our algebraic perspective does not offer any insights on the problem of where to find such a cohomology theory in geometry. Nevertheless, it seems inevitable that a geometric understanding of elliptic cohomology will eventually emerge. The resulting interaction between algebraic topology, number theory, mathematical physics, and classical geometry will surely prove to be an excellent source of interesting mathematics in years to come.

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On Voevodsky's Algebraic K-Theory Spectrum

Ivan Panin, Konstantin Pimenov, and Oliver Röndigs

Abstract Under a certain normalization assumption we prove that the $\mathbf{P}^1$-spectrum BGL of Voevodsky which represents algebraic K-theory is unique over $\mathrm{Spec}(\mathbb{Z})$. Following an idea of Voevodsky, we equip the $\mathbf{P}^1$-spectrum BGL with the structure of a commutative $\mathbf{P}^1$-ring spectrum in the motivic stable homotopy category. Furthermore, we prove that under a certain normalization assumption this ring structure is unique over $\mathrm{Spec}(\mathbb{Z})$. For an arbitrary Noetherian scheme S of finite Krull dimension we pull this structure back to obtain a distinguished monoidal structure on BGL. This monoidal structure is relevant for our proof of the motivic Conner–Floyd theorem (Panin et al., Invent Math 175:435–451, 2008). It has also been used to obtain a motivic version of Snaith's theorem (Gepner and Snaith, arXiv:0712.2817v1 [math.AG]).

1 Preliminaries

This paper is concerned with results in motivic homotopy theory, which was put on firm foundations by Morel and Voevodsky in [MV] and [V]. Due to technical reasons explained below, the setup in [MV], as well as other model categories used in motivic homotopy theory, are inconvenient for our purposes, so we decided to pursue a slightly different approach. We refer to the Sect. A for the basic terminology, notation, constructions, definitions, and results concerning motivic homotopy theory. For a Noetherian scheme S of finite Krull dimension we write $\mathbf{M}(S)$, $\mathbf{M}_\bullet(S)$, $\mathrm{H}_\bullet(S)$ and $\mathrm{SH}(S)$ for the category of motivic spaces, the category of pointed motivic spaces, the pointed motivic homotopy category and the stable motivic homotopy category over S. These categories are equipped with symmetric monoidal structures. In particular, a symmetric monoidal structure $(\wedge, \mathbb{I})$ is constructed on the motivic stable homotopy category and its basic properties are proved. This structure is used extensively over the present text.

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Let S be a regular scheme, and let $K^0(S)$ denote the Grothendieck group of vector bundles over S. Morel and Voevodsky proved in [MV, Theorem 4.3.13] that the Thomason–Trobaugh K-theory [TT] is represented in the pointed motivic homotopy category $\mathrm{H}_\bullet(S)$ by the space $\mathbb{Z}\times\mathrm{Gr}$ pointed by $(0, x_0)$. Here Gr is the union of the finite Grassmann varieties $\bigcup_{n=0}^{\infty}\mathrm{Gr}(n,2n)$, considered as motivic spaces. There is a unique element $\xi_\infty \in K^0(\mathbb{Z}\times\mathrm{Gr})$ which corresponds to the identity morphism $\mathrm{id}\colon \mathbb{Z}\times\mathrm{Gr}\to\mathbb{Z}\times\mathrm{Gr}$. It follows that there exists a unique morphism

$$\mu_\otimes\colon (\mathbb{Z}\times\mathrm{Gr})\wedge(\mathbb{Z}\times\mathrm{Gr})\to\mathbb{Z}\times\mathrm{Gr}$$

in $\mathrm{H}_\bullet(S)$ such that the composition $(\mathbb{Z}\times\mathrm{Gr})\times(\mathbb{Z}\times\mathrm{Gr})\to(\mathbb{Z}\times\mathrm{Gr})\wedge(\mathbb{Z}\times\mathrm{Gr})\xrightarrow{\mu_\otimes}\mathbb{Z}\times\mathrm{Gr}$ represents the element $\xi_\infty\otimes\xi_\infty$ in $K^0\big((\mathbb{Z}\times\mathrm{Gr})\times(\mathbb{Z}\times\mathrm{Gr})\big)$ (see Lemma B.1). Let $e_\otimes\colon S^0\to\mathbb{Z}\times\mathrm{Gr}$ be the map which corresponds to the point $(1,x_0)\in\mathbb{Z}\times\mathrm{Gr}$. The triple

$$(\mathbb{Z}\times\mathrm{Gr},\mu_\otimes,e_\otimes) \tag{1}$$

is a commutative monoid in $\mathrm{H}_\bullet(S)$.

Using this fact, Voevodsky constructed in [V] a $\mathbf{P}^1$-spectrum

$$\mathrm{BGL}=(\mathcal{K}_0,\mathcal{K}_1,\mathcal{K}_2,\dots)$$

with structure maps $e_i\colon\mathcal{K}_i\wedge\mathbf{P}^1\to\mathcal{K}_{i+1}$ such that:

1. There is a motivic weak equivalence $w\colon\mathbb{Z}\times\mathrm{Gr}\to\mathcal{K}_0$, and for all i one has $\mathcal{K}_i=\mathcal{K}_0$ and $e_i=e_0$.
2. The morphism

$$\mathbb{Z}\times\mathrm{Gr}\times\mathbf{P}^1\xrightarrow{\mathrm{can}}(\mathbb{Z}\times\mathrm{Gr})\wedge\mathbf{P}^1\xrightarrow{w\wedge\mathbf{P}^1}\mathcal{K}_i\wedge\mathbf{P}^1\xrightarrow{e_i}\mathcal{K}_{i+1}\xrightarrow{w^{-1}}\mathbb{Z}\times\mathrm{Gr}$$

   in $\mathrm{H}_\bullet(S)$ represents the element $\xi_\infty\otimes([\mathcal{O}(-1)]-[\mathcal{O}])\in K^0(\mathbb{Z}\times\mathrm{Gr}\times\mathbf{P}^1)$.
3. The adjoint $\mathcal{K}_i\to\Omega_{\mathbf{P}^1}(\mathcal{K}_{i+1})$ of e_i is a motivic weak equivalence.

With this spectrum in hand given a smooth X over S we may identify $K^0(X)$ with $\mathrm{BGL}^{2i,i}(X)$ as follows

$$K^0(X)=\mathrm{Hom}_{\mathrm{H}_\bullet(S)}(X_+,\mathbb{Z}\times\mathrm{Gr})=\mathrm{Hom}_{\mathrm{H}_\bullet(S)}(X_+,\mathcal{K}_i)=\mathrm{BGL}^{2i,i}(X) \tag{2}$$

Our first aim is to recall Voevodsky's construction to show that this spectrum is essentially unique. This has also been obtained in [R]. *Our second and more important aim is to give a commutative monoidal structure to the* $\mathbf{P}^1$*-spectrum* BGL *which respects the naive multiplicative structure on the functor* $X\mapsto\mathrm{BGL}^{2*,*}(X)$. To be more precise, we construct a product

$$\mu_{\mathrm{BGL}}\colon\mathrm{BGL}\wedge\mathrm{BGL}\to\mathrm{BGL} \tag{3}$$

in the stable motivic homotopy category $\mathrm{SH}(S)$ such that for any $X \in \mathcal{S}m/S$ the diagram

$$\begin{array}{ccc} K^0(X) \times K^0(X) & \xrightarrow{\otimes} & K^0(X) \\ \cong \downarrow & & \downarrow \cong \\ \mathrm{BGL}^{2i,i}(X) \times \mathrm{BGL}^{2j,j}(X) & \xrightarrow{\mu_{\mathrm{BGL}}} & \mathrm{BGL}^{2(i+j),i+j}(X) \end{array}$$

commutes. We show in Theorem 2.2.1 that *there is a unique product* $\mu_{\mathrm{BGL}} \in \mathrm{Hom}_{\mathrm{SH}(\mathbb{Z})}$ $(\mathrm{BGL} \wedge \mathrm{BGL}, \mathrm{BGL})$ satisfying this property. This induces a product $\mu_{\mathrm{BGL}} \in \mathrm{Hom}_{\mathrm{SH}(S)}(\mathrm{BGL} \wedge \mathrm{BGL}, \mathrm{BGL})$ for an arbitrary regular scheme S by pullback along the structural morphism $S \to \mathrm{Spec}(\mathbb{Z})$. As well, we show that the product is associative, commutative and unital. The resulting multiplicative structure on the bigraded theory $\mathrm{BGL}^{*,*}$ coincides with the Waldhausen multiplicative structure on the Thomason–Trobaugh K-theory.

1.1 Recollections on Motivic Homotopy Theory

The basic definitions, constructions and model structures used in the text are given in Sect. A. The word "model structure" is used in its modern sense and thus refers to a "closed model structure" as originally defined by Quillen. Let S be a Noetherian finite-dimensional scheme. A *motivic space over* S is a simplicial presheaf on the site $\mathcal{S}m/S$ of smooth quasi-projective S-schemes. A *pointed motivic space over* S is a pointed simplicial presheaf on the site $\mathcal{S}m/S$. We write $\mathbf{M}_\bullet(S)$ for the category of pointed motivic spaces over S. A *closed motivic model structure* $\mathbf{M}^{\mathrm{cm}}_\bullet(S)$ is constructed in Theorem A.17. The adjective "closed" refers to the fact that closed embeddings in $\mathcal{S}m/S$ are forced to become cofibrations. The resulting homotopy category $\mathrm{H}^{\mathrm{cm}}_\bullet(S)$ obtained in Theorem A.17 is called *the motivic homotopy category* of S. By Theorem A.19 it is equivalent to the Morel–Voevodsky $\mathbf{A}^1$-homotopy category [MV], and we may drop the superscript in $\mathrm{H}^{\mathrm{cm}}_\bullet(S)$ for convenience. The closed motivic model structure has the properties that:

1. For any closed S-point $x_0 \colon S \hookrightarrow X$ in a smooth S-scheme, the pointed motivic space (X, x_0) is cofibrant in $\mathbf{M}^{\mathrm{cm}}_\bullet(S)$ (Lemma A.10).
2. A morphism $f \colon S \to S'$ of base schemes induces a left Quillen functor $f^*\colon \mathbf{M}^{\mathrm{cm}}_\bullet(S') \to \mathbf{M}^{\mathrm{cm}}_\bullet(S)$ (Theorem A.17).
3. Taking complex points is a left Quillen functor $\mathbf{R}_{\mathbb{C}} \colon \mathbf{M}^{\mathrm{cm}}_\bullet(\mathbb{C}) \to \mathbf{Top}_\bullet$ (Theorem A.23).

Conditions 2 and 3 do not hold for the Morel–Voevodsky model structure, condition 1 fails for the so-called projective model structure [DRØ, Theorem 2.12]. For a morphism $f: A \to B$ of pointed motivic spaces we will write $[f]$ for the class of f in $\mathrm{H}_\bullet(S)$.

We will consider $\mathbf{P}^1$ as a pointed motivic space over S pointed by $\infty: S \hookrightarrow \mathbf{P}^1$. A $\mathbf{P}^1$-*spectrum* E over S consists of a sequence $E_0, E_1, \ldots$ of pointed motivic spaces over S, together with structure maps $\sigma_n: E_n \wedge \mathbf{P}^1 \to E_{n+1}$. A map of $\mathbf{P}^1$-spectra is a sequence of maps of pointed motivic spaces which is compatible with the structure maps. Let $\mathrm{SH}(S)$ denote the homotopy category of $\mathbf{P}^1$-spectra, as described in Sect. A.5. By Theorem A.30 it is canonically equivalent to the motivic stable homotopy category constructed in [V] and [J]. As we will see below there exists an essentially unique $\mathbf{P}^1$-spectrum BGL over $S = \mathrm{Spec}(\mathbb{Z})$ satisfying properties 1. and 2. from Sect. 1. In the following, we will construct BGL in a slightly different way than Voevodsky did originally in [V]. In order to achieve this, we begin with a description of the known monoidal structure on the Thomason–Trobaugh K-theory [TT].

1.2 A Construction of BGL

Let S be a regular scheme. For every S-scheme X consider the category $\mathrm{Big}(X)$ of big vector bundles over X (see for instance [FS] for the definition and basic properties). The assignments $X \mapsto \mathrm{Big}(X)$ and $(f: Y \to X) \mapsto f^*: \mathrm{Big}(X) \to \mathrm{Big}(Y)$ form a functor from schemes to the category of small categories. The reason is that there is an equality $(f \circ g)^* = g^* \circ f^*$, not just a unique natural isomorphism. In what follows we will always consider the Waldhausen K-theory space of X as the space obtained by applying Waldhausen's $\mathcal{S}_\bullet$-construction [W] to the category $\mathrm{Big}(X)$ rather than to the category $\mathrm{Vect}(X)$ of usual vector bundles on X. This has the advantage that the assignment taking an S-scheme X to the Waldhausen K-theory space of X becomes a functor on the category of S-schemes, and in particular a pointed motivic space over S. In what follows a category $\mathcal{S}m\mathcal{O}p/S$ will be useful as well. Its objects are pairs (X, U) with an $X \in \mathcal{S}m/S$ and an open subscheme U in X. Morphisms (X, U) to (Y, V) are morphisms of X to Y in $\mathcal{S}m/S$ which take U to V.

Let $\mathbb{K}^W$ be the pointed motivic space defined in Example A.12. It has the properties that it is fibrant in $\mathbf{M}_\bullet^{\mathrm{cm}}(S)$ and that $\mathbb{K}^W(X)$ is naturally weakly equivalent to the Waldhausen K-theory space associated to the category of big vector bundles on X. For $X \in \mathcal{S}m/S$ the simplicial set $\mathbb{K}^W(X)$ is thus a Kan simplicial set having the Waldhausen K-theory groups $K_*^W(X)$ as its homotopy groups. These K-theory groups coincide with Quillen's higher K-theory groups [Qu, TT, Theorem 1.11.2]. We write $K_*(X)$ for $K_*^W(X)$. It follows immediately from the adjunction isomorphism

$$\mathrm{Hom}_{\mathrm{H}_\bullet(S)}(S^{p,0} \wedge X_+, \mathbb{K}^W) \cong \mathrm{Hom}_{\mathrm{H}_\bullet}\big(S^p, \mathbb{K}^W(X)\big) = K_p(X) \qquad (1)$$

that $\mathbb{K}^W$, regarded as an object in the motivic homotopy category $\mathrm{H}_\bullet(S)$ (see Theorem A.17) represents Quillen K-theory on $\mathcal{S}m/S$. Here $S^n = S^{n,0}$ denotes the n-fold smash product of the constant simplicial presheaf $\Delta^1/\partial\Delta^1$ with itself.

For a pointed motivic space A set

$$K_p(A) := \mathrm{Hom}_{\mathrm{H}_\bullet(S)}(S^{p,0} \wedge A, \mathbb{K}^W).$$

For $X \in \mathcal{S}m/S$ and a closed subset $Z \hookrightarrow X$, $K_n(X\ on\ Z)$ denotes the n-th Thomason–Trobaugh K-group of perfect complexes on X with support on Z [TT, Definition 3.1]. For $A = X/(X - Z)$ with an $X \in \mathcal{S}m/S$ and a closed subset $Z \subset X$, there is an isomorphism $K_p(A) \cong K_p(X\ on\ Z)$ natural in the pair $(X, X - Z)$ (see [TT, Theorem 5.1]). It follows immediately that $\mathbb{K}^W$, regarded as an object in the motivic homotopy category $\mathrm{H}_\bullet(S)$ (see Theorem A.17) represents the Thomason–Trobaugh $K(X\ on\ Z)$-theory on $\mathcal{S}m\mathcal{O}p/S$. The known monoidal structure [TT, (3.15.4)] on the Thomason–Trobaugh $K(X\ on\ Z)$-theory coincides with the one induced by the Waldhausen monoid $(\mathbb{K}^W, \mu^W, e^W)$ described below.

Using the notation of Example A.12, consider the diagram

$$K^W(X) \wedge K^W(X) = \Omega^1_s(W_1(X)) \wedge \Omega^1_s(W_1(X)) \xrightarrow{m} \Omega^2_s(W_2(X)) \xleftarrow{ad} K^W(X) \quad (2)$$

with the Waldhausen multiplication m and the adjunction weak equivalence ad described in [W, p. 342]. The diagram defines a morphism

$$\mathbb{K}^W \wedge \mathbb{K}^W \xrightarrow{\mu^W} \mathbb{K}^W \quad (3)$$

in $\mathrm{H}_\bullet(S)$ which is the Waldhausen multiplication on $\mathbb{K}^W$. Together with the unit $e^W\colon S^0 \to \mathbb{K}^W$ it forms the Waldhausen monoid $(\mathbb{K}^W, \mu^W, e^W)$.

By [MV, Theorem 4.3.13] there is an isomorphism $\psi\colon \mathbb{Z} \times \mathrm{Gr} \to \mathbb{K}^W$ in $\mathrm{H}_\bullet(S)$. The pointed motivic space $\big(\mathbb{Z} \times \mathrm{Gr}, (0, x_0)\big)$ is closed cofibrant by Lemma A.10. Let $b^W\colon \mathbf{P}^1 \to \mathbb{K}^W$ be a morphism in $\mathrm{H}_\bullet(S)$ representing the class $[\mathcal{O}(-1)] - [\mathcal{O}]$ in the kernel of the homomorphism $\infty^*\colon K_0(\mathbf{P}^1) \to K_0(S)$.

Definition 1.2.1. Choose a pointed motivic space $\mathcal{K}$, together with a weak equivalence $i\colon \mathbb{Z} \times \mathrm{Gr} \to \mathcal{K}$ in $\mathbf{M}^{\mathrm{cm}}_\bullet(S)$, as well as a morphism $\epsilon\colon \mathcal{K} \wedge \mathbf{P}^1 \to \mathcal{K}$ in $\mathbf{M}^{\mathrm{cm}}_\bullet(S)$ which descends to

$$\mu^W \circ (\mathrm{id} \wedge b^W)\colon \mathbb{K}^W \wedge \mathbf{P}^1 \to \mathbb{K}^W$$

under the identification of $\mathcal{K}$ with $\mathbb{K}^W$ in $\mathrm{H}_\bullet(S)$ via the isomorphism $\psi \circ [i]^{-1}$. Define BGL as the $\mathbf{P}^1$-spectrum of the form $(\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2, \dots)$ with $\mathcal{K}_i = \mathcal{K}$ for all i and with the structure maps $e_i\colon \mathcal{K}_i \wedge \mathbf{P}^1 \to \mathcal{K}_{i+1}$ equal to the map $\epsilon\colon \mathcal{K} \wedge \mathbf{P}^1 \to \mathcal{K}$.

$\mathbf{P}^1$-spectra as described in Definition 1.2.1 will be used extensively below.

Remark 1.2.2. The Voevodsky spectrum **BGL** is obtained if $\mathcal{K} = Ex^{\mathbf{A}^1}(\mathbb{Z} \times \mathrm{Gr})$, $i\colon \mathbb{Z} \times \mathrm{Gr} \to Ex^{\mathbf{A}^1}(\mathbb{Z} \times \mathrm{Gr})$ is the Voevodsky fibrant replacement morphism in the model structure described in [V, Theorem 3.7] and the structure map

$$e\colon Ex^{\mathbf{A}^1}(\mathbb{Z} \times \mathrm{Gr}) \wedge \mathbf{P}^1 \to Ex^{\mathbf{A}^1}(\mathbb{Z} \times \mathrm{Gr})$$

described in [V, Sect. 6.2]. By [V, Theorem 3.6] and Note A.20, the map $\mathbb{Z} \times \mathrm{Gr} \to Ex^{\mathbf{A}^1}(\mathbb{Z} \times \mathrm{Gr})$ is also a weak equivalence in $\mathbf{M}^{\mathrm{cm}}_{\bullet}(S)$. In particular, **BGL** is an example of a $\mathbf{P}^1$-spectrum as described in Definition 1.2.1.

Remark 1.2.3. The Waldhausen structure of a commutative monoid on $\mathbb{K}^W$ in $\mathrm{H}_{\bullet}(S)$ induces via the isomorphism $\psi \circ [i]^{-1}$ the structure of a commutative monoid $(\mathcal{K}, \bar{\mu}, \bar{e}))$ on the motivic space $\mathcal{K}$ in $\mathrm{H}_{\bullet}(S)$ such that $\psi \circ [i]^{-1}$ is an isomorphism of monoids. The composition of the inclusion $\mathbf{P}^1 = \mathrm{Gr}(1,2) \hookrightarrow \{0\} \times \mathrm{Gr} \hookrightarrow \mathbb{Z} \times \mathrm{Gr}$ and the weak equivalence i is denoted $b \colon \mathbf{P}^1 \to \mathcal{K}$. Clearly $[\epsilon] = \bar{\mu} \circ [(\mathrm{id} \wedge b)]$ in $\mathrm{H}_{\bullet}(S)$.

Lemma 1.2.4. *Given $\mathcal{K}$, $i \colon \mathbb{Z}\times\mathrm{Gr} \to \mathcal{K}$ and $\epsilon \colon \mathcal{K}\wedge\mathbf{P}^1 \to \mathcal{K}$ fulfilling the conditions of Definition 1.2.1, there exist $\mathcal{K}'$, $i' \colon \mathbb{Z}\times\mathrm{Gr} \to \mathcal{K}'$, $\epsilon' \colon \mathcal{K}'\wedge\mathbf{P}^1 \to \mathcal{K}'$ and $q \colon \mathcal{K}' \to \mathcal{K}$ such that:*

- *i' and ϵ' fulfil the condition of Definition 1.2.1.*
- *$\mathcal{K}'$ is cofibrant in $\mathbf{M}^{\mathrm{cm}}_{\bullet}(S)$.*
- *q is a weak equivalence and $q \circ \epsilon' = \epsilon \circ (q \wedge \mathrm{id})$.*

Thus the $\mathbf{P}^1$-spectra BGL' *and* BGL *are weakly equivalent via the morphism given by the sequence of maps of pointed motivic spaces $q, q, q, \ldots$.*

Proof. Decompose i as $q \circ i'$ with a trivial cofibration $i' \colon \mathbb{Z}\times\mathrm{Gr} \to \mathcal{K}'$ and a fibration $q \colon \mathcal{K}' \to \mathcal{K}$. Note that q is a weak equivalence since so are i' and i. The pointed motivic space $\mathcal{K}'$ is cofibrant in $\mathbf{M}^{\mathrm{cm}}_{\bullet}(S)$, because so is $\mathbb{Z} \times \mathrm{Gr}$. Furthermore i' is a weak equivalence. It remains to construct ϵ'. Consider the commutative diagram of pointed motivic spaces

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \mathcal{K}' \arrow[d, "q"] \\ \mathcal{K}' \wedge \mathbf{P}^1 \arrow[r, "\epsilon\circ(q\wedge\mathrm{id})"] & \mathcal{K} \end{tikzcd}$$

The left vertical arrow is a cofibration and the right hand side one is a trivial fibration. Thus there exists a map $\epsilon' \colon \mathcal{K}' \wedge \mathbf{P}^1 \to \mathcal{K}'$ of pointed motivic spaces making the diagram commutative.

Remark 1.2.5. Let $f \colon S' \to S$ be a morphism of schemes and let $\mathrm{BGL} = (\mathcal{K}, \mathcal{K}, \mathcal{K}, \ldots)$ be a $\mathbf{P}^1$-spectrum over S as described in Definition 1.2.1. Suppose further that $\mathcal{K}$ is cofibrant in $\mathbf{M}^{\mathrm{cm}}_{\bullet}(S)$. The $\mathbf{P}^1$-spectrum $f^*(\mathrm{BGL})$ over S' is given by $(\mathcal{K}', \mathcal{K}', \mathcal{K}', \ldots)$, where $\mathcal{K}' = f^*\mathcal{K}$ and the structure map is

$$\epsilon' \colon f^*\mathcal{K} \wedge \mathbf{P}^1_{S'} \cong f^*(\mathcal{K} \wedge \mathbf{P}^1_S) \xrightarrow{f^*(\epsilon)} f^*\mathcal{K}$$

Since $f^* \colon \mathbf{M}^{\mathrm{cm}}_S \to \mathbf{M}^{\mathrm{cm}}_{S'}$ is a left Quillen functor by Theorem A.17, $f^*(\mathrm{BGL})$ satisfies the conditions of Definition 1.2.1 in $M_{\bullet}(S')$ and $H_{\bullet}(S')$ provided that S' is regular. If S' is noetherian finite dimensional, then by [V, Theorem 6.9] and Remark 1.4.5 $f^*(\mathrm{BGL})$ represents the homotopy invariant K-theory as introduced in [We].

It will be proved in Sect. 1.4 that in the case $S = \mathrm{Spec}(\mathbb{Z})$ there is essentially just one $\mathbf{P}^1$-spectrum BGL in $\mathrm{SH}(S)$. In the next section, we will construct a monoidal structure on BGL regarded as an object in the stable homotopy category $\mathrm{SH}(S)$. In the case of $S = \mathrm{Spec}(\mathbb{Z})$ such a monoidal structure is unique. Pulling it back via the structural morphism $S' \xrightarrow{f} \mathrm{Spec}(\mathbb{Z})$ we get a monoidal structure on $f^*(\mathrm{BGL})$ in $\mathrm{SH}(S')$ for an arbitrary Noetherian finite-dimensional base scheme S'.

To complete this section we prove certain properties of BGL. It turns out that if $\mathcal{K}$ is fibrant in $\mathbf{M}^{\mathrm{cm}}_{\bullet}(S)$, then BGL is stably fibrant as a $\mathbf{P}^1$-spectrum. In other words, BGL is an $\Omega_{\mathbf{P}^1}$-spectrum which represents the Thomason–Trobaugh K-theory on $\mathcal{S}m/S$. For $X \in \mathcal{S}m/S$ we abbreviate $\mathrm{BGL}^{p,q}(X_+)$ as $\mathrm{BGL}^{p,q}(X)$, which forces us to write $\mathrm{BGL}^{p,q}(X, x_0)$, for a pointed S-scheme (X, x_0).

Lemma 1.2.6. *Let $X \in \mathcal{S}m/S$ and $n \geq 0$. The adjoint of the structure map $\epsilon\colon \mathcal{K} \wedge \mathbf{P}^1 \to \mathcal{K}$ induces an isomorphism*

$$\mathrm{Hom}_{H_\bullet(S)}(S^{n,0} \wedge X_+, \mathcal{K}_i) \to \mathrm{Hom}_{H_\bullet(S)}(S^{n,0} \wedge X_+ \wedge \mathbf{P}^1, \mathcal{K}_{i+1}).$$

In particular, if $\mathcal{K}$ is fibrant in $\mathbf{M}^{\mathrm{cm}}_{\bullet}(S)$, then BGL *is stably fibrant.*

Proof. Recall that for $Y \in \mathcal{S}m/S$ and a closed subset $Z \hookrightarrow Y$, $K_n(Y\ on\ Z)$ denotes the n-th Thomason–Trobaugh K-group of perfect complexes on Y with support on Z. It may be obtained as the n-th homotopy group of the homotopy fiber of the map $\mathbb{K}^W(Y) \to \mathbb{K}^W(Y \smallsetminus Z)$. Abbreviate $\mathrm{Hom}_{\mathrm{H}_\bullet(S)}(-,-)$ as $[-,-]$. For each smooth X over S the map

$$\begin{aligned} K_n(X) &= [S^{n,0}X_+, \mathcal{K}_i] \to [S^{n,0}X_+ \wedge \mathbf{P}^1, \mathcal{K}_i \wedge \mathbf{P}^1] \\ &\to [S^{n,0}X_+ \wedge \mathbf{P}^1, \mathcal{K}_{i+1}] \cong K_n(X \times \mathbf{P}^1\ on\ X \times \{\infty\}) \end{aligned}$$

induced by the structure map e_i coincides with the multiplication by the class $[\mathcal{O}(-1)] - [\mathcal{O}]$ in $K_0(\mathbf{P}^1\ on\ \{\infty\})$. This multiplication is known to be an isomorphism for the Thomason–Trobaugh K-groups, by the projective bundle theorem [TT, Theorem 4.1] for $X \times \mathbf{P}^1$. Whence the Lemma.

In the following statement, the notation $\Sigma^\infty_{\mathbf{P}^1} A(-i)$ will be used for the $\mathbf{P}^1$-spectrum $\mathrm{Fr}_i A = (*, \dots, *, A, A \wedge \mathbf{P}^1, \dots,)$ associated to a pointed motivic space A in Example A.26. Note that $\Sigma^\infty_{\mathbf{P}^1} A(-i) \cong \Sigma^\infty_{\mathbf{P}^1} A \wedge S^{-2i,-i}$ in $\mathrm{SH}(S)$, as mentioned in Notation A.40.

Corollary 1.2.7. *For each pointed motivic space A over S the adjunction map*

$$\mathrm{Hom}_{\mathrm{H}_\bullet(S)}(A, \mathcal{K}_0) \to \mathrm{Hom}_{\mathrm{SH}(S)}(\Sigma^\infty_{\mathbf{P}^1} A, \mathrm{BGL})$$

is an isomorphism. In particular, for every smooth scheme X over S and each closed subscheme Z in X one has $K_p(X\ on\ Z) = \mathrm{BGL}^{-p,0}\big(X/(X \smallsetminus Z)\big)$. The family of these isomorphisms form an isomorphism $Ad : K_* \to \mathrm{BGL}^{-*,0}$ of cohomology theories on the category $\mathcal{S}m\mathcal{O}p/S$ in the sense of [PS]. Moreover the *adjunction map* $[A, \mathcal{K}_i] \to [\Sigma^\infty_{\mathbf{P}^1}(A)(-i), \mathrm{BGL}]$ *is an isomorphism. In particular, for every smooth scheme* X *over* S *and each closed subscheme* Z *in* X *one has* $K_p(X\ on\ Z) = \mathrm{BGL}^{2i-p,i}(X/(X \smallsetminus Z))$.

The family of pairings $\mathcal{K}_i \wedge \mathcal{K}_j \xrightarrow{\mu_{ij}} \mathcal{K}_{i+j}$ in $\mathrm{H}_\bullet(S)$ with $\mu_{ij} = \bar{\mu}$ from Remark 1.2.3 defines a family of pairings

$$\cup\colon \mathrm{BGL}^{p,i}(A) \otimes \mathrm{BGL}^{q,j}(B) \to \mathrm{BGL}^{p+i,q+j}(A \wedge B) \tag{4}$$

for pointed motivic spaces A and B. We will refer to the latter as the *naive product structure* on the functor $\mathrm{BGL}^{*,*}$ on the category $\mathbf{M}_*(S)$. It has the following property.

Corollary 1.2.8. *The isomorphism* $Ad\colon K_* \to \mathrm{BGL}^{-*,0}$ *of cohomology theories on* $\mathcal{S}m\mathcal{O}p/S$ *is an isomorphism of ring cohomology theories in the sense of [PS].*

1.3 The Periodicity Element

The aim of this section is to construct an element $\beta \in \mathrm{BGL}^{2,1}(S)$, to show that it is invertible and to check that for any pointed motivic space A one has

$$\mathrm{BGL}^{*,0}(A)[\beta, \beta^{-1}] = \mathrm{BGL}^{*,*}(A)$$

(the Laurent polynomials over $\mathrm{BGL}^{*,0}(A)$). We will use the naive product structure on BGL described just above Corollary 1.2.8.

Definition 1.3.1. Set $\beta := [S^0 \xrightarrow{\bar{e}} \mathcal{K} = \mathcal{K}_1] \in \mathrm{BGL}^{2,1}(S)$, where $\bar{e}$ is the unit of the monoid $\mathcal{K}$ (see Remark 1.2.3).

Lemma 1.3.2. *Let* $b\colon \mathbf{P}^1 \hookrightarrow \mathcal{K}$ *be the map described in Remark 1.2.3. It represents the element* $[\mathcal{O}(-1)] - [\mathcal{O}]$ *in* $\mathrm{BGL}^{0,0}(\mathbf{P}^1, \infty) = \mathrm{Ker}\big(\infty^*\colon K_0(\mathbf{P}^1) \to K_0(S)\big)$. *There is a relation*

$$\beta \cup \big([\mathcal{O}(-1)] - [\mathcal{O}]\big) = \Sigma_{\mathbf{P}^1}(1) \in \mathrm{BGL}^{2,1}(\mathbf{P}^1, \infty), \tag{1}$$

where $\Sigma_{\mathbf{P}^1}$ *is the suspension isomorphism and* $1 \in \mathrm{BGL}^{0,0}(S)$ *is the unit. There is another relation*

$$\beta \cup \big([\mathcal{O}(1)] - [\mathcal{O}]\big) = -\Sigma_{\mathbf{P}^1}(1) \in \mathrm{BGL}^{2,1}(\mathbf{P}^1, \infty). \tag{2}$$

Proof. The element $\Sigma_{\mathbf{P}^1}(1)$ is represented by the morphism

$$S^0 \wedge \mathbf{P}^1 \xrightarrow{\bar{e} \wedge \mathrm{id}} \mathcal{K}_0 \wedge \mathbf{P}^1 \xrightarrow{\mathrm{id} \wedge b} \mathcal{K}_0 \wedge \mathcal{K}_1 \xrightarrow{\mu_{01}} \mathcal{K}_1,$$

where μ_{ij} is defined just above (4), and $\bar{e}$ is the unit of the monoid $\mathcal{K} = \mathcal{K}_0$. The element $\beta \cup \big([\mathcal{O}(-1)] - [\mathcal{O}]\big)$ is represented by the morphism

$$S^0 \wedge \mathbf{P}^1 \xrightarrow{\bar{e}\wedge b} \mathcal{K}_1 \wedge \mathcal{K}_0 \xrightarrow{\mu_{10}} \mathcal{K}_1.$$

Since $\mathcal{K}_0 = \mathcal{K} = \mathcal{K}_1$ one has $(\mathrm{id} \wedge b) \circ (\bar{e} \wedge \mathrm{id}) = \bar{e} \wedge b$. This implies the relation (1) since $\mu_{10} = \mu_{01}$. Relation (2) follows from the first one since $[\mathcal{O}(1)] - [\mathcal{O}] = -[\mathcal{O}(-1)] + [\mathcal{O}]$ in $K^0(\mathbf{P}^1)$.

Lemma 1.3.3. *Let $u \in \mathrm{BGL}^{-2,-1}(S)$ be the unique element such that $\Sigma_{\mathbf{P}^1}(u) = [\mathcal{O}(-1)] - [\mathcal{O}]$ in $\mathrm{BGL}^{0,0}(\mathbf{P}^1, \infty)$. Then $\beta \cup u = 1$.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{BGL}^{2,1}(S) \otimes \mathrm{BGL}^{0,0}(\mathbf{P}^1, \infty) \arrow[r, "\cup"] & \mathrm{BGL}^{2,1}(\mathbf{P}^1, \infty) \\
\mathrm{BGL}^{2,1}(S) \otimes \mathrm{BGL}^{-2,-1}(S) \arrow[r, "\cup"] \arrow[u, "\mathrm{id}\otimes\Sigma_{\mathbf{P}^1}"] & \mathrm{BGL}^{0,0}(S). \arrow[u, "\Sigma_{\mathbf{P}^1}"']
\end{tikzcd}$$

Now the Lemma follows from the relation (1).

Definition 1.3.4. For $\mathbf{P}^1$-spectra E and F set $E^{\mathrm{alg}}(F) = \oplus_{-\infty}^{+\infty} E^{2i,i}(F)$.

Proposition 1.3.5. *For every pointed motivic space A the map*

$$\mathrm{BGL}^{*,0}(A) \otimes_{K_0(S)} \mathrm{BGL}^{\mathrm{alg}}(S) \to \mathrm{BGL}^{*,*}(A) \tag{3}$$

given by $a \otimes b \mapsto a \cup b$ is a ring isomorphism and $\mathrm{BGL}^{\mathrm{alg}}(S) = K_0(S)[\beta, \beta^{-1}]$ is the Laurent polynomial ring. One can rewrite this ring isomorphism as

$$\mathrm{BGL}^{*,0}(A)[\beta, \beta^{-1}] \cong \mathrm{BGL}^{*,*}(A) \tag{4}$$

Proof. In fact, $\mathrm{BGL}^{*,0}(A) \xrightarrow{\cup\beta} \mathrm{BGL}^{*+2,1}(A)$ is an isomorphism since β is invertible. Since $\mathrm{BGL}^{0,0}(S) = K^0(S)$ the map (3) is a ring isomorphism.

Using the isomorphism $Ad\colon K_* \to \mathrm{BGL}^{-*,0}$ of ring cohomology theories from Corollary 1.2.8 we get the following statement.

Corollary 1.3.6. *For every $X \in \mathcal{S}m/S$ and every closed subset $Z \hookrightarrow X$ one has*

$$K_{-*}(X \; on \; Z)[\beta, \beta^{-1}] \cong \mathrm{BGL}^{*,*}_Z(X). \tag{5}$$

The family of these isomorphisms form an isomorphism of ring cohomology theories on $\mathcal{S}m\mathcal{O}p/S$ in the sense of [PS]. As well, there is an isomorphism

$$K_{-*}(X \; on \; Z) = \mathrm{BGL}^{*,*}(X/X-Z)/(\beta+1)\mathrm{BGL}^{*,*}(X/X-Z). \tag{6}$$

The family of these isomorphisms form an isomorphism of ring cohomology theories on $\mathcal{S}m\mathcal{O}p/S$ in the same sense.

1.4 Uniqueness of BGL

We prove in this section that, at least over $S = \mathrm{Spec}(\mathbb{Z})$, a $\mathbf{P}^1$-spectrum BGL as described in Definition 1.2.1 is essentially unique regarded as an object in the stable homotopy category $\mathrm{SH}(S)$. This has also been obtained in [R].

Let BGL be a $\mathbf{P}^1$-spectrum as described in Definition 1.2.1. Recall that this involves the choice of a weak equivalence $i\colon \mathbb{Z}\times \mathrm{Gr} \to \mathcal{K}$ and a structure map $\epsilon\colon \mathcal{K}\wedge \mathbf{P}^1 \to \mathcal{K}$. Let BGL' be a possibly different $\mathbf{P}^1$-spectrum. More precisely, take a pointed motivic space $\mathcal{K}'$ together with a weak equivalence $i'\colon \mathbb{Z}\times \mathrm{Gr} \to \mathcal{K}'$ and with a morphism $\epsilon'\colon \mathcal{K}' \wedge \mathbf{P}^1 \to \mathcal{K}'$ which descends to

$$\mu^W \circ (\mathrm{id} \wedge b^W)\colon \mathbb{K}^W \wedge \mathbf{P}^1 \to \mathbb{K}^W$$

under the identification of $\mathcal{K}'$ with $\mathbb{K}^W$ in $\mathrm{H}_\bullet(S)$ via the isomorphism $\psi \circ [i']^{-1}$. Let BGL' be the $\mathbf{P}^1$-spectrum of the form $(\mathcal{K}'_0, \mathcal{K}'_1, \mathcal{K}'_2, \dots)$ with $\mathcal{K}'_i = \mathcal{K}'$ for all i, and with the structure maps $\epsilon'_i\colon \mathcal{K}'_i \wedge \mathbf{P}^1 \to \mathcal{K}'_{i+1}$ equal to the map $\epsilon'\colon \mathcal{K}' \wedge \mathbf{P}^1 \to \mathcal{K}'$.

Proposition 1.4.1. *Let* $S = \mathrm{Spec}(\mathbb{Z})$. *There exists a unique morphism* $\theta\colon \mathrm{BGL} \to \mathrm{BGL}'$ *in* $\mathrm{SH}(S)$ *such that for every integer* $i \geq 0$ *the diagram*

$$\begin{tikzcd} \Sigma^\infty_{\mathbf{P}^1}\mathcal{K}_i(-i) \arrow[r, "u_i"] \arrow[d, "\Sigma^\infty_{\mathbf{P}^1}\phi_i(-i)"'] & \mathrm{BGL} \arrow[d, "\theta"] \\ \Sigma^\infty_{\mathbf{P}^1}\mathcal{K}'_i(-i) \arrow[r, "u'_i"] & \mathrm{BGL}' \end{tikzcd}$$

commutes in $\mathrm{SH}(S)$, *where* $\phi_i = i' \circ i^{-1} \in [\mathcal{K}_i, \mathcal{K}'_i]_{\mathrm{H}_\bullet(S)}$ *and* u_i, u'_i *are the canonical morphisms. Similarly, there exists a unique morphism* $\theta'\colon \mathrm{BGL}' \to \mathrm{BGL}$ *in* $\mathrm{SH}(S)$ *such that for every integer* $i \geq 0$ *the diagram*

$$\begin{tikzcd} \Sigma^\infty_{\mathbf{P}^1}\mathcal{K}'_i(-i) \arrow[r, "u'_i"] \arrow[d, "\Sigma^\infty_{\mathbf{P}^1}\phi'_i(-i)"'] & \mathrm{BGL}' \arrow[d, "\theta'"] \\ \Sigma^\infty_{\mathbf{P}^1}\mathcal{K}_i(-i) \arrow[r, "u_i"] & \mathrm{BGL} \end{tikzcd}$$

commutes in $\mathrm{SH}(S)$, *where* $\theta_i = i \circ (i')^{-1} \in [\mathcal{K}'_i, \mathcal{K}_i]_{\mathrm{H}_\bullet(S)}$.

Proof. Consider the exact sequence

$$0 \to \varprojlim{}^1 \mathrm{BGL}^{2i-1,i}(\mathcal{K}'_i) \to \mathrm{BGL}^{0,0}(\mathrm{BGL}') \to \varprojlim \mathrm{BGL}^{2i,i}(\mathcal{K}'_i) \to 0$$

from Lemma A.34. The family of elements $(u_i \circ \Sigma^\infty_{\mathbf{P}^1}\theta'_i(-i))$ is an element of the group $\varprojlim \mathrm{BGL}^{2i,i}(\mathcal{K}'_i)$. Thus there exists the required morphism θ'. To prove its uniqueness, observe that the $\varprojlim^1$-group vanishes by Proposition 1.6.1. Whence

$\mathrm{BGL}^{0,0}(\mathrm{BGL}') = \varprojlim \mathrm{BGL}^{2i,i}(\mathcal{K}'_i)$ and θ' is indeed unique. By symmetry there also exists a unique morphism θ with the required property.

Proposition 1.4.2. *Let $S = \mathrm{Spec}(\mathbb{Z})$. The morphism $\theta\colon \mathrm{BGL} \to \mathrm{BGL}'$ is the inverse of $\theta'\colon \mathrm{BGL}' \to \mathrm{BGL}$ in $\mathrm{SH}(S)$, and in particular an isomorphism.*

Proof. The composite morphism $\theta' \circ \theta\colon \mathrm{BGL} \to \mathrm{BGL}$ has the property that for every integer $i \geq 0$ the diagram

$$\begin{tikzcd}
\Sigma^\infty_{\mathbf{P}^1}\mathcal{K}_i(-i) \arrow[r, "u_i"] \arrow[d, "\mathrm{id}"'] & \mathrm{BGL} \arrow[d, "\theta'\circ\theta"] \\
\Sigma^\infty_{\mathbf{P}^1}\mathcal{K}_i(-i) \arrow[r, "u_i"] & \mathrm{BGL}
\end{tikzcd}$$

commutes. However, the identity morphism $\mathrm{id}\colon \mathrm{BGL} \to \mathrm{BGL}$ has the same property. Thus $\theta' \circ \theta = \mathrm{id}.$ by the uniqueness in Proposition 1.4.1, and similarly $\theta \circ \theta' = \mathrm{id}.$

Remark 1.4.3. The isomorphisms θ and θ' are monoid isomorphisms provided that BGL and BGL' are equipped with the monoidal structures given by Theorem 2.2.1. This follows from the fact that both i and i' are isomorphisms of monoids in $H_\bullet(S)$.

Remark 1.4.4. There exists a unique morphism $e\colon (\mathbb{Z}\times\mathrm{Gr})\wedge\mathbf{P}^1 \to \mathbb{Z}\times\mathrm{Gr}$ in $\mathrm{H}_\bullet(S)$ such that the diagram

$$\begin{tikzcd}
(\mathbb{Z}\times\mathrm{Gr})\wedge\mathbf{P}^1 \arrow[r, "e"] \arrow[d, "\psi\wedge\mathrm{id}"'] & \mathbb{Z}\times\mathrm{Gr} \arrow[d, "\psi"] \\
\mathbb{K}^W\wedge\mathbf{P}^1 \arrow[r, "e^W"] & \mathbb{K}^W
\end{tikzcd}$$

commutes in $\mathrm{H}_\bullet(S)$, where $e^W = \mu^W \circ (\mathrm{id}\wedge b^W)$ and ψ is described right above Definition 1.2.1. The diagram

$$\begin{tikzcd}
(\mathbb{Z}\times\mathrm{Gr})\wedge\mathbf{P}^1 \arrow[r, "e"] \arrow[d, "i\wedge\mathrm{id}"'] & \mathbb{Z}\times\mathrm{Gr} \arrow[d, "i"] \\
\mathcal{K}\wedge\mathbf{P}^1 \arrow[r, "\epsilon"] & \mathcal{K}
\end{tikzcd}$$

then commutes as well. That is, ϵ descends to e in $\mathrm{H}_\bullet(S)$.

We will need an observation concerning the morphism e. Let τ_n be the tautological bundle over the Grassmann variety $\mathrm{Gr}(n,2n)$. By Lemma B.7 there exists a unique element $\xi_\infty \in \mathrm{K}_0(\mathbb{Z}\times\mathrm{Gr})$ such that for any positive integer n and any $0 \leq m \leq n$ one has $\xi|_{\{m\}\times\mathrm{Gr}(n,2n)} = [\tau_n] - n + m$. By Lemmas B.8 and B.2 the element $\xi_\infty \otimes ([\mathcal{O}(-1)] - [\mathcal{O}])$ belongs to the subgroup $\mathrm{K}_0((\mathbb{Z}\times\mathrm{Gr})\wedge\mathbf{P}^1)$ of the group $\mathrm{K}_0(\mathbb{Z}\times\mathrm{Gr}\times\mathbf{P}^1)$. The isomorphism $\psi\colon \mathbb{Z}\times\mathrm{Gr} \to \mathbb{K}^W$ in $\mathrm{H}_\bullet(S)$ represents the element ξ_∞ in $\mathrm{K}_0((\mathbb{Z}\times\mathrm{Gr}))$. Now the definitions of e^W and e show that

$$e^*(\xi_\infty) = \xi_\infty \otimes ([\mathcal{O}(-1)] - [\mathcal{O}])$$

in $\mathrm{K}_0((\mathbb{Z} \times \mathrm{Gr}) \wedge \mathbf{P}^1)$.

Remark 1.4.5. Let S be a regular scheme. Given two triples $(\mathcal{K}_1, i_1, \epsilon_1)$ and $(\mathcal{K}_2, i_2, \epsilon_2)$ fulfilling the conditions of Definition 1.2.1, there exists a zig-zag of weak equivalences of triples connecting these two. In particular, there exists a zig-zag of levelwise weak equivalences of $\mathbf{P}^1$-spectra over S connecting BGL_1 and BGL_2. It follows that the $\mathbf{P}^1$-spectra BGL_1 and BGL_2 associated to these triples are "naturally" isomorphic in $\mathrm{SH}(S)$. This shows that the strength of Proposition 1.4.1 is in its uniqueness assertion.

1.5 Preliminary Computations I

In this section we prepare for the next section, in which we show that certain $\varprojlim^1$-groups vanish. Let BGL be the $\mathbf{P}^1$-spectrum defined in Definition 1.2.1. We will identify in this section the functors $\mathrm{BGL}^{0,0}$ and $\mathrm{BGL}^{2i,i}$ on the category $\mathrm{H}_\bullet(S)$ via the iterated $(2,1)$-periodicity isomorphism as follows:

$$\mathrm{BGL}^{0,0}(A) \cong \mathrm{Hom}_{\mathrm{H}_\bullet(S)}(A, \mathcal{K}_0) = \mathrm{Hom}_{\mathrm{H}_\bullet(S)}(A, \mathcal{K}_i) \cong \mathrm{BGL}^{2i,i}(A). \tag{1}$$

Similarly,

$$\begin{aligned} \mathrm{BGL}^{-1,0}(A) &\cong \mathrm{Hom}_{\mathrm{H}_\bullet(S)}(S^{1,0} \wedge A, \mathcal{K}_0) \\ &= \mathrm{Hom}_{\mathrm{H}_\bullet(S)}(S^{1,0} \wedge A, \mathcal{K}_i) \cong \mathrm{BGL}^{2i-1,i}(A) \end{aligned} \tag{2}$$

These identifications respect the naive product structure (4) on the functor $\mathrm{BGL}^{*,*}$. In particular, the following diagram commutes for every pointed motivic space A over S.

$$\begin{array}{ccc} \mathrm{BGL}^{-1,0}(S) \otimes \mathrm{BGL}^{0,0}(A) & \longrightarrow & \mathrm{BGL}^{-1,0}(A) \\ \cong \downarrow & & \downarrow \cong \\ \mathrm{BGL}^{-1,0}(S) \otimes \mathrm{BGL}^{2i,i}(A) & \longrightarrow & \mathrm{BGL}^{2i-1,i}(A) \end{array} \tag{3}$$

Remark 1.4.20. The identification (2) of $\mathrm{BGL}^{-1,0}(X)$ with $\mathrm{BGL}^{2i-1,i}(X)$ coincides with the periodicity isomorphism $\mathrm{BGL}^{-1,0}(X) \xrightarrow{\cup\beta^i} \mathrm{BGL}^{2i-1,i}(X)$.

Lemma 1.5.1. *Let* $S = \mathrm{Spec}(\mathbb{Z})$. *For every integer* i *the map*

$$\mathrm{BGL}^{-1,0}(S) \otimes \mathrm{BGL}^{2i,i}(\mathcal{K}) \to \mathrm{BGL}^{2i-1,i}(\mathcal{K})$$

induced by the naive product structure is an isomorphism. The same holds if $\mathcal{K} \wedge \mathbf{P}^1$ *replaces* $\mathcal{K}$.

Proof. The commutativity of the diagram (3) shows that it suffices to consider the case $i = 0$. Furthermore we may replace the pointed motivic space $\mathcal{K}$ with $\mathbb{Z} \times \mathrm{Gr}$ since the map $i\colon \mathbb{Z} \times \mathrm{Gr} \to \mathcal{K} = \mathcal{K}$ is a weak equivalence. The functor isomorphism $K_* \to \mathrm{BGL}^{-*,0}$ is a ring cohomology isomorphism by Corollary 1.2.8. Thus it remains to check that the map

$$K_1(S) \otimes K_0(\mathbb{Z} \times \mathrm{Gr}) \to K_1(\mathbb{Z} \times \mathrm{Gr})$$

is an isomorphism. For a set M and a smooth S-scheme X we will write $M \times X$ for the disjoint union $\bigsqcup_M X$ of M copies of X in the category of motivic spaces over S. Let $[-n, n]$ be the set of integers with absolute value $\leq n$. By Lemmas B.7 and B.3 it suffices to check that the natural map

$$A \otimes \varprojlim K_0\big([-n,n] \times \mathrm{Gr}(n,2n)\big) \to \varprojlim A \otimes K_0\big([-n,n] \times \mathrm{Gr}(n,2n)\big)$$

is an isomorphism, where $A = K_1(S)$. This is the case since $K_1(S)$ is a finitely generated abelian group (it is just $\mathbb{Z}/2\mathbb{Z}$). The assertion concerning $\mathcal{K} \wedge \mathbf{P}^1$ is proved similarly using Lemmas B.8 and B.5 instead.

To state the next lemma, consider the scheme morphism $f\colon \mathrm{Spec}(\mathbb{C}) \to S = \mathrm{Spec}(\mathbb{Z})$, the pull-back functor $f^*\colon \mathrm{SH}(\mathbb{Z}) \to \mathrm{SH}(\mathbb{C})$ described in Proposition A.47, and the topological realization functor $\mathbf{R}_{\mathbb{C}}\colon \mathrm{SH}(\mathbb{C}) \to \mathrm{SH}_{\mathbb{C}\mathbf{P}^1}$ described in Sect. A.7. Set $r = \mathbf{R}_{\mathbb{C}} \circ f^*\colon \mathrm{SH}(S) \to \mathrm{SH}_{\mathbb{C}\mathbf{P}^1}$. The functor r will be called for short the realization functor below in this section.

Lemma 1.5.2. *Let* $\mathbb{B}\mathrm{U}$ *be the periodic complex K-theory* $\mathbb{C}\mathbf{P}^1$*-spectrum with terms* $\mathbb{Z} \times \mathrm{BU}$. *There is a zigzag* $\mathbb{B}\mathrm{U} \xleftarrow{\sim} E \xrightarrow{\sim} r\mathrm{BGL}$ *of levelwise weak equivalences of* $\mathbb{C}\mathbf{P}^1$*-spectra.*

Proof. This follows from Remark 1.4.4, Example A.46 and the fact that Grassmann varieties pull back.

Lemma 1.5.3. *Let* $X \in \mathcal{S}m/S$, *where* $S = \mathrm{Spec}(\mathbb{Z})$, *and let* $X_0 \subset X_1 \subset \cdots \subset X_n = X$ *be a filtration by closed subsets such that for every integer* $i \geq 0$ *the* S*-scheme* $X_i - X_{i-1}$ *is isomorphic to a disjoint union of several copies of the affine space* $\mathbf{A}^i_S$. *The map* $\mathrm{BGL}^{0,0}(X) \to (r\mathrm{BGL})^0(rX)$ *is an isomorphism.*

Proof. Consider the class $\mathcal{R}$ of $\mathbf{P}^1$-spectra E such that the homomorphism $\mathrm{BGL}^{0,0}(E) \to r\mathrm{BGL}^0(rE)$ is an isomorphism. It contains $S^{0,0}$ because in this case we obtain the isomorphism $\mathrm{BGL}^{0,0}(S^{0,0}) \cong K^0(\mathbb{Z}) \cong \mathbb{Z} \cong K^0_{\mathrm{top}}(S^0)$ which identifies the class of an algebraic resp. complex topological vector bundle over $\mathrm{Spec}(\mathbb{Z})$ resp. $\bullet$ with its rank. The $(2,1)$-periodicity isomorphism for BGL described in Remark 1.4.20 and the Bott periodicity isomorphism for $r\mathrm{BGL}$ are compatible by Example A.46. This implies that $S^{2m,m} \in \mathcal{R}$ for all $m \in \mathbb{Z}$. Finally, if $E \to F \to G \to S^{1,0} \wedge E$ is a distinguished triangle in $\mathrm{SH}(S)$ such that E and G are in $\mathcal{R}$, then so is F.

For $i \geq 0$ write $U^i := X \smallsetminus X_i$, so that U^i is an open subset of U^{i-1}. In particular we have $U^n = \emptyset$ and $U^{-1} = X$. The closed subscheme $X_i \smallsetminus X_{i-1} = X_i \cap U^{i-1} \hookrightarrow U^{i-1}$ is isomorphic to a disjoint union of m_i copies of affine spaces $\mathbf{A}^i$, and is in particular smooth over S. Furthermore the normal bundle is trivial. The homotopy purity theorem [MV, Theorem 3.2.29] supplies a distinguished triangle

$$\Sigma^\infty_{\mathbf{P}^1} U^i_+ \to \Sigma^\infty_{\mathbf{P}^1} U^{i-1}_+ \to \Sigma^\infty_{\mathbf{P}^1} U^{i-1}/U^i \cong \vee_{j=1}^{m_i} S^{2(n-i),(n-i)}$$

of $\mathbf{P}^1$-spectra. Since $\mathcal{R}$ contains $\Sigma^\infty_{\mathbf{P}^1} U^n = \bullet$ we obtain inductively that $\mathcal{R}$ contains $\Sigma^\infty_{\mathbf{P}^1} U^{-1} = \Sigma^\infty_{\mathbf{P}^1} X_+$.

Lemma 1.5.4. *Let* $S = \mathrm{Spec}(\mathbb{Z})$ *and let* $r\colon \mathrm{SH}(\mathbb{Z}) \to \mathrm{SH}_{\mathbb{C}\mathbf{P}^1}$ *be the topological realization functor. Then for every integer* i *the realization homomorphism* $\mathrm{BGL}^{2i,i}(\mathcal{K}) \to (r\mathrm{BGL})^{2i}(r\mathcal{K})$ *is an isomorphism.*

Proof. Clearly it suffices to prove the case $i = 0$. We may replace the pointed motivic space $\mathcal{K}_i$ with $\mathbb{Z} \times \mathrm{Gr}$ as in the proof of Lemma 1.5.1. It remains to check that the topological realization homomorphism $\mathrm{BGL}^{0,0}(\mathrm{Gr}) \to (r\mathrm{BGL})^0(r\mathrm{Gr})$ is an isomorphism.

Since $\mathrm{Gr}(n, 2n)$ has a filtration satisfying the condition of Lemma 1.5.3, we see that the map $\mathrm{BGL}^{0,0}(\mathrm{Gr}(n, 2n)) \to (r\mathrm{BGL})^0(\mathrm{Gr}(n, 2n))$ is an isomorphism for every n. To conclude the statement for $\mathrm{Gr} = \cup\mathrm{Gr(n, 2n)}$, use the short exact sequence from Lemma A.34. In the resulting diagram

$$\begin{array}{ccccc}
\varprojlim^1 \mathrm{BGL}^{-1,0}\big(\mathrm{Gr}(n,2n)\big) & \longrightarrow & \mathrm{BGL}^{0,0}(\mathrm{Gr}) & \longrightarrow & \varprojlim \mathrm{BGL}^{0,0}\big(\mathrm{Gr}(n,2n)\big) \\
\downarrow & & \downarrow & & \downarrow \\
\varprojlim^1 r\mathrm{BGL}^{-1,0}\big(r\mathrm{Gr}(n,2n)\big) & \longrightarrow & r\mathrm{BGL}^{0,0}(r\mathrm{Gr}) & \longrightarrow & \varprojlim r\mathrm{BGL}^{0,0}\big(r\mathrm{Gr}(n,2n)\big)
\end{array}$$

the map on the right hand side is then an isomorphism. Furthermore one concludes from [Sw, Theorem 16.32] that

$$\varprojlim^1 r\mathrm{BGL}^{-1,0}\big(r\mathrm{Gr}(n,2n)\big) = \varprojlim^1 K^1_{\mathrm{top}}\big(r\mathrm{Gr}(n,2n)\big) = 0.$$

On the other hand $\varprojlim^1 \mathrm{BGL}^{-1,0}\big(\mathrm{Gr}(n,2n)\big) = \varprojlim^1 K_1\big(\mathrm{Gr}(n,2n)\big) = 0$ by Lemma B.6. The result follows.

Lemma 1.5.5. *Let* $\mathbb{B}^0\mathrm{U}$ *be the sub-spectrum of* $\mathbb{B}\mathrm{U}$ *with the* n*-th term equal to the connected component* BU *of the topological space* $\mathbb{Z} \times \mathrm{BU}$ *containing the basepoint* $\bullet$. *The inclusion* $\mathbb{B}^0\mathrm{U} \to \mathbb{B}\mathrm{U}$ *is a weak equivalence of* $\mathbb{C}\mathbf{P}^1$*-spectra.*

Proof. One has to check that the inclusion induces an isomorphism on stable homotopy groups. This follows because the structure map $(\mathbb{Z} \times BU) \wedge \mathbb{C}\mathbf{P}^1 \to \mathbb{Z} \times BU$ factors over $\{0\} \times BU$.

Lemma 1.5.6. *There exists a sub-spectrum* $\mathbb{B}^f\mathrm{U}$ *of the* $\mathbb{C}\mathbf{P}^1$*-spectrum* $\mathbb{B}\mathrm{U}$ *with the* n*-th term* $\mathrm{Gr}\big(b(n), 2b(n)\big)$ *such that the inclusion* $\mathbb{B}^f\mathrm{U} \to \mathbb{B}\mathrm{U}$ *is a stable equivalence.*

Proof. The sequence $b(n)$ will be constructed such that $b(n) \geq 2n+1$. Set $b(0)=1$. We may assume that the structure map $e_0\colon \mathrm{BU}\wedge \mathbb{C}\mathbf{P}^1 \to \mathrm{BU}$ is cellular. Since $r\mathrm{Gr}(b(0),2b(0))\wedge \mathbb{C}\mathbf{P}^1$ is a finite cell complex, it lands in a Grassmannian $r\mathrm{Gr}(b(1),2b(1))$ for some integer $b(1)\geq 2\cdot 1+1$. Continuing this process produces the required sequence of $b(n)$'s. The inclusions induce an isomorphism $\mathrm{colim}_{n\geq 0}\,\mathrm{Gr}\big(b(n),2b(n)\big)\cong \mathrm{Gr}$.

To observe that the inclusion $j\colon \mathbb{B}^f\mathrm{U}\to \mathbb{B}\mathrm{U}$ is then a stable equivalence, recall that the number of $2k$-cells in $\mathrm{Gr}(n,m)$ is given by the number of partitions of k into at most n subsets each of which has cardinality $\geq m-n$ [MS]. In particular, the $2k$-skeleton of BU coincides with the $2k$-skeleton of $r\mathrm{Gr}(k,2k)$. To prove the surjectivity of $\pi_i(j)$ choose an element $\alpha\in\pi_i\mathbb{B}\mathrm{U}$. It is represented by a cellular map $a\colon S^{i+2m}\to \mathrm{BU}$ for some m with $i+2m\geq 0$. We may choose m such that $m\geq i$. Thus a lands in $r\mathrm{Gr}(b(m),2b(m))$ and gives rise to an element in $\pi_i\mathbb{B}^f\mathrm{U}$ mapping to α. To prove the injectivity of $\pi_i(j)$, choose an element $\alpha\in\pi_i\mathbb{B}^f\mathrm{U}$ such that $\pi_i(j)(\alpha)=0$. We may represent α by some map $a\colon S^{i+2m}\to \mathrm{Gr}(b(m),2b(m))$ for some m with $i+2m>0$ and $m\geq i$. The composition

$$S^{i+2m}\xrightarrow{a}\mathrm{Gr}(b(m),2b(m))\hookrightarrow \mathrm{BU}$$

is nullhomotopic since $\pi_{i+2m}\mathrm{BU}\cong\pi_i\mathbb{B}\mathrm{U}$ via the homomorphism induced by the structure map. The nullhomotopy may be chosen to be cellular and thus lands in $\mathrm{Gr}(b(m),2b(m))$. This completes the proof.

1.6 Vanishing of Certain Groups I

Consider the stable equivalence $\mathrm{hocolim}_{i\geq 0}\,\Sigma^\infty_{\mathbf{P}^1}\mathcal{K}_i(-i)\cong \mathrm{BGL}$ [see (2)] and the induced short exact sequence

$$0\to \varprojlim{}^1\mathrm{BGL}^{2i-1,i}(\mathcal{K}_i)\to \mathrm{BGL}^{0,0}(\mathrm{BGL})\to \varprojlim \mathrm{BGL}^{2i,i}(\mathcal{K}_i)\to 0$$

We prove in this section the following result.

Proposition 1.6.1. *Let* $S=\mathrm{Spec}(\mathbb{Z})$*, then* $\varprojlim{}^1\mathrm{BGL}^{2i-1,i}(\mathcal{K}_i)=0$.

Proof. The connecting homomorphism in the tower of groups for the $\varprojlim^1$-term is the composite map

$$\mathrm{BGL}^{2i-1,i}(\mathcal{K}_i)\xleftarrow{\Sigma^{-1}_{\mathbf{P}^1}}\mathrm{BGL}^{2i+1,i+1}(\mathcal{K}_i\wedge\mathbf{P}^1)\xleftarrow{e_i^*}\mathrm{BGL}^{2i+1,i+1}(\mathcal{K}_{i+1})$$

where $\Sigma^{-1}_{\mathbf{P}^1}$ is the inverse to the $\mathbf{P}^1$-suspension isomorphism and e_i^* is the pull-back induced by the structure map e_i. Set $A=\mathrm{BGL}^{-1,0}(S)$ and consider the diagram

$$\begin{array}{ccccc}
\mathrm{BGL}^{2i-1,i}(\mathcal{K}) & \xleftarrow{\Sigma^{-1}_{\mathbf{P}^1}} & \mathrm{BGL}^{2i+1,i+1}(\mathcal{K}\wedge\mathbf{P}^1) & \xleftarrow{e_i^*} & \mathrm{BGL}^{2i+1,i+1}(\mathcal{K}) \\
\uparrow & & \uparrow & & \uparrow \\
A\otimes\mathrm{BGL}^{2i,i}(\mathcal{K}) & \xleftarrow{\mathrm{id}\otimes\Sigma^{-1}_{\mathbf{P}^1}} & A\otimes\mathrm{BGL}^{2(i+1),i+1}(\mathcal{K}\wedge\mathbf{P}^1) & \xleftarrow{\mathrm{id}\otimes e_i^*} & A\otimes\mathrm{BGL}^{2(i+1),i+1}(\mathcal{K})
\end{array}$$

where the vertical arrows are induced by the naive product structure on the functor $\mathrm{BGL}^{*,*}$. Clearly it commutes. Since S is regular, the vertical arrows are isomorphisms by Lemma 1.5.1. It follows that $\varprojlim^1\mathrm{BGL}^{2i-1,i}(\mathcal{K}_i) = \varprojlim^1(A\otimes\mathrm{BGL}^{2i,i}(\mathcal{K}_i))$ where in the last tower of groups the connecting maps are $\mathrm{id}\otimes(\Sigma^{-1}_{\mathbf{P}^1}\circ e_i^*)$. It remains to prove the following assertion.

Claim. The equality $\varprojlim^1(A\otimes\mathrm{BGL}^{2i,i}(\mathcal{K}_i)) = 0$ holds.

Since $S = \mathrm{Spec}(\mathbb{Z})$ one gets $A = \mathrm{BGL}^{-1,0}(S) = K_1(\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Thus $A\otimes\mathrm{BGL}^{2i,i}(\mathcal{K}_i) = \mathrm{BGL}^{2i,i}(\mathcal{K}_i)/m\mathrm{BGL}^{2i,i}(\mathcal{K}_i)$ with $m = 2$ and the connecting maps in the tower are just the mod-m reduction of the maps $\Sigma^{-1}_{\mathbf{P}^1}\circ e_i^*$. Now a chain of isomorphisms completes the proof of the Claim.

$$\begin{aligned}
\varprojlim{}^1\mathrm{BGL}^{2i,i}(\mathcal{K}_i)/m &\cong \varprojlim{}^1(r\mathrm{BGL})^{2i}(r\mathcal{K}_i)/m \cong \varprojlim{}^1\mathbb{B}\mathrm{U}^{2i}(r\mathcal{K}_i)/m \\
&\cong \varprojlim{}^1\mathbb{B}\mathrm{U}^{2i}(\mathbb{Z}\times\mathrm{BU})/m \cong \varprojlim{}^1\mathbb{B}\mathrm{U}^{2i}(\mathbb{Z}\times\mathrm{BU};\mathbb{Z}/m) \\
&\cong K^1_{\mathrm{top}}(\mathbb{B}\mathrm{U};\mathbb{Z}/m) \cong K^1_{\mathrm{top}}(\mathbb{B}^0\mathrm{U};\mathbb{Z}/m) \\
&\cong \varprojlim{}^1 K^{2i}_{\mathrm{top}}\big(\mathrm{Gr}(b(i),2b(i));\mathbb{Z}/m\big) = 0
\end{aligned}$$

The first isomorphism follows from Lemma 1.5.4. The second isomorphism is induced by the levelwise weak equivalence $\mathbb{B}\mathrm{U}\simeq r\mathrm{BGL}$ mentioned in Lemma 1.5.2. The third isomorphism is induced by the image of the weak equivalence $\mathcal{K}_i\simeq\mathbb{Z}\times\mathrm{Gr}$ under topological realization. The forth and fifth isomorphism hold since $\mathbb{B}\mathrm{U}^{2i+1}(\mathbb{Z}\times\mathrm{BU}) = 0$. The sixth isomorphism is induced by the stable equivalence $\mathbb{B}^0\mathrm{U}\simeq\mathbb{B}\mathrm{U}$ from Lemma 1.5.5, the seventh one is induced by the stable equivalence $\mathbb{B}^f\mathrm{U}\simeq\mathbb{B}^0\mathrm{U}$ from Lemma 1.5.6. The last one holds since all groups in the tower are finite.

2 Smash-Product, Pull-backs, Topological Realization

In this section we construct a smash product $\wedge$ of $\mathbf{P}^1$-spectra, check its basic properties and consider its behavior with respect to pull-back and realization functors. We follow here an idea of Voevodsky [V, Comments to Theorem 5.6] and use results of Jardine [J]. In several cases we will not distinguish notationally between a Quillen functor and its total derived functor, $\Sigma^\infty_{\mathbf{P}^1}$ being the most prominent example.

2.1 The Smash Product

Definition 2.1.1. Let $V := \mathcal{L}v\colon \mathrm{SH}(S) \to \mathrm{SH}^{\Sigma}(S)$ and $U := \mathcal{R}u\colon \mathrm{SH}^{\Sigma}(S) \to \mathrm{SH}(S)$ be the equivalence described in Remark A.39. For a pair of $\mathbf{P}^1$-spectra E and F set

$$E \wedge F := U(VE \wedge VF)$$

as in Remark A.39.

Proposition 2.1.2. *Let S be a Noetherian finite-dimensional base scheme. The smash product of $\mathbf{P}^1$-spectra over S induces a closed symmetric monoidal structure $(\wedge, \mathbb{I})$ on the motivic stable homotopy category $\mathrm{SH}(S)$ having the properties required by Theorem 5.6 of Voevodsky's congress talk [V]:*

1. *There is a canonical isomorphism $E \wedge \Sigma^{\infty}_{\mathbf{P}^1} A \cong (A \wedge E_i, \mathrm{id} \wedge e_i)$ for every pointed motivic space A and every $\mathbf{P}^1$-spectrum.*
2. *There is a canonical isomorphism $(\oplus E_\alpha) \wedge F \cong \oplus(E_\alpha \wedge F)$ for $\mathbf{P}^1$-spectra E_i, F.*
3. *Smashing with a $\mathbf{P}^1$-spectrum preserves distinguished triangles. To be more precise, if $E \xrightarrow{f} F \to \mathrm{cone}(f) \xrightarrow{\epsilon} E[1]$ is a distinguished triangle and G is a $\mathbf{P}^1$-spectrum, the sequence $E \wedge G \xrightarrow{f} F \wedge G \to \mathrm{cone}(f) \wedge G \xrightarrow{\epsilon} E \wedge G[1]$ is a distinguished triangle, where the last morphism is the composition of $\epsilon \wedge \mathrm{id}_G$ with the canonical isomorphism $E[1] \wedge G \to (E \wedge F)[1]$.*

Proof. Follows from Remark A.39 and Theorem A.38.

In the following we use that the homotopy colimit of a sequence $E = E_0 \to E_1 \to \cdots$ of morphisms in the homotopy category $\mathrm{SH}(S)$ may be computed in three steps:

1. Lift E to a sequence $E' = E'_0 \to E'_1 \to \cdots$ of cofibrations (in fact, arbitrary maps suffice) of $\mathbf{P}^1$-spectra.
2. Take the colimit $\mathrm{colim}_{i\geq 0} E'_i$ of E' in the category of $\mathbf{P}^1$-spectra.
3. Consider $\mathrm{colim}_{i\geq 0} E'_i$ as an object in $\mathrm{SH}(S)$.

Lemma 2.1.3. *Let $E = \mathrm{hocolim}_{i\geq 0} E_i$ be a sequential homotopy colimit of $\mathbf{P}^1$-spectra. For every $\mathbf{P}^1$-spectrum F there is an exact sequence of abelian groups*

$$0 \to \varprojlim{}^1 F^{p-1,q}(E_i) \to F^{p,q}(E) \to \varprojlim F^{p,q}(E_i) \to 0. \tag{1}$$

Proof. This is Lemma A.34.

By Lemma A.33, any $\mathbf{P}^1$-spectrum E can be expressed as the homotopy colimit

$$\mathrm{hocolim}\, \Sigma^{\infty}_{\mathbf{P}^1} E_i(-i) \cong E. \tag{2}$$

Corollary 2.1.4. *For two $\mathbf{P}^1$-spectra E and F there is a canonical short exact sequence*

$$0 \to \varprojlim^1 F^{p+2i-1,q+i}(E_i) \to F^{p,q}(E) \to \varprojlim F^{p+2i,q+i}(E_i) \to 0. \tag{3}$$

Corollary 2.1.5. *For a pair of spectra E and F and each spectrum G one has a canonical exact sequence of the form*

$$0 \to \varprojlim^1 G^{p+4i-1,q+2i}(E_i \wedge F_i) \to G^{p,q}(E \wedge F) \to \varprojlim G^{p+4i,q+2i}(E_i \wedge F_i) \to 0. \tag{4}$$

Proof. For a pair of spectra E and F one has a canonical isomorphism of the form

$$\operatorname{hocolim}\,(\Sigma^\infty_{\mathbf{P}^1}(E_i \wedge F_i)(-2i)) \cong E \wedge F \tag{5}$$

as deduced in Lemma A.42. The result follows from Corollary 2.1.3.

2.2 A Monoidal Structure on BGL

For a $\mathbf{P}^1$-spectrum E and an integer $i \geq 0$ $u_i\colon \Sigma^\infty_{\mathbf{P}^1} E_i(-i) \to E$ denotes the canonical map from Example A.26. Let BGL be the $\mathbf{P}^1$-spectrum defined in Definition 1.2.1. Recall that this involves the choice of a weak equivalence $i\colon \mathbb{Z} \times \mathrm{Gr} \to \mathcal{K}$ and a structure map $\epsilon\colon \mathcal{K} \wedge \mathbf{P}^1 \to \mathcal{K}$. Following Lemma 1.2.4 we may and will assume additionally that the pointed motivic space $\mathcal{K}$ is cofibrant. The aim of this section is to prove the following statement.

Theorem 2.2.1. *Assume that the pointed motivic space $\mathcal{K}$ is cofibrant. Consider the family of pairings $\mathcal{K}_i \wedge \mathcal{K}_j \xrightarrow{\mu_{ij}} \mathcal{K}_{i+j}$ in $H_\bullet(S)$ with $\mu_{ij} = \bar{\mu}$ from Remark 1.2.3. For $S = \mathrm{Spec}(\mathbb{Z})$ there is a unique morphism $\mu_{\mathrm{BGL}}\colon \mathrm{BGL} \wedge \mathrm{BGL} \to \mathrm{BGL}$ in the motivic stable homotopy category $\mathrm{SH}(S)$ such that for every i the diagram*

$$\begin{tikzcd}
\Sigma^\infty_{\mathbf{P}^1}\mathcal{K}_i(-i) \wedge \Sigma^\infty_{\mathbf{P}^1}\mathcal{K}_i(-i) \arrow[r, "\Sigma^\infty(\mu_{ii})"] \arrow[d, "u_i \wedge u_i"'] & \Sigma^\infty_{\mathbf{P}^1}\mathcal{K}_{2i}(-2i) \arrow[d, "u_{2i}"] \\
\mathrm{BGL} \wedge \mathrm{BGL} \arrow[r, "\mu_{\mathrm{BGL}}"] & \mathrm{BGL}
\end{tikzcd}$$

commutes. Let $e_{\mathrm{BGL}}\colon \mathbb{I} \to \mathrm{BGL}$ in $\mathrm{SH}(S)$ be adjoint to the unit $e_{\mathcal{K}}\colon S^{0,0} \to \mathcal{K}$. Then

$$(\mathrm{BGL}, \mu_{\mathrm{BGL}}, e_{\mathrm{BGL}})$$

is a commutative monoid in $\mathrm{SH}(S)$.

Proof. The morphism μ_{BGL} we are looking for is an element of the group $\mathrm{BGL}^{0,0}$ $(\mathrm{BGL} \wedge \mathrm{BGL})$. This group fits in the exact sequence

$$0 \to \varprojlim{}^1 \mathrm{BGL}^{4i-1,2i}(\mathcal{K}_i^{\wedge 2}) \to \mathrm{BGL}^{0,0}(\mathrm{BGL} \wedge \mathrm{BGL}) \to \varprojlim \mathrm{BGL}^{4i,2i}(\mathcal{K}_i^{\wedge 2}) \to 0$$

by Corollary 2.1.5. The family of elements $\{u_{2i} \circ \Sigma^\infty(\mu_{ii})\}$ is an element of the $\varprojlim$ group. The $\varprojlim^1$ group vanishes by Proposition 2.4.1 below, whence there exist a unique element μ_{BGL} whose image in the $\varprojlim$ group coincides with the element $\{u_{2i} \circ \Sigma^\infty(\mu_{ii})\}$. That morphism μ_{BGL} is the required one.

In fact, the identities $u_{2i} \circ \Sigma^\infty(\mu_{ii}) = \mu_{\mathrm{BGL}} \circ (u_i \wedge u_i)$ hold by the very construction of μ_{BGL}. The operation μ_{BGL} is associative because the group $\varprojlim^1 \mathrm{BGL}^{8i-1,4i}(\mathcal{K}_i \wedge \mathcal{K}_i \wedge \mathcal{K}_i)$ vanishes by Proposition 2.5.1. That μ_{BGL} is commutative follows from the vanishing of the group $\varprojlim^1 \mathrm{BGL}^{4i-1,2i}(\mathcal{K}_i \wedge \mathcal{K}_i)$ (see Proposition 2.4.1). The fact that e_{BGL} is a two-sided unit for the multiplication μ_{BGL} follows by Proposition 1.6.1, which shows that the group $\varprojlim^1 \mathrm{BGL}^{2i-1,i}(\mathcal{K}_i)$ vanishes.

Definition 2.2.2. Let S be a Noetherian finite-dimensional scheme, with $f : S \to \mathrm{Spec}(\mathbb{Z})$ being the canonical morphism. Let $f^* : \mathrm{SH}(\mathbb{Z}) \to \mathrm{SH}(S)$ be the strict symmetric monoidal pull-back functor from Proposition A.47. Set

$$\begin{aligned} \mu^S_{\mathrm{BGL}} &:= f^*(\mathrm{BGL}) \wedge f^*(\mathrm{BGL}) \xrightarrow{\mathrm{can}} f^*(\mathrm{BGL} \wedge \mathrm{BGL}) \xrightarrow{f^*(\mu_{\mathrm{BGL}})} f^*(\mathrm{BGL}) \\ e^S_{\mathrm{BGL}} &:= S^0 \xrightarrow{can} f^*(S^0) \xrightarrow{f^*(e_{\mathrm{BGL}})} f^*(\mathrm{BGL}) \end{aligned}$$

and $\mathrm{BGL}_S = f^*(\mathrm{BGL})$. Then $(\mathrm{BGL}_S, \mu^S_{\mathrm{BGL}}, e^S_{\mathrm{BGL}})$ is a commutative monoid in $\mathrm{SH}(S)$. Note that BGL_S satisfies the conditions from Definition 1.2.1 in $\mathbf{MS}(S)$ by Remark 1.2.5.

We will sometimes refer to a monoid in $\mathrm{SH}(S)$ as a $\mathbf{P}^1$-*ring spectrum*.

Corollary 2.2.3. *The multiplicative structure on the functor* $\mathrm{BGL}_S^{*,*}$ *induced by the pairing* μ^S_{BGL} *and the unit* e^S_{BGL} *coincides with the naive product structure (4).*

Proof. Follows from Theorem 2.2.1.

Corollary 2.2.4. *The functor isomorphism* $[X, \mathcal{K}_0] \to [\Sigma^\infty_{\mathbf{P}^1}(X), \mathrm{BGL}_S]$ *respects the multiplicative structures on both sides. In particular, the isomorphism* $Ad : K_* \to \mathrm{BGL}_S^{-*,0}$ *of cohomology theories on* $\mathcal{S}m\mathcal{O}p/S$ *is an isomorphism of ring cohomology theories in the sense of [PS].*

Proof. Follows from Theorem 2.2.1.

Remark 2.2.5. Let S be a finite dimensional Noetherian scheme, with $f : S \to \mathrm{Spec}(\mathbb{Z})$ being the canonical morphism. Then by [V, Theorem 6.9] the $\mathbf{P}^1$-spectrum BGL_S over S as defined in Definition 2.2.2 represents homotopy invariant K-theory as introduced in [We]. The triple $(\mathrm{BGL}_S, \mu^S_{\mathrm{BGL}}, e^S_{\mathrm{BGL}})$ is a distinguished monoidal structure on the $\mathbf{P}^1$-spectrum BGL_S.

2.3 Preliminary Computations II

Let BGL be the $\mathbf{P}^1$-spectrum defined in Definition 1.2.1. We will identify in this section the functors $\mathrm{BGL}^{0,0}$ and $\mathrm{BGL}^{2i,i}$, $\mathrm{BGL}^{-1,0}$ and $\mathrm{BGL}^{2i-1,i}$ on the motivic unstable category $\mathrm{H}_{\bullet}(S)$ as in Sect. 1.5.

Lemma 2.3.1. *Let* $S = \mathrm{Spec}(\mathbb{Z})$. *For every integer* i *the map*

$$\mathrm{BGL}^{-1,0}(S) \otimes \mathrm{BGL}^{2i,i}(\mathcal{K}_i \wedge \mathcal{K}_i) \to \mathrm{BGL}^{2i-1,i}(\mathcal{K}_i \wedge \mathcal{K}_i)$$

induced by the naive product structure is an isomorphism. The same holds if we replace $\mathcal{K}_i \wedge \mathcal{K}_i$ *by* $\mathcal{K}_i \wedge \mathcal{K}_i \wedge \mathbf{P}^1$ *or by* $\mathcal{K}_i \wedge \mathcal{K}_i \wedge \mathbf{P}^1 \wedge \mathbf{P}^1$.

Proof. Since diagram (3) commutes, it suffices to consider the case $i = 0$. Furthermore we may replace the pointed motivic space $\mathcal{K}_i$ with $\mathbb{Z} \times \mathrm{Gr}$ since the map $i : \mathbb{Z} \times \mathrm{Gr} \to \mathcal{K}_i = \mathcal{K}$ is a motivic weak equivalence. The functor isomorphism $\mathbb{K}_* \to \mathrm{BGL}^{-*,0}$ is an isomorphism of ring cohomology theories. Thus it remains to check that the map

$$K_1(S) \otimes K_0((\mathbb{Z} \times \mathrm{Gr}) \wedge (\mathbb{Z} \times \mathrm{Gr})) \to K_1((\mathbb{Z} \times \mathrm{Gr}) \wedge (\mathbb{Z} \times \mathrm{Gr}))$$

is an isomorphism. This can be checked by arguing as in the proof of Lemma 1.5.1 and using Lemmas B.8 and B.5. The cases of $\mathcal{K}_i \wedge \mathcal{K}_i \wedge \mathbf{P}^1$ and $\mathcal{K}_i \wedge \mathcal{K}_i \wedge \mathbf{P}^1 \wedge \mathbf{P}^1$ are proved by the same arguments.

Lemma 2.3.2. *Suppose that* $S = \mathrm{Spec}(\mathbb{Z})$, *and let* $r : \mathrm{SH}(S) \to \mathrm{SH}_{\mathbb{C}\mathbf{P}^1}$ *be the topological realization functor. Then for every integer* i *the homomorphism* $\mathrm{BGL}^{2i,i}(\mathcal{K} \wedge \mathcal{K}) \to (r\mathrm{BGL})^{2i}\big(r(\mathcal{K} \wedge \mathcal{K})\big) \cong (r\mathrm{BGL})^{2i}(r\mathcal{K} \wedge r\mathcal{K})$ *is bijective.*

Proof. Since the $(2,1)$-periodicity isomorphism for BGL described in Remark 1.4.20 and the Bott periodicity isomorphism for rBGL are compatible by Example A.46, it suffices to consider the case $i = 0$. We may replace the pointed motivic space $\mathcal{K}$ with $\mathbb{Z} \times \mathrm{Gr}$ as in the proof of Lemma 1.5.4. It remains to check that the realization homomorphism

$$\begin{aligned} \mathrm{BGL}^{0,0}((\mathbb{Z} \times \mathrm{Gr}) \wedge (\mathbb{Z} \times \mathrm{Gr})) &\to (r\mathrm{BGL})^0(r((\mathbb{Z} \times \mathrm{Gr}) \wedge (\mathbb{Z} \times \mathrm{Gr}))) \\ &= (r\mathrm{BGL})^0(r(\mathbb{Z} \times \mathrm{Gr}) \wedge r(\mathbb{Z} \times \mathrm{Gr})) \end{aligned}$$

is an isomorphism. By Example A.32 the $\mathbf{P}^1$-spectrum $\Sigma^\infty_{\mathbf{P}^1}((\mathbb{Z}\times\mathrm{Gr}) \wedge (\mathbb{Z}\times\mathrm{Gr}))$ is a retract of $\Sigma^\infty_{\mathbf{P}^1}(\mathbb{Z}\times\mathrm{Gr}\times\mathbb{Z}\times\mathrm{Gr})$ in $\mathrm{SH}(S)$, whence it suffices to consider the topological realization homomorphism for $\mathbb{Z}\times\mathrm{Gr}\times\mathbb{Z}\times\mathrm{Gr}$. Since the latter is an increasing union of the cellular S-schemes $[-n, n] \times \mathrm{Gr}(n, 2n) \times [-m, m] \times \mathrm{Gr}(m, 2m)$, the result follows with the help of Lemma 1.5.3 as in the proof of Lemma 1.5.4.

2.4 Vanishing of Certain Groups II

Consider the stable equivalence $\operatorname{hocolim} \Sigma^{\infty}_{\mathbf{P}^1}(\mathcal{K}\wedge\mathcal{K})(-2i) \cong \mathrm{BGL}\wedge\mathrm{BGL}$ displayed in (5) and the corresponding short exact sequence

$$0 \to \varprojlim{}^1 \mathrm{BGL}^{4i-1,2i}(\mathcal{K}^{\wedge 2}) \to \mathrm{BGL}^{0,0}(\mathrm{BGL}\wedge\mathrm{BGL}) \to \varprojlim \mathrm{BGL}^{4i,2i}(\mathcal{K}^{\wedge 2}) \to 0$$

from Corollary 2.1.5. We prove in this section the following result.

Proposition 2.4.1. *If* $S = \operatorname{Spec}(\mathbb{Z})$ *then* $\varprojlim{}^1 \mathrm{BGL}^{4i-1,2i}(\mathcal{K}\wedge\mathcal{K}) = 0$.

Proof. For a pointed motivic space A we abbreviate $A\wedge A$ as $A^{\wedge 2}$. The connecting homomorphism in the tower of groups forming the $\varprojlim{}^1$-term is the composite map

$$\begin{tikzcd}
\mathrm{BGL}^{4i-1,2i}(\mathcal{K}\wedge\mathcal{K}) & & \mathrm{BGL}^{4(i+1)-1,2(i+1)}(\mathcal{K}^{\wedge 2}) \arrow[dl, "(\epsilon\wedge\epsilon)^*"'] \\
& \mathrm{BGL}^{4(i+1)-1,2(i+1)}\big((\mathcal{K}\wedge\mathbf{P}^1)^{\wedge 2}\big) \arrow[ul, "(\Sigma\circ\Sigma)^{-1}\circ\mathrm{tw}"'] &
\end{tikzcd}$$

where Σ is the $\mathbf{P}^1$-suspension isomorphism, tw is induced by interchanging the two pointed motivic spaces in the middle of the four-fold smash product, and $\epsilon\colon \mathcal{K}\wedge\mathbf{P}^1 \to \mathcal{K}$ is the structure map of BGL. Set $A = \mathrm{BGL}^{-1,0}(S)$ and write B for BGL. Consider the diagram

$$\begin{tikzcd}
A\otimes \mathrm{B}^{4i+4,2i+2}(\mathcal{K}^{\wedge 2}) \arrow[r] \arrow[d, "\mathrm{id}\otimes(\epsilon\wedge\epsilon)^*"'] & \mathrm{B}^{4i+3,2i+2}(\mathcal{K}^{\wedge 2}) \arrow[d, "(\epsilon\wedge\epsilon)^*"] \\
A\otimes \mathrm{B}^{4i+4,2i+2}\big((\mathcal{K}\wedge\mathbf{P}^1)^{\wedge 2}\big) \arrow[r] \arrow[d, "\mathrm{id}\otimes(\Sigma\circ\Sigma)^{-1}\circ\mathrm{tw}"'] & \mathrm{B}^{4i+3,2i+2}\big((\mathcal{K}\wedge\mathbf{P}^1)^{\wedge 2}\big) \arrow[d, "(\Sigma\circ\Sigma)^{-1}\circ\mathrm{tw}"] \\
A\otimes \mathrm{B}^{4i,2i}(\mathcal{K}^{\wedge 2}) \arrow[r] & \mathrm{B}^{4i-1,2i}(\mathcal{K}^{\wedge 2})
\end{tikzcd}$$

where the horizontal arrows are induced by the naive product structure on the functor $\mathrm{BGL}^{*,*}$. Clearly it commutes. The horizontal arrows are isomorphisms by Lemma 2.3.1. It follows that $\varprojlim{}^1 \mathrm{BGL}^{4i-1,2i}(\mathcal{K}^{\wedge 2}) = \varprojlim{}^1\big(A\otimes \mathrm{BGL}^{4i,2i}(\mathcal{K}^{\wedge 2})\big)$ where in the last tower of groups the connecting maps are $\mathrm{id}\otimes\big((\Sigma\circ\Sigma)^{-1}_{\mathbf{P}^1}\circ\mathrm{tw})\circ(\epsilon\wedge\epsilon)^*\big)$. It remains to prove the following statement.

Claim. One has $\varprojlim{}^1(A\otimes \mathrm{BGL}^{4i,2i}(\mathcal{K}^{\wedge 2})) = 0$.

Since $A = \mathrm{BGL}^{-1,0}(S) = K_1(S) = \mathbb{Z}/2\mathbb{Z}$, there is an isomorphism $A\otimes \mathrm{BGL}^{4i,2i}(\mathcal{K}^{\wedge 2}) = \mathrm{BGL}^{4i,2i}(\mathcal{K}^{\wedge 2})/m\mathrm{BGL}^{4i,2i}(\mathcal{K}^{\wedge 2})$ with $m = 2$. The connecting maps in the tower are just the mod-m reduction of the maps $(\Sigma\circ\Sigma)^{-1}\circ\mathrm{tw}\circ(\epsilon\wedge\epsilon)^*$. Now a chain of isomorphisms completes the proof of the Claim.

$$
\begin{aligned}
\varprojlim{}^1 \mathrm{BGL}^{4i,2i}(\mathcal{K}^{\wedge 2})/m &\cong \varprojlim{}^1 (r\mathrm{BGL})^{4i}(r\mathcal{K}^{\wedge 2})/m \cong \varprojlim{}^1 \mathbb{B}\mathrm{U}^{4i}(r\mathcal{K}^{\wedge 2})/m \\
&\cong \varprojlim{}^1 \mathbb{B}\mathrm{U}^{4i}\big((\mathbb{Z}\times \mathrm{BU})\wedge(\mathbb{Z}\times \mathrm{BU})\big)/m \\
&\cong \varprojlim{}^1 \mathbb{B}\mathrm{U}^{4i}\big((\mathbb{Z}\times \mathrm{BU})\wedge(\mathbb{Z}\times \mathrm{BU});\mathbb{Z}/m\big) \\
&\cong K^1_{\mathrm{top}}(\mathbb{B}\mathrm{U}\wedge\mathbb{B}\mathrm{U};\mathbb{Z}/m) \cong K^1_{\mathrm{top}}(\mathbb{B}^0\mathrm{U}\wedge\mathbb{B}^0\mathrm{U};\mathbb{Z}/m) \\
&\cong \varprojlim{}^1 K^{4i}_{\mathrm{top}}\big(\mathrm{Gr}(b(i),2b(i))\wedge \mathrm{Gr}(b(i),2b(i));\mathbb{Z}/m\big) = 0
\end{aligned}
$$

The first isomorphism follows from Lemma 2.3.2. The second isomorphism is induced by the levelwise weak equivalence $\mathbb{B}\mathrm{U} \simeq r\mathrm{BGL}$ mentioned in Lemma 1.5.2. The third isomorphism is induced by the image of the weak equivalence $\mathcal{K}_i \simeq \mathbb{Z}\times\mathrm{Gr}$ under topological realization. The forth and fifth isomorphism hold since $\mathbb{B}\mathrm{U}^{4i+1}(\mathbb{Z}\times\mathrm{BU}) = 0$. The sixth isomorphism is induced by the stable equivalence $\mathbb{B}^0\mathrm{U}\simeq\mathbb{B}\mathrm{U}$ from Lemma 1.5.5, the seventh one is induced by the stable equivalence $\mathbb{B}^f\mathrm{U}\simeq\mathbb{B}^0\mathrm{U}$ from Lemma 1.5.6. The last one holds since all groups in the tower are finite.

2.5 Vanishing of Certain Groups III

Consider the stable equivalence

$$\mathrm{hocolim}\ \Sigma^\infty_{\mathbf{P}^1}(\mathcal{K}\wedge\mathcal{K}\wedge\mathcal{K})(-3i) \cong \mathrm{BGL}\wedge\mathrm{BGL}\wedge\mathrm{BGL}$$

from (5) and the induced short exact sequence

$$0 \to \varprojlim{}^1 \mathrm{BGL}^{8i-1,4i}(\mathcal{K}^{\wedge 3}) \to \mathrm{BGL}^{0,0}(\mathrm{BGL}^{\wedge 3}) \to \varprojlim \mathrm{BGL}^{8i,4i}(\mathcal{K}^{\wedge 3}) \to 0.$$

Proposition 2.5.1. *If* $S = \mathrm{Spec}(\mathbb{Z})$ *then* $\varprojlim{}^1 \mathrm{BGL}^{8i-1,4i}(\mathcal{K}\wedge\mathcal{K}\wedge\mathcal{K}) = 0$.

Proof. This is proved in the same way as Proposition 2.4.1.

2.6 BGL as an Oriented Commutative P^1-Ring Spectrum

Following Adams and Morel, we define an orientation of a commutative $\mathbf{P}^1$-ring spectrum. However we prefer to use a Thom class rather than a Chern class. Let $\mathbf{P}^\infty = \bigcup \mathbf{P}^n$ be the motivic space pointed by $\infty \in \mathbf{P}^1 \hookrightarrow \mathbf{P}^\infty$ and let $\mathcal{O}(-1)$ be the tautological line bundle over $\mathbf{P}^\infty$. It is also known as the Hopf bundle. If $V \to X$ is a vector bundle over $X \in \mathcal{S}m/S$, with zero section $z\colon X \hookrightarrow V$, let $\mathrm{Th}_X(V) = V/(V\smallsetminus z(X))$ be the Thom space of V, considered as a pointed motivic space over S. For example $\mathrm{Th}_X(\mathbf{A}^n_X) \simeq S^{2n,n}$. Define $\mathrm{Th}_{\mathbf{P}^\infty}\big(\mathcal{O}(-1)\big)$ as the obvious colimit of the Thom spaces $\mathrm{Th}_{\mathbf{P}^n}\big(\mathcal{O}(-1)\big)$.

Definition 2.6.1. Let E be a commutative $\mathbf{P}^1$-ring spectrum. An orientation of E is an element $th \in E^{2,1}(\mathrm{Th}_{\mathbf{P}^\infty}(\mathcal{O}(-1)))$ such that its restriction to the Thom space of the fibre over the distinguished point coincides with the element $\Sigma_{\mathbf{P}^1}(1) \in E^{2,1}(\mathrm{Th}(1)) = E^{2,1}(\mathbf{P}^1, \infty)$.

Remark 2.6.2. Let th be an orientation of E. Set $c := z^*(th) \in E^{2,1}(\mathbf{P}^\infty)$. Then [PY, Proposition 6.5.1] implies that $c|_{\mathbf{P}^1} = -\Sigma_{\mathbf{P}^1}(1)$. The class $th(\mathcal{O}(-1)) \in E^{2,1}(\mathrm{Th}_{\mathbf{P}^\infty}\, \mathcal{O}(-1))$ given by (2) below coincides with the element th by [PS, Theorem 3.5]. Thus another possible definition of an orientation of E is the following.

Definition 2.6.3. Let E be a commutative $\mathbf{P}^1$-ring spectrum. An orientation of E is an element $c \in E^{2,1}(\mathbf{P}^\infty)$ such that $c|_{\mathbf{P}^1} = -\Sigma_{\mathbf{P}^1}(1)$ (of course the element c should be regarded as the first Chern class of the Hopf bundle $\mathcal{O}(-1)$ on $\mathbf{P}^\infty$).

Remark 2.6.4. Let c be an orientation of the commutative $\mathbf{P}^1$-ring spectrum E. Consider the element $th(\mathcal{O}(-1)) \in E^{2,1}(\mathrm{Th}_{\mathbf{P}^\infty}\, \mathcal{O}(-1))$ given by (2) and set $th{=}th(\mathcal{O}(-1))$. It is straightforward to check that $th|_{\mathrm{Th}(1)} = \Sigma_{\mathbf{P}^1}(1)$. Thus th is an orientation of E. Clearly $c = z^*(th) \in E^{2,1}(\mathbf{P}^\infty)$, whence the two definitions of orientations of E are equivalent.

Example 2.6.5. Set $c^K = (-\beta) \cup \big([\mathcal{O}] - [\mathcal{O}(1)]\big) \in \mathrm{BGL}^{2,1}(\mathbf{P}^\infty)$. The relation (2) shows that c^K is an orientation of BGL. Consider $th(\mathcal{O}(-1)) \in \mathrm{BGL}^{2,1}(\mathrm{Th}_{\mathbf{P}^\infty}\, \mathcal{O}(-1))$ given by (2) below and set $th^K = th(\mathcal{O}(-1))$. The class th^K is the same orientation of BGL.

The orientation of BGL described in Example 2.6.5 has the following property. The map (6)

$$\mathrm{BGL}^{*,*} \to K_*$$

which takes β to -1 is an oriented morphism of oriented cohomology theories, provided that K_* is oriented via the Chern structure $L/X \mapsto [\mathcal{O}] - [L^{-1}] \in K_0(X)$.

2.7 BGL*,* *as an Oriented Ring Cohomology Theory*

An oriented $\mathbf{P}^1$-ring spectrum (E, c) defines an oriented cohomology theory on $\mathcal{S}m\mathcal{O}p$ in the sense of [PS, Definition 3.1] as follows. The restriction of the functor $E^{*,*}$ to the category $\mathcal{S}m\mathcal{O}p$ is a ring cohomology theory. By [PS, Theorem 3.35] it remains to construct a Chern structure on $E^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ in the sense of [PS, Definition 3.2]. The functor isomorphism $\mathrm{Hom}_{\mathrm{H}_\bullet(S)}(-, \mathbf{P}^\infty) \to \mathrm{Pic}(-)$ on the category $\mathcal{S}m/S$ provided by [MV, Theorem 4.3.8] takes the class of the canonical map $\mathbf{P}^\infty_+ \to \mathbf{P}^\infty$ to the class of the tautological line bundle $\mathcal{O}(-1)$ over $\mathbf{P}^\infty$. Now for a line bundle L over $X \in \mathcal{S}m/S$ set $c(L) = f_L^*(c) \in E^{2,1}(X)$, where the morphism $f_L\colon X_+ \to \mathbf{P}^\infty$ in $\mathrm{H}_\bullet(S)$ corresponds to the class $[L]$ of L in the group

$\mathrm{Pic}(X)$. Clearly, $c(\mathcal{O}(-1)) = c$. The assignment $L/X \mapsto c(L)$ is a Chern structure on $E^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ since $c|_{\mathbf{P}^1} = -\Sigma_{\mathbf{P}^1}(1) \in E^{2,1}(\mathbf{P}^1, \infty)$. With that Chern structure $E^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ is an oriented ring cohomology theory in the sense of [PS]. In particular, (BGL, c^K) defines an oriented ring cohomology theory on $\mathcal{S}m\mathcal{O}p$.

This Chern structure induces a theory of Thom classes $V/X \mapsto th(V) \in E^{2\mathrm{rank}(V),\mathrm{rank}(V)}\big(\mathrm{Th}_X(V)\big)$ on $E^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ in the sense of [PS, Definition 3.32] as follows. There is a unique theory of Chern classes $V \mapsto c_i(V) \in E^{2i,i}(X)$ such that for every line bundle L on X one has $c_1(L) = c(L)$. Now for a rank r vector bundle V over X consider the vector bundle $W := \mathbf{1} \oplus V$ and the associated projective vector bundle $\mathbf{P}(W)$ of lines in W. Set

$$\bar{t}h(V) = c_r(p^*(V) \otimes \mathcal{O}_{\mathbf{P}(W)}(1)) \in E^{2r,r}(\mathbf{P}(W)). \tag{1}$$

It follows from [PS, Corollary 3.18] that the support extension map

$$E^{2r,r}\big(\mathbf{P}(W)/(\mathbf{P}(W) \smallsetminus \mathbf{P}(\mathbf{1}))\big) \to E^{2r,r}\big(\mathbf{P}(W)\big)$$

is injective and $\bar{t}h(E) \in E^{2r,r}\big(\mathbf{P}(W)/(\mathbf{P}(W) \smallsetminus \mathbf{P}(\mathbf{1}))\big)$. Set

$$th(E) = j^*(\bar{t}h(E)) \in E^{2r,r}\big(\mathrm{Th}_X(V)\big), \tag{2}$$

where $j\colon \mathrm{Th}_X(V) \to \mathbf{P}(W)/(\mathbf{P}(W) \smallsetminus \mathbf{P}(\mathbf{1}))$ is the canonical motivic weak equivalence of pointed motivic spaces induced by the open embedding $V \hookrightarrow \mathbf{P}(W)$. The assignment $V/X \mapsto th(V)$ is a theory of Thom classes on $E^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ (see the proof of [PS, Theorem 3.35]). So the Thom classes are natural, multiplicative and satisfy the following Thom isomorphism property.

Theorem 2.7.1. *For a rank r vector bundle $p\colon V \to X$ on $X \in \mathcal{S}m/S$ the map*

$$- \cup th(V)\colon E^{*,*}(X) \to E^{*+2r,*+r}\big(\mathrm{Th}_X(V)\big)$$

is an isomorphism of two-sided $E^{*,*}(X)$-modules, where $- \cup th(V)$ is written for the composition map $\big(- \cup th(V)\big) \circ p^*$.

Proof. See [PS, Definition 3.32.(4)].

A Motivic Homotopy Theory

The aim of this section is to present details on the model structures we use to perform homotopical calculations. Our reference on model structures is [Ho]. For the convenience of the reader who is not familiar with model structures, we recall the basic features and purposes of the theory below, after discussing categorical prerequisites.

A.1 Categories of Motivic Spaces

Let S be a Noetherian separated scheme of finite Krull dimension (*base scheme* for short). The category of smooth quasi-projective S-schemes is denoted $\mathcal{S}m/S$. A smooth morphism is always of finite type. In particular, $\mathcal{S}m/S$ is equivalent to a small category.

The category of compactly generated topological spaces is denoted **Top**, the category of simplicial sets is denoted **sSet**. The set of n-simplices in K is K_n.

Definition A.1. A motivic space over S is a functor $A\colon \mathcal{S}m/S^{\mathrm{op}} \to \mathbf{sSet}$. The category of motivic spaces over S is denoted $\mathbf{M}(S)$.

For $X \in \mathcal{S}m/S$ the motivic space sending $Y \in \mathcal{S}m/S$ to the discrete simplicial set $\mathrm{Hom}_{\mathcal{S}m/S}(Y, X)$ is denoted X as well. More generally, any scheme X over S defines a motivic space X over S. Any simplicial set K defines a constant motivic space K. A pointed motivic space is a pair (A, a_0), where $a_0\colon S \to A$. Usually the basepoint will be omitted from the notation. The resulting category is denoted $\mathbf{M}_\bullet(S)$.

Definition A.2. A morphism $f\colon S \to S'$ of base schemes defines the functor

$$f_*\colon \mathrm{M}_\bullet(S) \to \mathbf{M}_\bullet(S')$$

sending A to $(Y \to S') \mapsto A(S \times_{S'} Y)$. Left Kan extension produces a left adjoint $f^*\colon \mathbf{M}_\bullet(S') \to \mathbf{M}_\bullet(S)$ of f_*.

If A is a motivic space, let A_+ denote the pointed motivic space $(A \coprod S, i)$, where $i\colon S \to A \coprod S$ is the canonical inclusion. The category $\mathbf{M}_\bullet(S)$ is closed symmetric monoidal, with smash product $A \wedge B$ defined by the sectionwise smash product

$$\big(A \wedge B\big)(X) := A(X) \wedge B(X) \tag{1}$$

and with internal hom $\underline{\mathrm{Hom}}_{\mathrm{M}_\bullet(\mathrm{S})}(A, B)$ defined by

$$\underline{\mathrm{Hom}}_{\mathrm{M}_\bullet(\mathrm{S})}(A, B)(x\colon X \to S)_n := \mathrm{Hom}_{\mathbf{M}_\bullet(S)}(A \wedge \Delta^n_+, x_* x^* B). \tag{2}$$

In particular, $\mathrm{M}_\bullet(S)$ is also enriched over the category of pointed simplicial sets, with enrichment $\mathbf{sSet}_\bullet(A, B) := \underline{\mathrm{Hom}}_{\mathrm{M}_\bullet(\mathrm{S})}(A, B)(S)$. The *mapping cylinder* of a map $f\colon A \to B$ is the pushout of the diagram

$$\begin{tikzcd} A \wedge \partial\Delta^1_+ \arrow[r, "\cong"] \arrow[d, "\rho"] & A \coprod A \arrow[r, "\mathrm{id}_A \coprod f"] & A \coprod B \arrow[d] \\ A \wedge \Delta^1_+ \arrow[rr] & & \mathrm{Cyl}(f) \end{tikzcd} \tag{3}$$

The composition of the canonical maps $A \hookrightarrow \mathrm{Cyl}(f) \to B$ is f.

The *pushout product* of two maps $f: A \to C$ and $g: B \to D$ of motivic spaces over S is the map $f \Box g: A \wedge D \cup_{A \wedge B} C \wedge B \to C \wedge D$ induced by the commutative diagram

$$\begin{tikzcd} A \wedge B \arrow[r] \arrow[d] & A \wedge D \arrow[d] \\ C \wedge B \arrow[r] & C \wedge D. \end{tikzcd} \tag{4}$$

The functor $f^*: \mathrm{M}_\bullet(S') \to \mathrm{M}_\bullet(S)$ induced by $f: S \to S'$ is strict symmetric monoidal in the sense that there are isomorphisms

$$f^*(A) \wedge f^*(B) \xrightarrow{\cong} f^*(A \wedge B) \quad \text{and} \quad f^*(S'_+) \xrightarrow{\cong} S_+ \tag{5}$$

which are natural in A and B. The isomorphisms (5) are induced by the corresponding isomorphisms for the strict symmetric monoidal pullback functor sending $X \in \mathcal{S}m_{S'}$ to $S \times_{S'} X \in \mathcal{S}m/S$. This ends the categorical considerations.

A.2 Model Categories

The basic purpose of a model structure is to give a framework for the construction of a homotopy category. Suppose $w\mathcal{C}$ is a class of morphisms in a category $\mathcal{C}$ one wants to make invertible. Call them weak equivalences. One can define the homotopy "category" of the pair $(\mathcal{C}, w\mathcal{C})$ to be the target of the universal "functor" $\Gamma: \mathcal{C} \to \mathrm{Ho}(\mathcal{C}, w\mathcal{C})$ such that every weak equivalence is mapped to an isomorphism. In general, this homotopy "category" may not be a category: it has hom-classes, but not necessarily hom-sets. If one requires the existence of two auxiliary classes of morphisms $f\mathcal{C}$ (the fibrations) and $c\mathcal{C}$ (the cofibrations), together with certain compatibility axioms, one does get a homotopy category $\mathrm{Ho}(\mathcal{C}, w\mathcal{C})$ and an explicit description of the hom-sets in it.

Theorem A.3 (Quillen). *Let $(w\mathcal{C}, f\mathcal{C}, c\mathcal{C})$ be a model structure on a bicomplete category $\mathcal{C}$. Then the universal functor to the homotopy category $\Gamma: \mathcal{C} \to \mathrm{Ho}(\mathcal{C}, w\mathcal{C})$ exists and is the identity on objects. The set of morphisms in $\mathrm{Ho}(\mathcal{C}, w\mathcal{C})$ from ΓA to ΓB is the set of morphisms in $\mathcal{C}$ from A to B modulo a homotopy equivalence relation, provided that $\emptyset \to A$ is a cofibration and $B \to *$ is a fibration.*

Here $\emptyset$ is the initial object and $*$ is the terminal object in $\mathcal{C}$. An object A resp. B as in Theorem A.3 is called *cofibrant* resp. *fibrant*. Every object ΓA in the homotopy category is isomorphic to an object ΓC, where C is both fibrant and cofibrant. A (co)fibration which is also a weak equivalence is usually called a *trivial* or *acyclic* (co)fibration.

To describe the standard way to construct model structures on a bicomplete category, one needs a definition.

Definition A.4. Let $f: A \to B$ and $g: C \to D$ be morphisms in $\mathcal{C}$. If every commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r] & D \end{tikzcd}$$

admits a morphism $h: B \to C$ such that the resulting diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, "h"] & D \end{tikzcd}$$

commutes (a *lift* for short), then f has the left lifting property with respect to g, and g has the right lifting property with respect to f.

Here is the standard way of constructing a model structure on a given bicomplete category. Choose the class of weak equivalences such that it contains all identities, is closed under retracts and satisfies the two-out-of-three axiom. Pick a set I (the *generating cofibrations*) and define a cofibration to be a morphism which is a retract of a transfinite composition of cobase changes of morphisms in I. Pick a set J (the *generating acyclic cofibrations*) of weak equivalences which are also cofibrations and define the fibrations to be those morphisms which have the right lifting property with respect to every morphism in J. Some technical conditions have to be fulfilled in order to conclude that this indeed is a model structure, which is then called *cofibrantly generated*. See [Ho, Theorem 2.1.19].

Example A.5. In **Top**, let the weak equivalences be the weak homotopy equivalences, and set

$$I = \{\partial D^n \hookrightarrow D^n\}_{n\geq 0} \quad J = \{D^n \times \{0\} \hookrightarrow D^n \times I\}_{n\geq 0}.$$

Then the fibrations are precisely the Serre fibrations, and the cofibrations are retracts of generalized cell complexes ("generalized" refers to the fact that cells do not have to be attached in order of dimension). In **sSet**, let the weak equivalences be those maps which map to (weak) homotopy equivalences under geometric realization. Set

$$I = \{\partial\Delta^n \hookrightarrow \Delta^n\}_{n\geq 0} \quad J = \{\Lambda^n_j \hookrightarrow \Delta^n\}_{n\geq 1, 0\leq j\leq n}$$

where Λ^n_j is the sub-simplicial set of $\partial\Delta^n$ obtained by removing the j-th face. Then the fibrations are precisely the Kan fibrations, and the cofibrations are the inclusions.

Example A.6. For the purpose of this paper, model structures on presheaf categories $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{sSet})$ with values in simplicial sets are relevant. There is a canonical one, due to Quillen, which is usually referred to as the *projective* model structure. It has as weak equivalences those morphisms $f\colon A \to B$ such that $f(c)\colon A(c) \to B(c)$ is a weak equivalence for every $c \in \mathrm{Ob}\mathcal{C}$ (the *objectwise* or *sectionwise* weak equivalences). Set

$$\begin{aligned} I &= \{\mathrm{Hom}_{\mathcal{S}}(-,c) \times (\partial\Delta^n \hookrightarrow \Delta^n)\}_{n\geq 0,\, c\in\mathrm{Ob}\mathcal{C}} \\ J &= \{\mathrm{Hom}_{\mathcal{S}}(-,c) \times (\Lambda^n_j \hookrightarrow \Delta^n)\}_{n\geq 1,\, 0\leq j\leq n,\, c\in\mathrm{Ob}\mathcal{C}} \end{aligned}$$

so that by adjointness, the fibrations are precisely the sectionwise Kan fibrations. There is another cofibrantly generated model structure with the same weak equivalences, due to Heller [He], such that the cofibrations are precisely the injective morphisms (whence the name *injective* model structure). The description of J involves the cardinality of the set of morphisms in $\mathcal{C}$ and is not explicit. Neither is the characterization of the fibrations.

The morphisms of model categories are called Quillen functors. A Quillen functor of model categories $\mathcal{M} \to \mathcal{N}$ is an adjoint pair $(F,G)\colon \mathcal{M} \to \mathcal{N}$ such that F preserves cofibrations and G preserves fibrations. This condition ensures that (F,G) induces an adjoint pair on homotopy categories $(\mathcal{L}F, \mathcal{R}G)$, where $\mathcal{L}F$ is the total left derived functor of F. A Quillen functor is a Quillen equivalence if the total left derived is an equivalence. For example, geometric realization is a strict symmetric monoidal left Quillen equivalence $|-|\colon \mathbf{sSet} \to \mathbf{Top}$, and similarly in the pointed setting.

If a model category has a closed symmetric monoidal structure as well, one has the following statement, proven in [Ho, Theorem 4.3.2].

Theorem A.7 (Quillen). *Let $\mathcal{C}$ be a bicomplete category with a model structure. Suppose that $(\mathcal{C}, \otimes, \mathbb{I})$ is closed symmetric monoidal. Suppose further that these structures are compatible in the following sense:*

- *The pushout product of two cofibrations is a cofibration.*
- *The pushout product of an acyclic cofibration with a cofibration is an acyclic cofibration.*

Then $A \otimes -$ is a left Quillen functor for all cofibrant objects $A \in \mathcal{C}$. In particular, there is an induced (total derived) closed symmetric monoidal structure on $\mathrm{Ho}(\mathcal{C}, w\mathcal{C})$.

One abbreviates the hypotheses of Theorem A.7 by saying that $\mathcal{C}$ is a symmetric monoidal model category. This ends our introduction to model category theory.

A.3 Model Structures for Motivic Spaces

To equip $\mathbf{M}_{\bullet}(S)$ with a model structure suitable for the various requirements (compatibility with base change, taking complex points, finiteness conditions, having the correct motivic homotopy category), we construct a preliminary model structure first. Start with the following construction, which is a special case of the considerations in [I]. Choose any $X \in Sm_S$ and a finite set

$$\{i^j : Z_j \rightarrowtail X\}_{j=1}^m$$

of closed embeddings in $\mathcal{S}m/S$. Regarding i^j as a monomorphism of motivic spaces, one may form the categorical union (not the categorical coproduct!) $i : \cup_{j=1}^m Z_j \hookrightarrow X$. That is, $\cup_{j=1}^m Z_j$ is the coequalizer in the category of motivic spaces of the diagram

$$\coprod_{j,j'} Z_j \times_X Z_{j'} \rightrightarrows \coprod_{j=1}^m Z_j \tag{1}$$

Call the resulting monomorphism $i : \cup_{j=1}^m Z_j \hookrightarrow X$ *acceptable*. The closed embedding $\emptyset \rightarrowtail X$ is acceptable as well. Consider the set Ace of acceptable monomorphisms. Let I_S^c be the set of pushout product maps

$$\{i_+ \square (\partial\Delta^n \hookrightarrow \Delta^n)_+\}_{i \in \mathrm{Ace},\, n \geq 0} \tag{2}$$

and let J_S^c be the set of pushout product maps

$$\{i_+ \square (\Lambda_j^n \hookrightarrow \Delta^n)_+\}_{i \in \mathrm{Ace},\, n \geq 1,\, 0 \leq j \leq n} \tag{3}$$

defined via diagram (4).

Definition A.8. A map $f : A \to B$ in $\mathbf{M}_\bullet(S)$ is a *schemewise weak equivalence* if $f : A(X) \to B(X)$ is a weak equivalence of simplicial sets for all $X \in \mathcal{S}m/S$. It is a *closed schemewise fibration* if $f : A \to B$ has the right lifting property with respect to J_S^c. It is a *closed cofibration* if it has the left lifting property with respect to all acyclic closed schemewise fibrations (closed schemewise fibrations which are also schemewise weak equivalences).

Theorem A.9. *The classes described in Definition A.8 are a closed symmetric monoidal model structure on* $\mathbf{M}_\bullet(S)$*, denoted* $\mathbf{M}_\bullet^{cs}(S)$*. A morphism* $f : S \to T$ *of base schemes induces a strict symmetric monoidal left Quillen functor* $f^* : \mathbf{M}_\bullet^{cs}(T) \to \mathbf{M}_\bullet^{cs}(S)$.

Proof. The existence of the model structure follows from [I]. The pushout product axiom follows, because the pushout product of two acceptable monomorphism is again acceptable. To conclude the last statement, it suffices to check that f^* maps any map in I_T^c resp. J_T^c to a closed cofibration resp. schemewise weak equivalence. In fact, if $i : \cup_{j=1}^m Z_j \hookrightarrow X$ is an acceptable monomorphism over T, then $f^*(i)$ is the acceptable monomorphism obtained from the closed embeddings

$$\{T \times_S Z_j \rightarrowtail T \times_S X.\}$$

Because f^* is strict symmetric monoidal and a left adjoint, it preserves the pushout product. Hence f^* even maps the set I_T^c to the set I_S^c, and likewise for J_T^c. The result follows.

The resulting homotopy category is equivalent – via the identity functor – to the usual homotopy category of the diagram category $\mathbf{M}_\bullet(S)$ (obtained via the projective model structure from Example A.6), since the weak equivalences are just the objectwise ones. The model structure $\mathbf{M}_\bullet^{cs}(S)$ has the following advantage over the projective model structure.

Lemma A.10. *Let $i\colon Z \rightarrowtail X$ be a closed embedding in $\mathcal{S}m/S$. Then the induced map $i_+\colon Z_+ \to X_+$ is a closed cofibration in $\mathbf{M}_\bullet(S)$. In particular, for any closed S-point $x_0\colon S \rightarrowtail X$ in a smooth S-scheme, the pointed motivic space (X, x_0) is closed cofibrant.*

Proof. The first statement follows, because $i_+ = i_+\Box(\partial\Delta_+^0 \hookrightarrow \Delta_+^0)$ is contained in the set of generating closed cofibrations. The second statement follows, because cofibrations are closed under cobase change.

Not all pointed motivic spaces are closed cofibrant. Let $(-)^{cs} \to \mathrm{Id}_{\mathbf{M}_\bullet(S)}$ denote a cofibrant replacement functor, for example the one obtained from applying the small object argument to I_S^c. That is, the map $A^{cs} \to A$ is a natural closed schemewise fibration and a schemewise weak equivalence, and A^{cs} is closed cofibrant. Dually, let $\mathrm{Id}_{\mathbf{M}_\bullet(S)} \to (-)^{cf}$ denote the fibrant replacement functor obtained by applying the small object argument to J_S^c. The closed schemewise fibrations may be characterized explicitly.

Lemma A.11. *A map $f\colon A \to B$ is a closed schemewise fibration if and only if the following two conditions hold:*

1. *$f(X)\colon A(X) \to B(X)$ is a Kan fibration for every $X \in \mathcal{S}m/S$.*
2. *For every finite set $\{Z_j \rightarrowtail X\}_{j=1}^m$ of closed embeddings in $\mathcal{S}m/S$, the induced map*

$$A(X) \to B(X) \times_{\mathbf{sSet}_\bullet(\cup_{j=1}^m Z_j, B)} \mathbf{sSet}_\bullet(\cup_{j=1}^m Z_j, A)$$

is a Kan fibration.

Proof. Follows by adjointness from the definition. Note that condition 1 is a special case of condition 2 by taking the empty family.

To obtain a motivic model structure, one localizes $\mathbf{M}_\bullet^{cs}(S)$ as follows. Recall that an elementary distinguished square (or simply *Nisnevich square*) is a pullback diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

in $\mathcal{S}m/S$, where j is an open embedding and p is an étale morphism inducing an isomorphism $Y \smallsetminus V \cong X \smallsetminus U$ of reduced closed subschemes. Say that a pointed motivic space C is *closed motivic fibrant* if it is closed schemewise fibrant, the map

$$C\big(X \times_S \mathbb{A}^1_S \xrightarrow{\mathrm{pr}} X\big)$$

is a weak equivalence of simplicial sets for every $X \in \mathcal{S}m/S$, the square

$$\begin{array}{ccc} C(V) & \longleftarrow & C(Y) \\ \uparrow & & \uparrow \\ C(U) & \longleftarrow & C(X) \end{array}$$

is a homotopy pullback square of simplicial sets for every Nisnevich square in $\mathcal{S}m/S$ and $C(\emptyset)$ is contractible.

Example A.12. Let $X \mapsto K^W(X)$ be the pointed motivic space sending $X \in \mathcal{S}m/S$ to the loop space of the first term of the Waldhausen K-theory spectrum $W(X)$ associated to the category of big vector bundles over X [FS], with isomorphisms as weak equivalences [W]. That is,

$$K^W(X) = \Omega_s W_1(X) = \Omega_s \operatorname{Sing} |w\mathcal{S}_\bullet(\mathrm{Vect}^{\mathrm{big}}(X), \mathrm{iso})|$$

By [TT, Theorem 1.11.7, Proposition 3.10] the space $K^W(X)$ has the same homotopy type as the zeroth space (see [TT, 1.5.3]) of the Thomason–Trobaugh K-theory spectrum $K^{naive}(X)$ of X as it is defined in [TT, Definition 3.2].

Since in our case S is regular, then so is X and thus X has an ample family of line bundles by [TT, Example 2.1.2]. It follows that the zeroth space of the Thomason–Trobaugh K-theory spectrum $K^{\mathrm{naive}}(X)$ has the same homotopy type as the zeroth space (see [TT, 1.5.3]) of the Thomason–Trobaugh K-theory spectrum $K(X\ on\ X)$ of X [TT, Theorem 1.11.7, Corollary 3.9] as it is defined in [TT, Definition 3.1].

Thus for a regular S and $X \in \mathcal{S}m/S$ the following results hold. The projection induces a weak equivalence $K^W(X) \to K^W(X \times_S \mathbf{A}^1_S)$ [TT, Proposition 6.8]. By [TT, Theorem 10.8] the square

$$\begin{array}{ccc} K^W(X) & \longrightarrow & K^W(U) \\ {\scriptstyle K^W(p)}\downarrow & & \downarrow \\ K^W(Y) & \longrightarrow & K^W(V) \end{array}$$

associated to a Nisnevich square in $\mathcal{S}m/S$ is a homotopy pullback square. Hence K^W is fibrant in the projective motivic model structure on $\mathbf{M}_\bullet(S)$. However, if $i\colon Z \hookrightarrow X$ is a closed embedding in $\mathcal{S}m/S$, the map

$$K^W(i)\colon K^W(X) \to K^W(Z)$$

is not necessarily a Kan fibration. In particular, K^W is not closed schemewise fibrant. Choose a closed cofibration which is also a schemewise weak equivalence $K^W \to \mathbb{K}^W$ such that $\mathbb{K}^W$ is closed schemewise fibrant. It follows immediately that $\mathbb{K}^W$ is closed motivic fibrant, and that $\mathbb{K}^W(X)$ has the homotopy type of the zero term of the Waldhausen K-theory spectrum of X.

Definition A.13. A map $f\colon A \to B$ is a *motivic weak equivalence* if the map

$$\mathbf{sSet}_\bullet(f^{cs}, C)\colon \mathbf{sSet}_\bullet(B^{cs}, C) \to \mathbf{sSet}_\bullet(A^{cs}, C)$$

is a weak equivalence of simplicial sets for every closed motivic fibrant C. It is a *closed motivic fibration* if it has the right lifting property with respect to all acyclic closed cofibrations (closed cofibrations which are also motivic equivalences).

Example A.14. Suppose that $f\colon A \to B$ is a map in $\mathbf{M}_\bullet(S)$ inducing weak equivalences $x^*f\colon x^*A \to x^*B$ of simplicial sets on all Nisnevich stalks $x^*\colon \mathbf{M}_\bullet(S) \to \mathbf{sSet}$. Then f is a motivic weak equivalence. If $f\colon A \to B$ is an $\mathbf{A}^1$-homotopy equivalence (for example, the projection of a vector bundle), then it is a motivic weak equivalence.

Example A.15. The canonical covering of $\mathbf{P}^1$ shows that it is motivic weakly equivalent as a pointed motivic space to the suspension $S^1 \wedge (\mathbf{A}^1 - \{0\}, 1)$, where $S^1 = \Delta^1/\partial\Delta^1$. Set $S^{1,0} := S^1$ and $S^{1,1} := (\mathbf{A}^1 - \{0\}, 1)$, and define

$$S^{p,q} := \big(S^{1,0}\big)^{\wedge p-q} \wedge \big(S^{1,1}\big)^{\wedge q} \quad \text{for} \quad p \geq q \geq 0$$

To generalize the example of $\mathbf{P}^1$, one can show that if $\mathbf{P}^{n-1} \hookrightarrow \mathbf{P}^n$ is a linear embedding, then $\mathbf{P}^n/\mathbf{P}^{n-1}$ is motivic weakly equivalent to $S^{2n,n}$.

To prove that the classes from Definition A.13 are part of a model structure, it is helpful to characterize the closed motivic fibrant objects via a lifting property. Let J_S^{cm} be the union of the set J_S^c from (3) and the set J_S^m of pushout products of maps $(\partial\Delta^n \hookrightarrow \Delta^n)_+$ with maps of the form

$$X_+ \xrightarrow{\ \mathrm{zero}_+\ } (\mathbf{A}^1_S \times_S X)_+ \tag{4}$$

$$U_+ \cup_{V_+} \mathrm{Cyl}(h_+) \longrightarrow \mathrm{Cyl}\big(U_+ \cup_{V_+} \mathrm{Cyl}(h_+) \to X_+\big)$$

$$* \longrightarrow \emptyset_+$$

where h is the open embedding appearing on top of a Nisnevich square

$$\begin{tikzcd} V \arrow[r, "h"] \arrow[d] & Y \arrow[d, "p"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

in $\mathcal{S}m/S$.

Lemma A.16. *A pointed motivic space C is closed motivic fibrant if and only if the map $C \to *$ has the right lifting property with respect to the set J_S^{cm}.*

Proof. This follows from adjointness, the Yoneda lemma and the construction of J_S^{cm}.

Theorem A.17. *The classes of motivic weak equivalences, closed motivic fibrations and closed cofibrations constitute a symmetric monoidal model structure on $\mathbf{M}_\bullet(S)$, denoted $\mathbf{M}_\bullet^{\mathrm{cm}}(S)$. The resulting homotopy category is denoted $\mathrm{H}_\bullet^{\mathrm{cm}}(S)$ and called the pointed motivic unstable homotopy category of S. A morphism $f\colon S \to S'$ of base schemes induces a strict symmetric monoidal left Quillen functor $f^*\colon M_\bullet^{\mathrm{cm}}(S') \to M_\bullet^{\mathrm{cm}}(S)$.*

Proof. The existence of the model structure follows by standard Bousfield localization techniques. Here are some details. The problem is that J_S^{cm} might be to small in order to characterize all closed motivic fibrations. Let κ be a regular cardinal strictly bigger than the cardinality of the set of morphisms in $\mathcal{S}m/S$. A motivic space A is κ-bounded if the union

$$\coprod_{n\geq 0, X\in\mathcal{S}m/S} A(X)_n$$

has cardinality $\leq \kappa$. Let J_S^κ be a set of isomorphism classes of acyclic monomorphisms whose target is κ-bounded. One may show that given an acyclic monomorphism $j\colon A \hookrightarrow B$ and a κ-bounded subobject $C \subseteq B$, there exists a κ-bounded subobject $C' \subseteq B$ containing C such that $j^{-1}(C') \hookrightarrow C'$ is an acyclic monomorphism. Via Zorn's lemma, one then gets that a map $f \in \mathbf{M}_\bullet(S)$ has the right lifting property with respect to all acyclic monomorphisms if (and only if) it has the right lifting property with respect to the set J_S^κ. Such a map is in particular a closed motivic fibration. Any given map $f\colon A \to B$ can now be factored (via the small object argument) as an acyclic monomorphism $j\colon A \hookrightarrow C$ followed by a closed motivic fibration. Factoring j as a closed cofibration followed by an acyclic closed schemewise fibration in the model structure of Theorem A.9 implies the existence of the model structure.

To prove that the model structure is symmetric monoidal, it suffices – by the corresponding statement, Theorem A.9 for $\mathbf{M}_\bullet^{cs}(S)$ – to check that the pushout product of a generating closed cofibration and an acyclic closed cofibration is again a motivic equivalence. However, from the fact that the injective motivic model structure is symmetric monoidal, one knows that motivic equivalences are closed under smashing with arbitrary motivic spaces [DRØ, Lemma 2.20]. The first sentence is now proven.

Concerning the third sentence, Theorem A.9 already implies that f^* preserves closed cofibrations. To prove that f^* is a left Quillen functor, it suffices (by Dugger's lemma [D, Corollary A2]) to check that it maps the set $J_{S'}^{\mathrm{cm}}$ to motivic weak equivalences in $\mathbf{M}_\bullet(S)$. One may calculate that $f^*(J_{S'}^{\mathrm{cm}}) = J_S^{\mathrm{cm}}$, whence the statement.

The closed motivic model structure is cofibrantly generated. As remarked in the proof of Theorem A.17, the set J_S^{cm} is perhaps not big enough to yield a full set of generating trivial cofibrations. Still the following Lemma, whose analog for the projective motivic model structure is [DRØ, Corollary 2.16], is valid.

Lemma A.18. *Motivic equivalences and closed motivic fibrations with closed motivic fibrant codomain are closed under filtered colimits.*

Proof. By localization theory [Hi, 3.3.16] and Lemma A.16, a map with closed motivic fibrant codomain is a closed motivic fibration if and only if it has the right lifting property with respect to J_S^{cm}. The domains and codomains of the maps in J_S^{cm} are finitely presentable, which implies the statement about closed motivic fibrations with closed motivic fibrant codomain. The statement about motivic equivalences follows, because also the domains of the generating closed cofibrations in I_S^c are finitely presentable. See [DRØ2, Lemma 3.5] for details.

Let $\mathcal{M}_\bullet(S)$ be the category of simplicial objects in the category of pointed Nisnevich sheaves on $\mathcal{S}m/S$. The functor mapping a (pointed) presheaf to its associated (pointed) Nisnevich sheaf determines by degreewise application a functor $a_{Nis}\colon \mathbf{M}_\bullet(S) \to \mathcal{M}_\bullet(S)$. Let $\mathcal{M}_\bullet(S) \xrightarrow{i} \mathbf{M}_\bullet(S)$ be the inclusion functor, the right adjoint of a_{Nis}.

Theorem A.19. *The pair (a_{Nis}, i) is a Quillen equivalence to the Morel–Voevodsky model structure. The functor a_{Nis} is strict symmetric monoidal. In particular, the total left derived functor of a_{Nis} is a strict symmetric monoidal equivalence*

$$\mathrm{H}_\bullet^{\mathrm{cm}}(S) \to \mathrm{H}_\bullet(S)$$

to the unstable pointed $\mathbf{A}^1$-homotopy category from [MV].

Proof. Recall that the cofibrations in the Morel–Voevodsky model structure are precisely the monomorphisms. Since every closed cofibration is a monomorphism and Nisnevich sheafification preserves these, a_{Nis} preserves cofibrations. The unit $A \to i\big(a_{\mathrm{Nis}}(A)\big)$ of the adjunction is an isomorphism on all Nisnevich stalks, hence a motivic weak equivalence by Example A.14 for every motivic space A. In particular, a_{Nis} maps schemewise weak equivalences as well as the maps in J_S^m described in (4) to weak equivalences. Let $\mathrm{Id}_{M_\bullet(S)} \to (-)^{\mathrm{fib}}$ be the fibrant replacement functor in $\mathbf{M}_\bullet^{\mathrm{cm}}(S)$ obtained from the small object argument applied to J_S^{cm}. Hence if f is a motivic weak equivalence, then f^{fib} is a schemewise weak equivalence. One concludes that a_{Nis} preserves all weak equivalences, thus is a left Quillen functor. Since the unit $A \to i\big(a_{\mathrm{Nis}}(A)\big)$ is a motivic weak equivalence for every A, the functor a_{Nis} is a Quillen equivalence.

Note A.20. Note that a map f in $\mathbf{M}_\bullet(S)$ is a motivic weak equivalence if and only if $a_{\mathrm{Nis}}(f)$ is a weak equivalence in the Morel–Voevodsky model structure on simplicial sheaves. Conversely, a map of simplicial sheaves is a weak equivalence if and only if it is a motivic weak equivalence when considered as a map of motivic spaces.

Remark A.21. Starting with the injective model structure on simplicial presheaves mentioned in Example A.6, there is a model structure $\mathbf{M}_\bullet^{\mathrm{im}}(S)$ on the category of pointed motivic spaces with motivic weak equivalences as weak equivalences and monomorphisms as cofibrations. It has the advantage that every object is cofibrant, but the disadvantage that it does not behave well under base change [MV, Example 3.1.22] or geometric realization (to be defined below). The identity functor is a left Quillen equivalence $\mathrm{Id}\colon \mathbf{M}_\bullet^{\mathrm{cm}}(S) \to \mathbf{M}_\bullet^{\mathrm{im}}(S)$, since the homotopy categories coincide.

Further, let $Spc_\bullet(S)$ be the category of pointed Nisnevich sheaves on $\mathcal{S}m/S$. Recall the cosimplicial smooth scheme over S whose value at n is

$$\Delta_S^n = \mathrm{Spec}\big(\mathcal{O}_S[X_0,\dots,X_n]/(\sum_{i=0}^{n} X_i = 1)\big) \tag{5}$$

The functor $Spc_\bullet(S) \to \mathcal{M}_\bullet(S)$ sending A to the simplicial object $\mathrm{Sing}_S(A)_n = A(- \times \Delta_S^n)$ has a left adjoint $|-|_S\colon \mathcal{M}_\bullet(S)$. It maps B to the coend

$$|B|_S = \int_{n\in\Delta} B_n \times \Delta_S^n$$

in the category of pointed Nisnevich sheaves. The following statement is proved in [MV].

Theorem A.22 (Morel–Voevodsy). *There is a model structure on the category $Spc_\bullet(S)$ such that the pair $(|-|_S, \mathrm{Sing}_S)$ is a Quillen equivalence to the Morel–Voevodsky model structure. The functor $|-|_S$ is strict symmetric monoidal. In particular, the total left derived functor of $|-|_S$ is a strict symmetric monoidal equivalence from Voevodsky's pointed homotopy category to the unstable pointed $\mathbf{A}^1$-homotopy category.*

A.4 Topological Realization

In the case where the base scheme is the complex numbers, there is a topological realization functor $\mathbf{R}_{\mathbb{C}}\colon \mathbf{M}_\bullet^{\mathrm{cm}}(\mathbb{C}) \to \mathbf{Top}_\bullet$ which is a strict symmetric monoidal left Quillen functor. It is defined as follows. If $X \in \mathcal{S}m_{\mathbb{C}}$, the set $X(\mathbb{C})$ of complex points is a topological space when equipped with the analytic topology. Call this topological space X^{an}. It is a smooth manifold, and in particular a compactly generated topological space. One may view $X \mapsto X^{\mathrm{an}}$ as a functor $\mathcal{S}m_{\mathbb{C}} \to \mathbf{Top}$. Note that if $i: Z \rightarrowtail X$ is a closed embedding in $\mathcal{S}m_{\mathbb{C}}$, then the resulting map i^{an} is the closed embedding of a smooth submanifold, and in particular a cofibration of compactly generated topological spaces. Every motivic space A is a canonical colimit

$$\operatorname*{colim}_{X\times\Delta^n\to A} X\times\Delta^n \xrightarrow{\cong} A$$

and one defines

$$\mathbf{R}_{\mathbb{C}}(A) := \operatorname*{colim}_{X\times\Delta^n\to A} X^{\mathrm{an}}\times|\Delta^n| \in \mathbf{Top}.$$

Observe that if A is pointed, then so is $\mathbf{R}_{\mathbb{C}}(A)$.

Theorem A.23. *The functor* $\mathbf{R}_{\mathbb{C}}: \mathbf{M}_{\bullet}^{\mathrm{cm}}(\mathbb{C}) \to \mathbf{Top}_{\bullet}$ *is a strict symmetric monoidal left Quillen functor.*

Proof. The right adjoint of $\mathbf{R}_{\mathbb{C}}$ maps the compactly generated pointed topological space Z to the pointed motivic space $\mathrm{Sing}_{\mathbb{C}}(Z)$ which sends $X \in \mathcal{S}m_{\mathbb{C}}$ to the pointed simplicial set $\mathbf{sSet}_{\mathbf{Top}}(X^{\mathrm{an}}, Z)$. To conclude that $\mathbf{R}_{\mathbb{C}}$ is strict symmetric monoidal, it suffices to observe that there is a canonical homeomorphism $(X\times Y)^{\mathrm{an}} \cong X^{\mathrm{an}}\times Y^{\mathrm{an}}$, and that geometric realization is strict symmetric monoidal.

Suppose now that $i: \cup_{j=1}^m Z_j \hookrightarrow X$ is an acceptable monomorphism. One computes $\mathbf{R}_{\mathbb{C}}(\cup_{j=1}^m Z_j)$ as the coequalizer

$$\coprod_{j,j'}(Z_j\times_X Z_{j'})^{\mathrm{an}} \rightrightarrows \coprod_{j=1}^{m} Z_j^{\mathrm{an}}.$$

in $\mathbf{Top}$. Every map $\left(Z_j\times_X Z_{j'}\right)^{\mathrm{an}} \to Z_j^{\mathrm{an}}$ is a closed embedding of smooth submanifolds of complex projective space. In particular, one may equip Z_j^{an} with a cell complex structure such that $\left(Z_j\times_X Z_{j'}\right)^{\mathrm{an}}$ is a subcomplex for every j'. Then $\mathbf{R}_{\mathbb{C}}(\cup_{j=1}^m Z_j)$ is the union of these subcomplexes, and in particular again a subcomplex. It follows that $\mathbf{R}_{\mathbb{C}}(i)$ is a cofibration of topological spaces. Since $\mathbf{R}_{\mathbb{C}}$ is compatible with pushout products, it maps the generating closed cofibrations to cofibrations of topological spaces.

To conclude that $\mathbf{R}_{\mathbb{C}}$ preserves trivial cofibrations as well, it suffices by Dugger's lemma [D, Corollary A2] to check that $\mathbf{R}_{\mathbb{C}}$ maps every map in $J_{\mathbb{C}}^m$ to a weak homotopy equivalence. In fact, since the domains and codomains of the maps $\partial\Delta^m \hookrightarrow \Delta^m$ are cofibrant, it suffices to check the latter for the maps in diagram (4). In the first case, one obtains the map $X^{\mathrm{an}} \hookrightarrow (\mathbf{A}_{\mathbb{C}}^1\times X)^{\mathrm{an}} \cong \mathbb{R}^2\times X^{\mathrm{an}}$, in the second case one obtains up to simplicial homotopy equivalence the canonical map $U^{\mathrm{an}} \cup_{p^{-1}(V)^{\mathrm{an}}} Y^{\mathrm{an}} \to X^{\mathrm{an}}$ for a Nisnevich square

$$\begin{tikzcd} V \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ U \arrow[r] & X \end{tikzcd}$$

This is in fact a homeomorphism of topological spaces. The result follows.

Suppose now that $R \hookrightarrow \mathbb{C}$ is a subring of the complex numbers. Let $f\colon \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(R)$ denote the resulting morphism of base schemes. The realization with respect to R (or better f) is defined as the composition

$$\mathbf{R}_R = \mathbf{R}_{\mathbb{C}} \circ f^*\colon \mathbf{M}_\bullet(R) \to \mathbf{M}_\bullet(\mathbb{C}) \to \mathbf{Top}_\bullet. \tag{1}$$

It is a strict symmetric monoidal Quillen functor. The most relevant case is $R = \mathbb{Z}$.

Example A.24. The topological realization of the Grassmannian $\mathrm{Gr}(m, n)$ (over any base with a complex point) is the complex Grassmannian with the usual topology. Since $\mathbf{R}_{\mathbb{C}}$ commutes with filtered colimits, $\mathbf{R}_{\mathbb{C}}(\mathrm{Gr})$ is the infinite complex Grassmannian, which in turn is the classifying space $\mathrm{B}U$ for the infinite unitary group. Because $\mathbf{R}_{\mathbb{C}}$ is a left Quillen functor, the topological realization of any closed cofibrant motivic space weakly equivalent to Gr is homotopy equivalent to $\mathrm{B}U$.

A.5 Spectra

Definition A.25. Let $\mathbf{P}^1_S$ denote the pointed projective line over S. The category $\mathbf{MS}(S)$ of $\mathbf{P}^1$-*spectra* over S has the following objects. A $\mathbf{P}^1$-*spectrum* E consists of a sequence $(E_0, E_1, E_2, \ldots)$ of pointed motivic spaces over S, and structure maps $\sigma_n^E\colon E_n \wedge \mathbf{P}^1 \to E_{n+1}$ for every $n \geq 0$. A map of $\mathbf{P}^1$-spectra is a sequence of maps of pointed motivic spaces which is compatible with the structure maps.

Example A.26. Any pointed motivic space B over S gives rise to a $\mathbf{P}^1_S$-suspension spectrum

$$\Sigma^\infty_{\mathbf{P}^1} B = (B, B \wedge \mathbf{P}^1, B \wedge \mathbf{P}^1 \wedge \mathbf{P}^1, \ldots)$$

having identities as structure maps. More generally, let $\mathrm{Fr}_n B$ denote the $\mathbf{P}^1$-spectrum having values

$$(\mathrm{Fr}_n B)_{n+m} = \begin{cases} B \wedge \mathbf{P}^{1\wedge m} & m \geq 0 \\ * & m < 0 \end{cases}$$

and identities as structure maps, except for $\sigma_{n-1}^{\mathrm{Fr}_n B}$. The functor $B \mapsto \mathrm{Fr}_n B$ is left adjoint to the functor sending the $\mathbf{P}^1$-spectrum E to E_n. We often write $\Sigma^\infty_{\mathbf{P}^1} B(-n)$ for $\mathrm{Fr}_n B$. For a $\mathbf{P}^1$-spectrum E let $u_n \colon \Sigma^\infty_{\mathbf{P}^1} E_n(-n) \to E$ be a map of $\mathbf{P}^1$-spectra adjoint to the identity map $E_n \to E_n$.

Remark A.27. In Definition A.25, one may replace $\mathbf{P}^1$ by any pointed motivic space A, giving the category $\mathbf{MS}_A(S)$ of A-spectra over S. Essentially the only relevant example for us is when A is weakly equivalent to the pointed projective line $\mathbf{P}^1$. The Thom space $T = \mathbf{A}^1/\mathbf{A}^1 - \{0\}$ of the trivial line bundle over S admits motivic weak equivalences

$$\mathbf{P}^1 \xrightarrow{\sim} \mathbf{P}^1/\mathbf{A}^1 \xleftarrow{\sim} T \tag{1}$$

The motivic space $\mathbf{P}^1$ itself is not always the ideal suspension coordinate. For example, the algebraic cobordism spectrum $\mathbf{MGL}$ naturally comes as a T-spectrum. In order to switch between T-spectra and $\mathbf{P}^1$-spectra, consider the following general construction. A map $\phi \colon A \to B$ induces a functor $\phi^* \colon \mathbf{MS}_B(S) \to \mathbf{MS}_A(S)$ sending the B-spectrum $(E_0, E_1, \ldots, \sigma_n^E)$ to the A-spectrum

$$(E_0, E_1, \ldots) \quad \text{with structure maps} \quad \sigma_n^{\phi^* E} = \sigma_n^E \circ (E_n \wedge \phi)$$

Its left adjoint $\phi_\sharp$ maps the A-spectrum $(F_0, F_1, \ldots, \sigma_n^F)$ to the B-spectrum

$$\Big(F_0, B \wedge F_0 \cup_{A \wedge F_0} F_1, B \wedge \big(B \wedge F_0 \cup_{A \wedge F_0} F_1\big) \cup_{A \wedge F_1} F_2\big), \ldots\Big) \tag{2}$$

having the canonical maps as structure maps. Note that for the purpose of constructing a model structure on A-spectra over S, the pointed motivic space A has to be cofibrant in the model structure under consideration.

The next goal is to construct a model structure on $\mathbf{MS}(S)$ having the motivic stable homotopy category as its homotopy category.

Definition A.28. Let $\Omega_{\mathbf{P}^1} = \underline{\mathrm{Hom}}_{\mathbf{M}_\bullet(S)}(\mathbf{P}^1_S, -)$ denote the right adjoint of $\mathbf{P}^1 \wedge -$. For a $\mathbf{P}^1$-spectrum E with structure maps $\sigma_n^E \colon E_n \wedge \mathbf{P}^1 \to E_{n+1}$, let $\omega_n^E \colon E_n \to \Omega_{\mathbf{P}^1} E_{n+1}$ denote the adjoint structure map. A $\mathbf{P}^1$-spectrum E is *closed stably fibrant* if:

- E_n is closed motivic fibrant for every $n \geq 0$.
- $\omega_n^E \colon E_n \to \Omega_{\mathbf{P}^1} E_{n+1}$ is a motivic weak equivalence for every $n \geq 0$.

Any $\mathbf{P}^1$-spectrum E admits a closed stably fibrant replacement. First replace E by a levelwise fibrant $\mathbf{P}^1$-spectrum E^ℓ as follows. Let $E_0^\ell = E_0^{\mathrm{fib}}$ for a fibrant replacement in $\mathbf{M}_\bullet^{\mathrm{cm}}(S)$. Given $E_n \to E_n^\ell$, set

$$E_{n+1}^\ell \coloneqq \Big(E_n^\ell \wedge \mathbf{P}^1 \cup_{E_n \wedge \mathbf{P}^1} E_{n+1}\Big)^{\mathrm{fib}}$$

which yields a levelwise motivic weak equivalence $E \to E^\ell$ of $\mathbf{P}^1$-spectra. To continue, observe that the adjoint structure maps of any $\mathbf{P}^1$-spectrum F may be viewed as a natural transformation

$$q\colon F \to Q(F)$$

where $Q(F)$ is the $\mathbf{P}^1$-spectrum with terms $\Omega_{\mathbf{P}^1} F_1, \Omega_{\mathbf{P}^1} F_2, \ldots$ and structure maps

$$\Omega_{\mathbf{P}^1}(\omega^F_{n+1})\colon \Omega_{\mathbf{P}^1} F_{n+1} \to \Omega^2_{\mathbf{P}^1} F_{n+2}.$$

Define $Q^\infty(E)$ as the colimit of the sequence

$$E^\ell \xrightarrow{q} Q(E^\ell) \xrightarrow{Q(q)} Q^2(E^\ell) \longrightarrow \cdots.$$

Definition A.29. A map $f\colon E \to F$ of $\mathbf{P}^1$-spectra is a *stable equivalence* if the map $Q^\infty(f)_n$ is a weak equivalence for every $n \geq 0$. It is a *closed stable fibration* if $f_n\colon E_n \to F_n$ is a closed motivic fibration and the induced map $E_n \to F_n \times_{Q^\infty(E)_n} Q^\infty(F)_n$ is a motivic weak equivalence for every $n \geq 0$. It is a *closed cofibration* if $f_n\colon E_n \to F_n$ and $F_n \wedge \mathbf{P}^1 \cup_{E_n \wedge \mathbf{P}^1} E_{n+1} \to F_{n+1}$ are closed cofibrations for every $n \geq 0$.

Theorem A.30. *The classes from Definition A.29 are a model structure on the category of* $\mathbf{P}^1$*-spectra, denoted* $\mathbf{MS}^{\mathrm{cm}}(S)$*. The identity functor on* $\mathbf{P}^1$*-spectra from* $\mathbf{MS}^{\mathrm{cm}}(S)$ *to Jardine's stable model structure is a left Quillen equivalence. In particular, the homotopy category* $\mathrm{Ho}\big(\mathbf{MS}^{\mathrm{cm}}(S)\big)$ *is equivalent to the motivic stable homotopy category* $\mathrm{SH}(S)$.

Proof. Recall that $\mathbf{P}^1$ is closed cofibrant by Lemma A.10. The existence of the model structure follows as in [J, Theorem 2.9]. Moreover, the stable equivalences coincide with the ones in [J], because so do the stabilization constructions and the unstable weak equivalences. Since every closed cofibration of motivic spaces is in particular a monomorphism, $\mathrm{Id}_{\mathbf{MS}(S)}$ is a left Quillen equivalence. Note that the closed cofibrations are generated by the set

$$\{\mathrm{Fr}_m(g)\}_{m\geq 0,\, g\in I^c_S} \tag{3}$$

with I^c_S defined in (2). One may also describe a set of generating acyclic cofibrations.

Remark A.31. We will identify the homotopy category $\mathrm{Ho}\big(\mathbf{MS}^{\mathrm{cm}}(S)\big)$ with $\mathrm{SH}(S)$ via the equivalence from Theorem A.30. Note that one may form the smash product of a motivic space A and a $\mathbf{P}^1$-spectrum E by setting $(A \wedge E)_n := A \wedge E_n$ and $\sigma^{A\wedge E}_n := A \wedge \sigma^E_n$. If A is closed cofibrant, $A \wedge -$ is a left Quillen functor by Theorem A.17. Because $\mathbf{P}^1 \wedge -$ is a Quillen equivalence on Jardine's stable model structure by results in [J, Sect. 3.4], Theorem A.30 implies that $\mathbf{P}^1 \wedge -\colon \mathbf{MS}^{\mathrm{cm}}(S) \to \mathbf{MS}^{\mathrm{cm}}(S)$ is a left Quillen equivalence. Since $\mathbf{P}^1 \simeq S^{2,1} = S^1 \wedge (\mathbf{A}^1 - \{0\}, 1)$, also $S^1 \wedge -$ is a left Quillen equivalence. It induces the shift functor in the triangulated structure on $\mathrm{SH}(S)$. The triangles are those which are isomorphic to the image of

$$E \xrightarrow{f} F \longrightarrow F/E \xleftarrow{\sim} E \wedge \Delta^1 \cup_E F \longrightarrow S^1 \wedge E$$

in $\mathrm{SH}(S)$, where $f\colon E \to F$ is an inclusion of $\mathbf{P}^1$-spectra. As well, one has sphere spectra $S^{p,q} \in \mathrm{SH}(S)$ for all integers $p, q \in \mathbb{Z}$.

Example A.32. Since $\mathrm{SH}(S)$ is an additive category, the canonical map $E \vee F \to E \times F$ is a stable equivalence. In the special case of $\mathbf{P}^1$-suspension spectra, the canonical map factors as

$$\Sigma^\infty_{\mathbf{P}^1} A \vee \Sigma^\infty_{\mathbf{P}^1} B \cong \Sigma^\infty_{\mathbf{P}^1}(A \vee B) \to \Sigma^\infty_{\mathbf{P}^1}(A \times B) \to \Sigma^\infty_{\mathbf{P}^1} A \times \Sigma^\infty_{\mathbf{P}^1} B$$

which shows that $\Sigma^\infty_{\mathbf{P}^1}(A \times B)$ contains $\Sigma^\infty_{\mathbf{P}^1}(A \wedge B)$ as a retract in $\mathrm{SH}(S)$. Thus it is even a direct summand. The latter can be deduced as follows. The (reduced) *join* $(A, a_0) * (B, b_0)$ is defined as the pushout in the diagram

$$\begin{tikzcd}
A \times B \times \partial\Delta^1 \cup \{a_0\} \times \{b_0\} \times \Delta^1 \arrow[r] \arrow[d] & A \vee B \arrow[d] \\
A \times B \times \Delta^1 \arrow[r] & A * B.
\end{tikzcd}$$

of pointed motivic spaces over S. Attaching $A \wedge (\Delta^1, 0)$ and $B \wedge (\Delta^1, 0)$ to $A * B$ via $A \vee B$ produces a pointed motivic space C which is equipped with a sectionwise weak equivalence $C \to (A \times B) \wedge S^{1,0}$. Collapsing $\{a_0\} * B$ and $A * \{b_0\}$ inside C yields a sectionwise weak equivalence $C \to (A \wedge B \wedge S^{1,0}) \vee (A \wedge S^{1,0}) \vee (B \wedge S^{1,0})$. Since $S^{1,0}$ is invertible in $\mathrm{SH}(S)$, one gets a splitting $\Sigma^\infty_{\mathbf{P}^1}(A \times B) \simeq \big(\Sigma^\infty_{\mathbf{P}^1}(A \wedge B)\big) \vee \big(\Sigma^\infty_{\mathbf{P}^1} A\big) \vee \big(\Sigma^\infty_{\mathbf{P}^1} B\big)$ in $\mathrm{SH}(S)$.

For a $\mathbf{P}^1$-spectrum E let $\mathrm{Tr}_n E$ denote the $\mathbf{P}^1$-spectrum with

$$\big(\mathrm{Tr}_n E\big)_m = \begin{cases} E_m & m \leq n \\ E_n \wedge (\mathbf{P}^1)^{\wedge m-n} = \big(\mathrm{Fr}_n E_n\big)_m & m \geq n \end{cases}$$

and with the obvious structure maps. The structure maps of E determine maps $\mathrm{Tr}_n E \to \mathrm{Tr}_{n+1} E$ such that $E = \mathrm{colim}_n \mathrm{Tr}_n E$. The canonical map $\mathrm{Fr}_n E_n \to \mathrm{Tr}_n E$ adjoint to the identity id: $E_n \to E_n$ is an identity in all levels $\geq n$, and in particular a stable equivalence. The identity id: $E_n \wedge (\mathbf{P}^1)^{\wedge n} \to \big(\mathrm{Fr}_0 E_n\big)_n$ leads by adjointness to the map

$$\mathrm{Fr}_n(E_n) \wedge (\mathbf{P}^1)^{\wedge n} \xrightarrow{\cong} \mathrm{Fr}_n\big(E_n \wedge (\mathbf{P}^1)^{\wedge n}\big) \longrightarrow \mathrm{Fr}_0 E_n \tag{4}$$

and hence to a map $\mathrm{Fr}_n E_n \to \Omega^n_{\mathbf{P}^1}\big(\mathrm{Fr}_0 E_n\big)$. Since the map (4) is an isomorphism in all levels $\geq n$, it is a stable equivalence. Because $\Omega_{\mathbf{P}^1}$ is a Quillen equivalence, the map $\mathrm{Fr}_n E_n \to \Omega^n_{\mathbf{P}^1}\big((\mathrm{Fr}_0 E_n)^{\mathrm{fib}}\big)$ is a stable equivalence as well if E_n is closed cofibrant. In fact, the condition on E_n can be removed since Fr_n preserves all weak equivalences. This leads to the following statement.

Lemma A.33. *Any $\mathbf{P}^1$-spectrum E is the colimit of a natural sequence*

$$\mathrm{Tr}_0 E \longrightarrow \mathrm{Tr}_1 E \longrightarrow \mathrm{Tr}_2 E \longrightarrow \cdots \tag{5}$$

of $\mathbf{P}^1$-spectra in which the n-th term is naturally stably equivalent to $\mathrm{Fr}_n E_n = \Sigma^\infty_{\mathbf{P}^1} E_n(-n)$ from Example A.26, and also to $\Omega^n_{\mathbf{P}^1}\big((\Sigma^\infty_{\mathbf{P}^1} E_n)^f\big)$.

One may use the description in Lemma A.33 for computations as follows. Say that a $\mathbf{P}^1$-spectrum F is *finite* if it is stably equivalent to a $\mathbf{P}^1$-spectrum F' such that $* \to E'$ is obtained by attaching finitely many cells from the set (3).

Lemma A.34. *Let $D(0) \to D(1) \to D(2) \to \cdots$ be a sequence of maps of $\mathbf{P}^1$-spectra, with colimit $D(\infty)$:*

1. *Suppose that F is a finite $\mathbf{P}^1$-spectrum. The canonical map*

$$\operatorname*{colim}_{i\geq 0} \mathrm{Hom}_{\mathrm{SH}(S)}\big(F, D(i)\big) \to \mathrm{Hom}_{\mathrm{SH}(S)}\big(F, D(\infty)\big)$$

   is an isomorphism.
2. *For any $\mathbf{P}^1$-spectrum E there is a canonical short exact sequence*

$$0 \to \varprojlim_{i\geq 0}{}^1 \big[S^{1,0} \wedge D(i), E\big] \to \big[D(\infty), E\big] \to \varprojlim_{i\geq 0}\big[D(i), E\big] \to 0 \tag{6}$$

   of abelian groups, where $[-,-]$ denotes $\mathrm{Hom}_{\mathrm{SH}(S)}(-,-)$.

Proof. Observe first that stable equivalences and closed stable fibrations are detected by the functor Q^∞ which is defined in Definition A.29 as a sequential colimit. Lemma A.18 implies that stable equivalences and closed stable fibrations with closed stably fibrant codomain are closed under filtered colimits. Thus by Theorem A.3 one may compute

$$\mathrm{Hom}_{\mathrm{SH}(S)}(F, \operatorname*{colim}_{i\geq 0} D(i)) \cong \mathrm{Hom}_{\mathbf{MS}(S)}(F, \operatorname*{colim}_{i\geq 0} D(i)^{\mathrm{fib}})/\simeq$$

for any cofibrant $\mathbf{P}^1$-spectrum F, where $\simeq$ denotes the equivalence relation "simplicial homotopy". This implies statement 1 because $\mathrm{Hom}_{\mathbf{MS}(S)}(F,-)$ commutes with filtered colimits if $* \to F$ is obtained by attaching finitely many cells.

To prove the second statement, let C be the coequalizer of the diagram

$$\bigvee_{i\geq 0} D(i) \underset{g}{\overset{f}{\rightrightarrows}} \bigvee_{i\geq 0} \Delta^1_+ \wedge D(i)$$

where f resp. g is defined on the i-th summand $D(i)$ as $D(i) = 1_+ \wedge D(i) \hookrightarrow \Delta^1_+ \wedge D(i)$ resp. $D(i) \to D(i+1) = 0_+ \wedge D(i+1) \hookrightarrow \Delta^1_+ \wedge D(i+1)$. The canonical map $C \to \operatorname{colim}_{i\geq 0} D(i)$ induced by the composition $\Delta^1_+ \wedge D(i) \to D(i) \to \operatorname{colim}_{i\geq 0} D(i)$ is a weak equivalence. In the stable homotopy category, which is additive, one may take the difference of f and g, and thus describe $\operatorname{colim}_{i\geq 0} D(i)$ via the distinguished triangle

$$\bigvee_{i\geq 0} D(i) \xrightarrow{f-g} \bigvee_{i\geq 0} D(i) \longrightarrow \operatorname*{colim}_{i\geq 0} D(i) \longrightarrow \bigvee_{i\geq 0} S^{1,0} \wedge D(i). \tag{7}$$

Applying $[-,E] := \operatorname{Hom}_{\mathrm{SH}(S)}(-,E)$ to the triangle (7) produces a long exact sequence

$$\cdots \xleftarrow{f^*-g^*} \prod_{i\geq 0}\big[D(i),E\big] \longleftarrow \big[D(\infty),E\big] \longleftarrow \prod_{i\geq 0}\big[S^{1,0}\wedge D(i),E\big] \longleftarrow \cdots$$

which may be split into the short exact sequence

$$0 \longleftarrow \varprojlim_{i\geq 0}\big[D(i),E\big] \longleftarrow \big[D(\infty),E\big] \longleftarrow \varprojlim^1_{i\geq 0}\big[S^{1,0}\wedge D(i),E\big] \longleftarrow 0.$$

A.6 Symmetric Spectra

There seems to be no reasonable (i.e. symmetric monoidal) smash product for $\mathbf{P}^1$-spectra inducing a decent symmetric monoidal smash product on $\mathrm{SH}(S)$. This will be solved as in [HSS] and [J].

Definition A.35. A *symmetric* $\mathbf{P}^1$*-spectrum* E over S consists of a sequence $(E_0, E_1, \ldots)$ of pointed motivic spaces over S, together with group actions $(\Sigma_n)_+ \wedge E_n \to E_n$ and structure maps $\sigma^E_n \colon E_n \wedge \mathbf{P}^1 \to E_{n+1}$ for all $n \geq 0$. Iterations of these structure maps are required to be as equivariant as they can, using the permutation action of Σ_n on $(\mathbf{P}^1)^{\wedge n}$. A map of symmetric $\mathbf{P}^1$-spectra is a sequence of maps of pointed motivic spaces which is compatible with all the structure (group actions and structure maps). Call the resulting category $\mathbf{MSS}(S)$.

Example A.36. Analogous to Example A.26, the n-th shifted suspension spectrum $\mathrm{Fr}^\Sigma_n A$ of a pointed motivic space A has as values

$$(\mathrm{Fr}^\Sigma_n B)_{m+n} = \begin{cases} (\Sigma_{m+n})_+ \wedge_{\Sigma_m\times\{1\}} A \wedge (\mathbf{P}^1)^{\wedge m} & m \geq 0 \\ * & m < 0 \end{cases}$$

where the m-th fold smash product $(\mathbf{P}^1)^{\wedge m}$ carries the natural permutation action.

Every symmetric $\mathbf{P}^1$-spectrum determines a $\mathbf{P}^1$-spectrum by forgetting the symmetric group actions. Call the resulting functor $u \colon \mathbf{MSS}(S) \to \mathbf{MS}(S)$. It has a left adjoint v, which is characterized uniquely up to unique isomorphism by the fact that

$$v\big(\mathrm{Fr}_n A\big) = \mathrm{Fr}_n^{\Sigma} A. \tag{1}$$

The smash product $E \wedge F$ of two symmetric $\mathbf{P}^1$-spectra E and F is constructed as follows. Set $(E \wedge F)_n$ as the coequalizer of the diagram

$$\coprod_{r+1+s=n} (\Sigma_n)_+ \wedge_{\Sigma_r \times \Sigma_1 \times \Sigma_s} E_r \wedge \mathbf{P}^1 \wedge F_s \underset{\sigma_r^E \wedge (F_s \circ \mathrm{twist})}{\overset{\sigma_r^E \wedge F_s}{\rightrightarrows}} \coprod_{r+s=n} (\Sigma_n)_+ \wedge_{\Sigma_r \times \Sigma_s} E_r \wedge F_s \tag{2}$$

where the coequalizer is taken in the category of pointed Σ_n-motivic spaces. The structure map $\sigma_n^{E \wedge F}$ is induced by the structure maps $\sigma_0^F, \ldots, \sigma_n^F$ of F. One may provide natural coherence isomorphisms for associativity, commutativity and unitality, where the unit is $\mathbb{I}_S = (S_+, \mathbf{P}^1, \mathbf{P}^1 \wedge \mathbf{P}^1, \ldots, (\mathbf{P}^1)^{\wedge n}, \ldots)$ with the obvious permutation action and identities as structure maps.

We proceed with the homotopy theory of symmetric $\mathbf{P}^1$-spectra, as in [J]. As one deduces from [Ho2, Theorem 7.2], there exists a cofibrant replacement functor $(-)^{\mathrm{cof}} \to \mathrm{Id}_{\mathbf{MSS}(S)}$ for the model structure on symmetric $\mathbf{P}^1$-spectra with levelwise weak equivalences and levelwise fibrations.

Definition A.37. A map $\phi\colon E \to F$ of symmetric $\mathbf{P}^1$-spectra is a *levelwise acyclic fibration* if $\phi_n\colon E_n \to F_n$ is an acyclic closed motivic fibration of pointed motivic spaces over S for all $n \geq 0$. A map $\phi\colon E \to F$ of symmetric $\mathbf{P}^1$-spectra is a *closed cofibration* if it has the left lifting property with respect to all levelwise acyclic fibrations. A levelwise fibrant symmetric $\mathbf{P}^1$-spectrum E is *closed stably fibrant* if the adjoint $E_n \to \underline{\mathrm{Hom}}_{\mathbf{M}_{\bullet}(S)}(\mathbf{P}^1, E_{n+1})$ of the structure map is a weak equivalence for every $n \geq 0$. A map $\phi\colon E \to F$ is a *stable equivalence* if the map

$$\mathbf{sSet}_{\mathbf{MSS}(S)}(\phi^{\mathrm{cof}}, G)\colon \mathbf{sSet}_{\mathbf{MSS}(S)}(F^{\mathrm{cof}}, G) \to \mathbf{sSet}_{\mathbf{MSS}(S)}(E^{\mathrm{cof}}, G)$$

is a weak equivalence of pointed motivic spaces for all closed stably fibrant symmetric $\mathbf{P}^1$-spectra G. The *closed stable fibrations* are then defined by the right lifting property.

Theorem A.38 (Jardine). *The classes of stable equivalences, closed cofibrations and closed stable fibrations from Definition A.37 constitute a symmetric monoidal model structure on* $\mathbf{MSS}(S)$. *The forgetful functor* $u\colon \mathbf{MSS}^{\mathrm{cm}}(S) \to \mathbf{MS}^{\mathrm{cm}}(S)$ *is a right Quillen equivalence.*

Proof. The proof of the first statement follows as in [J, Theorem 4.15, Prop. 4.19]. Note that the stable equivalences in $\mathbf{MSS}^{\mathrm{cm}}(S)$ and in Jardine's model structure $\mathbf{MSS}^{\mathrm{Jar}}(S)$ coincide, since the unstable weak equivalences do so by Remark A.21. In particular, the identity $\mathrm{Id}\colon \mathbf{MSS}^{\mathrm{cm}}(S) \to \mathbf{MSS}^{\mathrm{Jar}}(S)$ is a left Quillen equivalence. The closed cofibrations in $\mathbf{MSS}^{\mathrm{cm}}(S)$ are generated by the inclusions

$$\{\mathrm{Fr}_m^{\Sigma}(g)\}_{m \geq 0,\, g \in I_S^c} \tag{3}$$

with I_S^c defined in (2). By formula (1), the left adjoint v of u sends the generating cofibrations to the generating cofibrations. It follows that v is a left Quillen functor. Since $v\colon \mathbf{MS}^{\mathrm{Jar}}(S) \to \mathbf{MSS}^{\mathrm{Jar}}(S)$ is a Quillen equivalence by [J, Theorem 4.31], so is the functor $v\colon \mathbf{MS}^{\mathrm{cm}}(S) \to \mathbf{MSS}^{\mathrm{cm}}(S)$.

Remark A.39. Let $\mathcal{R}u\colon \mathrm{SH}^{\Sigma}(S) := \mathrm{Ho}\big(\mathbf{MSS}^{\mathrm{cm}}(S)\big) \to \mathrm{SH}(S)$ be the total right derived functor of u, having $\mathcal{L}v$ as a left adjoint (and left inverse). Since $\mathcal{L}u$ is an equivalence, the category $\mathrm{Ho}(\mathbf{MS}^{\mathrm{cm}}(S))$ inherits a closed symmetric monoidal product $\wedge$ by setting

$$E \wedge F := \mathcal{R}u\big(\mathcal{L}v(E) \wedge \mathcal{L}v(F)\big)$$

In other words, if E and F are closed cofibrant $\mathbf{P}^1$-spectra, their smash product in $\mathrm{SH}(S)$ is given by the $\mathbf{P}^1$-spectrum $u\big((v(E) \wedge v(F))^{\mathrm{fib}}\big)$. The unit is $\mathcal{R}u\big(\mathcal{L}v(\mathbb{I})\big) \cong \mathbb{I}$, the sphere $\mathbf{P}^1$-spectrum.

Notation A.40. *For $\mathbf{P}^1$-spectra E and F over S define the E-cohomology and the E-homology of F as*

$$E^{p,q}(F) = \mathrm{Hom}_{\mathrm{SH}(S)}(F, S^{p,q} \wedge E) \tag{4}$$

$$E_{p,q}(F) = \mathrm{Hom}_{\mathrm{SH}(S)}(S^{p,q}, F \wedge E) \tag{5}$$

for all $p, q \in \mathbb{Z}$. In the special case $F = \Sigma^{\infty}_{\mathbf{P}^1} A$, where A is a pointed motivic space over S one writes $E^{p,q}(A)$ and $E_{p,q}(A)$ instead. Note that there is an isomorphism $\mathrm{Fr}_i A \cong S^{-2i,-i} \wedge \mathrm{Fr}_0 A$ in $\mathrm{SH}(S)$.

Remark A.41. Since (v, u) is a Quillen adjoint pair of stable model categories the total derived pair respects in particular the triangulated structures. The functor u preserves all colimits, thus both $\mathcal{L}v$ and $\mathcal{R}u$ preserve arbitrary coproducts.

Lemma A.42. *Let E and F be $\mathbf{P}^1$-spectra. Then $E \wedge F \in \mathrm{SH}(S)$ may be obtained as the sequential colimit of a sequence whose n-th term is stably equivalent to $\Omega^{2n}_{\mathbf{P}^1}(\Sigma^{\infty}_{\mathbf{P}^1} E_n \wedge F_n)^{\mathrm{fib}}$. Thus $E \wedge F \cong hocolim \Sigma^{\infty}_{\mathbf{P}^1}(E_n \wedge F_n)(-2n)$ in $\mathrm{SH}(S)$.*

Proof. Here $E \wedge F \in \mathrm{SH}(S)$ refers to the smash product of symmetric $\mathbf{P}^1$-spectra associated to closed cofibrant replacements $E^{\mathrm{cof}} \to E$ and $F^{\mathrm{cof}} \to F$. Because these two maps are levelwise weak equivalences, we may assume that both E and F are closed cofibrant. By Lemma A.33 E and F can be expressed as sequential colimits of their truncations. Since v preserves colimits, $v(E)$ is the sequential colimit of the diagram

$$v(\mathrm{Tr}_0 E) = v(\mathrm{Fr}_0 E_0) = \mathrm{Fr}^{\Sigma}_0 E_0 \to v(\mathrm{Tr}_1 E) \to \cdots$$

and similarly for F. The stable equivalence $\mathrm{Fr}_n E_n \to \mathrm{Tr}_n E$ of cofibrant $\mathbf{P}^1$-spectra induces a stable equivalence $v(\mathrm{Fr}_n E_n) = \mathrm{Fr}^{\Sigma}_n E_n \to v(\mathrm{Tr}_n E)$. Since smashing with a symmetric $\mathbf{P}^1$-spectrum preserves colimits, one has

$$v(\mathrm{Tr}_m E) \wedge \operatorname*{colim}_n v(\mathrm{Tr}_n F) \cong \operatorname*{colim}_n \bigl(v(\mathrm{Tr}_m E) \wedge v(\mathrm{Tr}_n F)\bigr)$$

for every n. It follows that $v(E) \wedge v(F)$ is the filtered colimit of the diagram sending (m,n) to $v(\mathrm{Tr}_m E) \wedge v(\mathrm{Tr}_n F)$. Since the diagonal is a final subcategory in $\mathbb{N} \times \mathbb{N}$, there is a canonical isomorphism $\mathrm{colim}_n\, v(\mathrm{Tr}_n E) \wedge v(\mathrm{Tr}_n F) \cong v(E) \wedge v(F)$. Theorem A.38 says that $\mathbf{MSS}^{\mathrm{cm}}(S)$ is symmetric monoidal, thus the canonical map

$$\mathrm{Fr}^{\Sigma}_{2n}(E_n \wedge F_n) \cong \mathrm{Fr}^{\Sigma}_n E_n \wedge \mathrm{Fr}^{\Sigma}_n F_n \to v(\mathrm{Tr}_n E) \wedge v(\mathrm{Tr}_n F) \tag{6}$$

is a stable equivalence. Let $\mathrm{Fr}^{\Sigma}_{2n}(E_n \wedge F_n) \to \Omega^{2n}_{\mathbf{P}^1} \mathrm{Fr}^{\Sigma}_0(E_n \wedge F_n)$ be the canonical map which is adjoint to the unit $E_n \wedge F_n \to \Omega^{2n}_{\mathbf{P}^1}(\mathbf{P}^1)^{\wedge 2n} \wedge (E_n \wedge F_n)$. As in the case of $\mathbf{P}^1$-spectra, the map

$$\mathrm{Fr}^{\Sigma}_{2n}(E_n \wedge F_n) \to \Omega^{2n}_{\mathbf{P}^1} \mathrm{Fr}^{\Sigma}_0(E_n \wedge F_n) \to \Omega^{2n}_{\mathbf{P}^1} \bigl(\mathrm{Fr}^{\Sigma}_0(E_n \wedge F_n)\bigr)^{\mathrm{fib}}$$

is a stable equivalence. It follows that $v(E) \wedge v(F)$ is the colimit of a sequence whose n-th term is stably equivalent to $\Omega^{2n}_{\mathbf{P}^1}\bigl(\mathrm{Fr}^{\Sigma}_0(E_n \wedge F_n)\bigr)^{\mathrm{fib}}$. Hence it also follows that a fibrant replacement of $v(E) \wedge v(F)$ may be obtained as the colimit of a sequence of closed stably fibrant symmetric $\mathbf{P}^1$-spectra whose n-th term is stably equivalent to $\Omega^{2n}_{\mathbf{P}^1}\bigl(\mathrm{Fr}^{\Sigma}_0(E_n \wedge F_n)\bigr)^{\mathrm{fib}}$. Since the forgetful functor u preserves colimits, stable equivalences of closed stably fibrant symmetric $\mathbf{P}^1$-spectra and $\Omega_{\mathbf{P}^1}$, the $\mathbf{P}^1$-spectrum $u\bigl(v(E) \wedge v(F)\bigr)$ is the colimit of a sequence of $\mathbf{P}^1$-spectra whose n-th term is stably equivalent to $\Omega^{2n}_{\mathbf{P}^1} u\Bigl(\bigl(\mathrm{Fr}^{\Sigma}_0(E_n \wedge F_n)\bigr)^{\mathrm{fib}}\Bigr)$. The map

$$\mathrm{Fr}_0(E_n \wedge F_n) \to u\Bigl(\bigl(v\mathrm{Fr}_0(E_n \wedge F_n)\bigr)^{\mathrm{fib}}\Bigr)$$

is a stable equivalence because (v,u) is a Quillen equivalence by Theorem A.38, whence the result.

As in the case of non-symmetric spectra, one may change the suspension coordinate as in Remark A.27. If A is a pointed motivic space over S, let $\mathbf{MSS}_A(S)$ denote the category of symmetric A-spectra over S.

Lemma A.43. *A map $A \to B$ in $\mathbf{M}_{\bullet}(S)$ induces a strict symmetric monoidal functor $\mathbf{MSS}_A(S) \to \mathbf{MSS}_B(S)$ having a right adjoint. If the map is a motivic weak equivalence of closed cofibrant pointed motivic spaces, this pair is a Quillen equivalence.*

Proof. This is quite formal. For a proof consider [Ho2, Theorem 9.4].

Because the change of suspension coordinate functors are lax symmetric monoidal, they preserve (commutative) monoid objects, that is, (commutative) symmetric ring spectra. Recall that a functor is lax symmetric monoidal if it is equipped with natural structure maps as in (5) which are not necessarily isomorphisms.

A.7 Stable Topological Realization

Let $\mathrm{Sp} = \mathrm{Sp}(\mathbf{Top}, \mathbb{C}\mathbf{P}^1)$ be the category of $\mathbb{C}\mathbf{P}^1$-spectra (in $\mathbf{Top}$). An object in Sp is thus a sequence of pointed compactly generated topological spaces $E_0, E_1, \ldots$ with structure maps $E_n \wedge \mathbb{C}\mathbf{P}^1 \to E_{n+1}$. The model structure on Sp is obtained as follows: Cofibrations are generated by

$$\{\mathrm{Fr}_m^{\mathbf{Top}}(|\partial\Delta^n \hookrightarrow \Delta^n|_+)\}_{m,n\geq 0}$$

so that every $\mathbb{C}\mathbf{P}^1$-spectrum E has a cofibrant replacement $E^{\mathrm{cof}} \to E$ mapping to E via a levelwise acyclic Serre fibration. For any $\mathbb{C}\mathbf{P}^1$-spectrum E and any $n \in \mathbb{Z}$ let $\pi_n E$ be the colimit of the sequence

$$\pi_{n+2m} E_m \to \pi_{n+2m+2} E_m \wedge \mathbb{C}\mathbf{P}^1 \to \pi_{n+2(m+1)} E_{m+1} \to \cdots$$

where $m \geq 0$ and $n+2m \geq 0$. It is called the n-th stable homotopy group of E. Note that homotopy groups of non-degenerately based compactly generated topological spaces commute with filtered colimits. If E is cofibrant, E_n is in particular non-degenerately based for all $n \geq 0$. A map $f: E \to F$ of $\mathbb{C}\mathbf{P}^1$-spectra is a *stable equivalence* if the induced map $\pi_n f: \pi_n E^{\mathrm{cof}} \to \pi_n F^{\mathrm{cof}}$ is an isomorphism for all $n \in \mathbb{Z}$. It is a *stable fibration* if it has the right lifting property with respect to all stable acyclic cofibrations.

Similarly, one may form the category $\mathrm{Sp}^{\Sigma} = \mathrm{Sp}^{\Sigma}(\mathbf{Top}, \mathbb{C}\mathbf{P}^1)$ of symmetric $\mathbb{C}\mathbf{P}^1$-spectra in $\mathbf{Top}$. Cofibrations are generated by

$$\{\mathrm{Fr}_m^{\mathbf{Top},\Sigma}(|\partial\Delta^n \hookrightarrow \Delta^n|_+)\}_{m,n\geq 0}$$

and a symmetric $\mathbb{C}\mathbf{P}^1$-spectrum is *stably fibrant* if its underlying $\mathbb{C}\mathbf{P}^1$-spectrum is stably fibrant. A map $f: E \to F$ of symmetric $\mathbb{C}\mathbf{P}^1$-spectra is a *stable equivalence* if the induced map $\mathbf{sSet}_{\mathrm{Sp}^{\Sigma}}(f^{\mathrm{cof}}, G)$ of simplicial sets of maps is an isomorphism for all stably fibrant symmetric $\mathbb{C}\mathbf{P}^1$-spectra G. It is a *stable fibration* if it has the right lifting property with respect to all stable acyclic cofibrations.

Theorem A.44. *Stable equivalences, stable fibrations and cofibrations form (symmetric monoidal) model structures on the categories of (symmetric)* $\mathbb{C}\mathbf{P}^1$*-spectra in* $\mathbf{Top}$. *The functor forgetting the symmetric group actions is a right Quillen equivalence. There is a zig-zag of strict symmetric monoidal left Quillen equivalences connecting* $\mathrm{Sp}^{\Sigma}(\mathbf{Top}, \mathbb{C}\mathbf{P}^1)$ *and* $\mathrm{Sp}^{\Sigma}(\mathbf{Top}, S^1)$. *In particular, the homotopy category of (symmetric)* $\mathbb{C}\mathbf{P}^1$*-spectra is equivalent as a closed symmetric monoidal and triangulated category to the stable homotopy category.*

Proof. The statement about the model structures follows as in [HSS] if one replaces S^1 by $\mathbb{C}\mathbf{P}^1$ and simplicial sets by compactly generated topological spaces. The same holds for the statement about the functor forgetting the symmetric group actions. To construct the zig-zag, consider the corresponding stable model structure on the category of symmetric S^1-spectra in the category $\mathrm{Sp}^{\Sigma}(\mathbf{Top}, \mathbb{C}\mathbf{P}^1)$, which is isomorphic as a symmetric monoidal model category to the category of symmetric $\mathbb{C}\mathbf{P}^1$-spectra in the category $\mathrm{Sp}^\Sigma(\mathbf{Top}, S^1)$ of topological symmetric S^1-spectra. The suspension spectrum functors give a zig-zag

$$\begin{tikzcd} \mathrm{Sp}^\Sigma(\mathbf{Top}, \mathbb{C}\mathbf{P}^1) \arrow[d] & \mathrm{Sp}^\Sigma(\mathbf{Top}, S^1) \arrow[d] \\ \mathrm{Sp}^\Sigma\big(\mathrm{Sp}^\Sigma(\mathbf{Top}, \mathbb{C}\mathbf{P}^1), S^1\big) \arrow[r, "\cong"] & \mathrm{Sp}^\Sigma\big(\mathrm{Sp}^\Sigma(\mathbf{Top}, S^1), \mathbb{C}\mathbf{P}^1\big) \end{tikzcd} \tag{1}$$

of strict symmetric monoidal left Quillen functors. Since $\mathbb{C}\mathbf{P}^1 \wedge -$ is a left Quillen equivalence on the left hand side in the zig-zag (1) and $S^1 \wedge S^1 \cong \mathbb{C}\mathbf{P}^1$, $S^1 \wedge -$ is a left Quillen equivalence on the left hand side as well. By [Ho2, Theorem 9.1], the arrow pointing to the right in the zig-zag (1) is a Quillen equivalence. A similar argument works for the arrow on the right hand side, which completes the proof.

Given a $\mathbf{P}^1$-spectrum E over $\mathbb{C}$, one gets a $\mathbb{C}\mathbf{P}^1$-spectrum $\mathbf{R}_\mathbb{C}(E) = (\mathbf{R}_\mathbb{C} E_0, \mathbf{R}_\mathbb{C} E_1, \dots)$ with structure maps $\mathbf{R}_\mathbb{C}(E_n) \wedge \mathbf{P}^1 \cong \mathbf{R}_\mathbb{C}(E_n \wedge \mathbf{P}^1) \to \mathbf{R}_\mathbb{C}(E_{n+1})$. The right adjoint for the resulting functor $\mathbf{R}_\mathbb{C}\colon \mathbf{MS}(\mathbb{C}) \to \mathrm{Sp}$ is also obtained by a levelwise application of $\mathrm{Sing}_\mathbb{C}$. The same works for symmetric $\mathbf{P}^1$-spectra over $\mathbb{C}$.

Theorem A.45. *The functors* $\mathbf{R}_\mathbb{C}\colon \mathbf{MS}(\mathbb{C}) \to \mathrm{Sp}$ *and* $\mathbf{R}_\mathbb{C}\colon \mathbf{MSS}(\mathbb{C}) \to \mathrm{Sp}^\Sigma$ *are left Quillen functors, the latter being strict symmetric monoidal.*

Proof. Since the diagrams

$$\begin{tikzcd} \mathbf{MSS}(\mathbb{C}) \arrow[r, "u"] \arrow[d, "\mathrm{Sing}_\mathbb{C}"'] & \mathbf{MS}(\mathbb{C}) \arrow[r, "E \mapsto E_n"] \arrow[d, "\mathrm{Sing}_\mathbb{C}"'] & \mathbf{M}_\bullet(\mathbb{C}) \arrow[d, "\mathrm{Sing}_\mathbb{C}"] \\ \mathrm{Sp}^\Sigma \arrow[r, "u"] & \mathrm{Sp} \arrow[r, "E \mapsto E_n"] & \mathbf{Top}_\bullet \end{tikzcd}$$

commute, $\mathbf{R}_\mathbb{C}$ preserves the generating cofibrations by Theorem A.23. Then Dugger's Lemma [D, Corollary A.2] implies that $\mathbf{R}_\mathbb{C}$ is a left Quillen functor, because $\mathrm{Sing}_\mathbb{C}$ preserves weak equivalences and fibrations between fibrant objects. The fact that $\mathbf{R}_\mathbb{C}\colon \mathbf{MSS}(\mathbb{C}) \to \mathrm{Sp}^\Sigma$ is strict symmetric monoidal follows from the definition of the smash product (2).

Example A.46. Let BGL be the $\mathbf{P}^1$-spectrum over $\mathbb{C}$ constructed in Sect. 1.2. Its n-th term is a pointed motivic space $\mathcal{K}$ weakly equivalent to $\mathbb{Z} \times \mathrm{Gr}$. One may assume that $\mathcal{K}$ is closed cofibrant. Then by Theorem A.45 the n-th term of $\mathbf{R}_\mathbb{C}(\mathrm{BGL})$ is weakly equivalent to $\mathrm{B}U$. To show that the $\mathbb{C}\mathbf{P}^1$-spectrum $\mathbf{R}_\mathbb{C}(\mathrm{BGL})$ is the one representing complex K-theory, it suffices to check that the structure map $\mathcal{K} \wedge \mathbf{P}^1 \to \mathcal{K}$ realizes to the structure map $\mathrm{B}U \wedge \mathbb{C}\mathbf{P}^1 \to \mathrm{B}U$ of complex K-theory. Consider the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{H}_\bullet(\mathbb{C})}(\mathbb{Z}\times\mathrm{Gr},\mathbb{Z}\times\mathrm{Gr}) \arrow[r, "\cong"] \arrow[d] & K_0^{\mathrm{alg}}(\mathbb{Z}\times\mathrm{Gr}) \arrow[d] \\
\mathrm{Hom}_{\mathrm{Ho}(\mathbf{Top}_\bullet)}\big(\mathbf{R}_{\mathbb{C}}(\mathbb{Z}\times\mathrm{Gr}),\mathbf{R}_{\mathbb{C}}(\mathbb{Z}\times\mathrm{Gr})\big) \arrow[r, "\cong"] & K_0^{\mathrm{top}}\big(\mathbf{R}_{\mathbb{C}}(\mathbb{Z}\times\mathrm{Gr})\big)
\end{tikzcd}$$

where the vertical map on the left hand side is induced by $\mathbf{R}_{\mathbb{C}}$ and the vertical map on the right hand side is induced by the passage from algebraic to topological complex vector bundles. The upper horizontal isomorphism sends the identity to the class ξ_∞ described in Remark 1.4.4 via tautological vector bundles over Grassmannians. The right vertical map sends ξ_∞ to the class ζ_∞ obtained via the corresponding tautological bundles, viewed as complex topological vector bundles. Since ζ_∞ is the image of $\mathrm{id}_{\mathbf{R}_{\mathbb{C}}(\mathbb{Z}\times\mathrm{Gr})}$ under the lower horizontal isomorphism, the diagram commutes at the identity. By naturality, it follows that the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{H}_\bullet(\mathbb{C})}(A,\mathbb{Z}\times\mathrm{Gr}) \arrow[r, "\cong"] \arrow[d] & K_0^{\mathrm{alg}}(A) \arrow[d] \\
\mathrm{Hom}_{\mathrm{Ho}(\mathbf{Top}_\bullet)}\big(\mathbf{R}_{\mathbb{C}}(A),\mathbf{R}_{\mathbb{C}}(\mathbb{Z}\times\mathrm{Gr})\big) \arrow[r, "\cong"] & K_0^{\mathrm{top}}\big(\mathbf{R}_{\mathbb{C}}(A)\big)
\end{tikzcd}$$

commutes for every pointed motivic space A over $\mathbb{C}$. In particular, the structure map of BGL which corresponds to $(\xi_\infty)\otimes([\mathcal{O}(-1)]-[\mathcal{O}])$ maps to the structure map of the complex K-theory spectrum, since it corresponds to the "same" class, viewed as a difference of complex topological vector bundles.

Proposition A.47. *A morphism $f\colon S \to S'$ of base schemes induces a strict symmetric monoidal left Quillen functor*

$$f^*\colon \mathbf{MSS}(S') \to \mathbf{MSS}(S)$$

such that $\big(f^*(E)\big)_n = f^*(E_n)$.

Proof. The structure maps of $f^*(E)$ are defined via the canonical map

$$f^*(E)_n \wedge \mathbf{P}^1_{S'} \cong f^*(E_n \wedge \mathbf{P}^1_S) \xrightarrow{f^*(\sigma_n^E)} f^*(E_{n+1}) = f^*(E)_{n+1}.$$

It follows that f^* has the functor f_* as right adjoint, where $\big(f_*(E)\big)_n = f_*(E_n)$ and

$$\begin{tikzcd}
\sigma_n^{f_*E} = f_*E_n \wedge \mathbf{P}^1_{S'} \arrow[r] & f_*E_n \wedge f_*f^*\mathbf{P}^1_{S'} \arrow[r, "\cong"] & f_*E_n \wedge f_*\mathbf{P}^1_S \arrow[d] \\
& f_*E_{n+1} & f_*(E_n \wedge \mathbf{P}^1_S) \arrow[l, "f_*\sigma_n^E"']
\end{tikzcd}$$

Theorem A.17 and Dugger's lemma imply that f^* preserves cofibrations and f_* preserves fibrations. Since $f^*\colon \mathbf{M}_\bullet(S') \to \mathbf{M}_\bullet(S)$ is strict symmetric monoidal and preserves all colimits, then so is $f^*\colon \mathbf{MSS}(S') \to \mathbf{MSS}(S)$ by the definition of the smash product (2).

In particular, any complex point $f\colon \mathrm{Spec}(\mathbb{C}) \to S$ of a base scheme S induces a strict symmetric monoidal left Quillen functor

$$\mathbf{MSS}(S) \to \mathbf{MSS}(\mathbb{C}) \to \mathrm{Sp}^{\Sigma}(\mathbf{Top}, \mathbb{C}\mathbf{P}^1) \tag{2}$$

to the category of topological $\mathbb{C}\mathbf{P}^1$-spectra.

B Some Results on K-Theory

B.1 Cellular Schemes

Suppose that S is a regular base scheme. Recall that an S-cellular scheme is an S-scheme X equipped with a filtration $X_0 \subset X_1 \subset \cdots \subset X_n = X$ by closed subsets such that for every integer $i \geq 0$ the S-scheme $X_i \smallsetminus X_{i-1}$ is a disjoint union of several copies of the affine space $\mathbf{A}^i_S$. We do not assume that X is connected. *A pointed S-cellular scheme* is an S-cellular scheme equipped with a closed S-point $x\colon S \hookrightarrow X$ such that $x(S)$ is contained in one of the open cells (a cell which is an open subscheme of X). The examples we are interested in are Grassmannians, projective lines and their products.

Lemma B.1. *Let (X, x) and (Y, y) be pointed motivic spaces. Then the sequence*

$$0 \to K_i(X \wedge Y) \to K_i(X \times Y) \to K_i(X \vee Y) \to 0$$

is short exact and the natural map

$$K_i(X) \oplus K_i(Y) \to K_i(X \vee Y)$$

is an isomorphism.

Proof. The exactness of the sequence follows from the isomorphism $K_i \cong \mathrm{BGL}^{-i,0}$ and Example A.32. The isomorphism is formal, given the isomorphism $K_i \cong \mathrm{BGL}^{-i,0}$.

Corollary B.2. *Let (X, x) and (Y, y) be pointed smooth S-schemes. Let $a \in K_0(X)$ and $b \in K_0(Y)$ be such that $x^*(a) = 0 = y^*(b)$ in $K_0(S)$. Then the element $a \otimes b \in K_0(X \times Y)$ belongs to the subgroup $K_0(X \wedge Y)$.*

Proof. Since $a \otimes b$ vanishes on $x(S) \times Y$ and on $X \times x(S)$ it follows that $a \otimes b$ vanishes on $X \vee Y$. Whence $a \otimes b \in K_0(X \wedge Y)$ by Lemma B.1.

We list further useful statements.

Lemma B.3. *Let X be a smooth S-cellular scheme. Then the map*

$$K_r(S) \otimes_{K_0(S)} K_0(X) \to K_r(X)$$

is an isomorphism and $K_0(X)$ is a free $K_0(S)$-module of rank equal to the number of cells.

The Lemma easily follows from a slightly different claim which we consider as a well-known one.

Claim. Under the assumption of the Lemma the map of Quillen's K-groups

$$K_r(S) \otimes_{K_0(S)} K'_0(X_j) \to K'_r(X_j)$$

is an isomorphism and $K'_0(X_j)$ is a free $K_0(S)$-module of the expected rank.

Lemma B.4. *Let (X, x) and (Y, y) be pointed smooth S-cellular schemes. Then the map*

$$K_i(S) \otimes_{K_0(S)} K_0(X \vee Y) \to K_i(X \vee Y)$$

is an isomorphism and $K_0(X \vee Y)$ is a projective $K_0(S)$-module.

Lemma B.5. *Let (X, x) and (Y, y) be pointed smooth S-cellular schemes. Then the map*

$$K_i(S) \otimes_{K_0(S)} K_0(X \wedge Y) \to K_i(X \wedge Y)$$

is an isomorphism and $K_0(X \wedge Y)$ is a projective $K_0(S)$-module.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
K_i(X \wedge Y) \arrow[r, "\alpha"] & K_i(X \times Y) \arrow[r, "\beta"] & K_i(X \vee Y) \\
K_i(S) \otimes K_0(X \wedge Y) \arrow[r, "\gamma"] \arrow[u, "\epsilon"] & K_i(S) \otimes K_0(X \times Y) \arrow[r, "\delta"] \arrow[u, "\rho"] & K_i(S) \otimes K_0(X \vee Y) \arrow[u, "\theta"]
\end{tikzcd}$$

in which K_i is written for K_i and the tensor product is taken over $K_0(S)$.

The sequence

$$0 \to K_i(X \wedge Y) \xrightarrow{\alpha} K_i(X \times Y) \xrightarrow{\beta} K_i(X \vee Y) \to 0$$

is short exact by Lemma B.1. In particular it is short exact for $i = 0$. Now the sequence

$$0 \to K_i(S) \otimes K_0(X \wedge Y) \xrightarrow{\gamma} K_i(S) \otimes K_0(X \times Y) \xrightarrow{\delta} K_i(S) \otimes K_0(X \vee Y) \to 0$$

is short exact since $K_0(X \vee Y)$ is a projective $K_0(S)$-module.

The arrows ρ and θ are isomorphisms by Lemmas B.3 and B.4 respectively. Whence ϵ is an isomorphism as well. Finally $K_0(X \wedge Y)$ is a projective $K_0(S)$-module since the sequence $0 \to K_0(X \wedge Y) \to K_0(X \times Y) \to K_0(X \vee Y) \to 0$ is short exact and $K_0(X \times Y)$ and $K_0(X \vee Y)$ are projective $K_0(S)$-modules.

As well we need to know that certain $\varprojlim^1$-groups vanish. Given a set M and a smooth S-scheme X, we write $M \times X$ for the disjoint union $\bigsqcup_M X$ of M copies of X in the category of motivic spaces over S. Recall that $[-n, n]$ is the set of integers with absolute value $\leq n$.

Lemma B.6.

$$\varprojlim^1 K_i(\mathrm{Gr}(n, 2n)) = 0$$
$$\varprojlim^1 K_i([-n, n] \times \mathrm{Gr}(n, 2n)) = 0$$
$$\varprojlim^1 K_i([-n, n] \times \mathrm{Gr}(n, 2n) \times [-n, n] \times \mathrm{Gr}(n, 2n)) = 0$$

Proof. This holds since all the bondings map in the towers are surjective.

Lemma B.7. *The canonical maps*

$$K_i(\mathrm{Gr}) \to \varprojlim K_i\big(\mathrm{Gr}(n, 2n)\big)$$

$$K_i(\mathrm{Gr} \times \mathrm{Gr}) \to \varprojlim K_i\big(\mathrm{Gr}(n, 2n) \times \mathrm{Gr}(n, 2n)\big)$$

are isomorphisms. A similar statement holds for the pointed motivic spaces $\mathbb{Z} \times \mathrm{Gr}$, $(\mathbb{Z} \times \mathrm{Gr}) \times \mathbf{P}^1$, $(\mathbb{Z} \times \mathrm{Gr}) \times (\mathbb{Z} \times \mathrm{Gr})$, $(\mathbb{Z} \times \mathrm{Gr}) \times \mathbf{P}^1 \times (\mathbb{Z} \times \mathrm{Gr}) \times \mathbf{P}^1$.

Proof. This follows from Lemma B.6.

Lemma B.8. *The canonical maps*

$$K_i(\mathrm{Gr} \wedge \mathrm{Gr}) \to \varprojlim K_i\big(\mathrm{Gr}(n, 2n) \wedge \mathrm{Gr}(n, 2n)\big)$$
$$K_i(\mathrm{Gr} \wedge \mathrm{Gr} \wedge \mathbf{P}^1) \to \varprojlim K_i\big(\mathrm{Gr}(n, 2n) \wedge \mathrm{Gr}(n, 2n) \wedge \mathbf{P}^1\big)$$

are isomorphisms. A similar statement holds for the pointed motivic spaces $(\mathbb{Z} \times \mathrm{Gr}) \wedge (\mathbb{Z} \times \mathrm{Gr})$, $(\mathbb{Z} \times \mathrm{Gr}) \wedge (\mathbb{Z} \times \mathrm{Gr}) \wedge \mathbf{P}^1$ *and* $(\mathbb{Z} \times \mathrm{Gr}) \wedge \mathbf{P}^1 \wedge (\mathbb{Z} \times \mathrm{Gr}) \wedge \mathbf{P}^1$.

Proof. It follows immediately from Lemmas B.7 and B.1.

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Chern Character, Loop Spaces and Derived Algebraic Geometry

Bertrand Toën and Gabriele Vezzosi

Abstract In this note we present a work in progress whose main purpose is to establish a categorified version of sheaf theory. We present a notion of *derived categorical sheaves*, which is a categorified version of the notion of complexes of sheaves of $\mathcal{O}$-modules on schemes, as well as its quasi-coherent and perfect versions. We also explain how ideas from derived algebraic geometry and higher category theory can be used in order to construct a Chern character for these categorical sheaves, which is a categorified version of the Chern character for perfect complexes with values in cyclic homology. Our construction uses in an essential way the *derived loop space* of a scheme X, which is a *derived scheme* whose theory of functions is closely related to cyclic homology of X. This work can be seen as an attempt to define algebraic analogs of elliptic objects and characteristic classes for them. The present text is an overview of a work in progress and details will appear elsewhere (see TV1 and TV2).

1 Motivations and Objectives

The purpose of this short note is to present, in a rather informal style, a construction of a Chern character for certain *sheaves of categories* rather than sheaves of modules (like vector bundles or coherent sheaves). This is part of a more ambitious project to develop a general theory of what we call *categorical sheaves*, in the context of algebraic geometry but also in topology, which is supposed to be a categorification of the theory of sheaf of modules. Our original motivations for starting such a project come from elliptic cohomology, which we now explain briefly.

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1.1 From Elliptic Cohomology to Categorical Sheaves

To any (complex oriented) generalized cohomology theory E_* (defined on topological spaces) is associated an integer called its chromatic level, which by definition is the height of the corresponding formal group. The typical generalized cohomology theory of height zero is singular cohomology and is represented by the Eleinberg–MacLane spectrum $H\mathbb{Z}$. The typical generalized cohomology theory of height 1 is complex K-theory which is represented by the spectrum $BU \times \mathbb{Z}$. A typical cohomology theory of height 2 is represented by an elliptic spectrum and is called elliptic cohomology. These elliptic cohomologies can be combined altogether into a spectrum tmf of *topological modular forms* (we recommend the excellent survey [Lu1] on the subject). The cohomology theories $H\mathbb{Z}$ and $BU \times \mathbb{Z}$ are rather well understood, in the sense that for a finite CW complex X is possible to describe the groups $[X, H\mathbb{Z}]$ and $[X, BU \times \mathbb{Z}]$ easily in terms of the topology of X. Indeed, $[X, H\mathbb{Z}] \simeq H^0(X, \mathbb{Z})$ is the group of continuous functions $X \longrightarrow \mathbb{Z}$. In the same way, $[X, BU \times \mathbb{Z}] = K_0^{top}(X)$ is the Grothendieck group of complex vector bundles on X. As far as we know, it is an open question to describe the group $[X, \mathsf{tmf}] = Ell_0(X)$, or the groups $[X, E]$ for some elliptic spectrum E, in similar geometric terms, e.g., as the Grothendieck group of some kind of geometric objects over X (for some recent works in this direction, see [Ba-Du-Ro] and [St-Te]).

It has already been observed by several authors that the chromatic level of the cohomology theories $H\mathbb{Z}$ and $BU \times \mathbb{Z}$ coincide with a certain *categorical level*. More precisely, $[X, H\mathbb{Z}]$ is the set of continuous functions $X \longrightarrow \mathbb{Z}$. In this description $\mathbb{Z}$ is a discrete topological space, or equivalently a set, or equivalently a 0-*category*. In the same way, classes in $[X, BU \times \mathbb{Z}]$ can be represented by finite dimensional complex vector bundles on X. A finite dimensional complex vector bundle on X is a continuous family of finite dimensional complex vector spaces, or equivalently a *continuous map* $X \longrightarrow \underline{Vect}$, where $\underline{Vect}$ is the *1-category* of finite dimensional complex vector spaces. Such an interpretation of vector bundles can be made rigorous if $\underline{Vect}$ is considered as a topological stack. It is natural to expect that $[X, \mathsf{tmf}]$ is related in one way or another to 2-categories, and that classes in $[X, \mathsf{tmf}]$ should be represented by certain continuous morphisms $X \longrightarrow 2-\underline{Vect}$, where now $2-\underline{Vect}$ is a 2-category (or rather a topological 2-stack). The notation $2-\underline{Vect}$ suggests here that $2-\underline{Vect}$ is a categorification of $\underline{Vect}$, which is itself a categorification of $\mathbb{Z}$ (or rather of $\mathbb{C}$). If we follow this idea further the typical generalized cohomology theory E of chromatic level n should itself be related to n-categories in the sense that classes in $[X, E]$ should be represented by continuous maps $X \longrightarrow n-\underline{Vect}$, where $n-\underline{Vect}$ is now a certain topological n-stack, which is supposed to be an n-categorification of the $(n-1)$-stack $(n-1)-\underline{Vect}$.

This purely formal observation relating the chromatic level to some, yet undefined, categorical level is in fact supported by at least two recent results. On the one hand, Rognes stated the so-called *red shift conjecture*, which from an intuitive point of view stipulates that if a commutative ring spectrum E is of chromatic level n then its K-theory spectrum $K(E)$ is of chromatic level $n+1$ (see [Au-Ro]). Some explicit computations of $K(BU \times \mathbb{Z})$ proves a major case of this conjecture for $n = 1$ (see [Ba-Du-Ro]). Moreover, $K(BU \times \mathbb{Z})$ can be seen to be the K-theory spectrum of the 2-category of complex 2-vector spaces (in the sense of Kapranov–Voevodsky). This clearly shows the existence of an interesting relation between elliptic cohomology and the notion of 2-*vector bundles* (parametrized version of the notion 2-vector spaces), even though the precise relation remains unclear at the moment. On the other hand, the fact that topological K-theory is obtained as the Grothendieck group of vector bundles implies the existence of equivariant K-theory by using equivariant vector bundles. It is important to notice here that the spectrum $BU \times \mathbb{Z}$ alone is not enough to reconstruct equivariant K-theory and that the fact that complex K-theory is obtained from a categorical construction is used in an essential way to define equivariant K-theory. Recently Lurie constructed not only equivariant versions but also 2-*equivariant versions* of elliptic cohomology (see [Lu1, Sect. 5.4]). This means that not only an action of a group can be incorporated in the definition of elliptic cohomology, but also an action of a 2-*group* (i.e., of a categorical group). Now, a 2-group G can not act in a very interesting way on an object in a 1-category, as this action would simply be induced by an action of the group $\pi_0(G)$. However, a 2-group can definitely act in an interesting manner on an object in a 2-category, since automorphisms of a given object naturally form a 2-group. The existence of 2-equivariant version of elliptic cohomology therefore suggests again a close relation between elliptic cohomology and 2-categories.

Perhaps it is also not surprising that our original motivation of understanding which are the geometric objects classified by elliptic cohomology, will lead us below to *loop spaces* (actually a derived version of them, better suited for algebraic geometry: see below). In fact, the chromatic picture of stable homotopy theory, together with the conjectural higher categorical picture recalled above, has a loop space ladder conjectural picture as well. A chromatic type $n+1$ cohomology theory on a space X should be essentially the same thing as a possibly equivariant chromatic type n cohomology theory on the free loop space LX, and the same is expected to be true for the geometric objects classified by these cohomology theories (see e.g., [An-Mo, p. 1]). This is already morally true for $n = 1$, according to Witten intuition of the geometry of the Dirac operator on the loop space ([Wi]): the elliptic cohomology on X is essentially the complex S^1-equivariant K-theory of LX. Actually, this last statement is not literally correct in general, but requires to stick to elliptic cohomology near ∞ and to the "small" loop space, as observed in [Lu1, Rmk. 5.10].[1]

[1] A possibly interesting point here is that our derived loop spaces have a closer relation to small loops than to actual loops.

1.2 Towards a Theory of Categorical Sheaves in Algebraic Geometry

The conclusion of the observation above is that there should exist an interesting notion of *categorical sheaves*, which are sheaves of categories rather than sheaves of vector spaces, useful for a geometric description of objects underlying elliptic cohomology. In this work we have been interested in this notion independently of elliptic cohomology and in the context of algebraic geometry rather than topology. Although our final motivations is to understand better elliptic cohomology, we have found the theory of categorical sheaves in algebraic geometry interesting in its own right and think that it deserves a fully independent development.

To be more precise, and to fix ideas, a categorical sheaf theory is required to satisfy the following conditions.

- For any scheme X there exists a 2-category $Cat(X)$, of categorical sheaves on X. The 2-category $Cat(X)$ is expected to be a symmetric monoidal 2-category. Moreover, we want $Cat(X)$ to be a categorification of the category $Mod(X)$ of sheaves of $\mathcal{O}_X$-modules on X, in the sense that there is a natural equivalence between $Mod(X)$ and the category of endomorphisms of the unit object in $Cat(X)$.
- The 2-category $Cat(X)$ comes equipped with monoidal sub-2-categories $Cat_{\mathrm{qcoh}}(X)$, $Cat_{\mathrm{coh}}(X)$, and $Cat_{\mathrm{parf}}(X)$, which are categorifications of the categories $QCoh(X)$, $Coh(X)$, and $Vect(X)$, of quasi-coherent sheaves, coherent sheaves, and vector bundles. The monoidal 2-category $Cat_{\mathrm{parf}}(X)$ is moreover expected to be rigid (i.e., every object is dualizable).
- For a morphism $f : X \longrightarrow Y$ of schemes, there is a 2-adjunction

$$f^* : Cat(Y) \longrightarrow Cat(X) \qquad Cat(Y) \longleftarrow Cat(X) : f_*.$$

  The 2-functors f^* and f_* are supposed to preserve the sub-2-categories $Cat_{\mathrm{qcoh}}(X)$, $Cat_{\mathrm{coh}}(X)$, and $Cat_{\mathrm{parf}}(X)$, under some finiteness conditions on f.
- There exists a notion of short exact sequence in $Cat(X)$, which can be used in order to define a Grothendieck group $K_0^{(2)}(X) := K_0(Cat_{\mathrm{parf}}(X))$ (or more generally a ring spectrum $K(Cat_{\mathrm{parf}}(X))$). This Grothendieck group is called the *secondary K-theory of X* and is expected to possess the usual functorialities in X (at least pull-backs and push-forwards along proper and smooth morphisms).
- There exists a Chern character

$$K_0^{(2)}(X) \longrightarrow H^{(2)}(X),$$

  for some *secondary cohomology group* $H^{(2)}(X)$. This Chern character is expected to be functorial for pull-backs and to satisfy some version of the Grothendieck–Riemann–Roch formula for push-forwards.

As we will see in Sect. 2, it is not clear how to develop a theory as above, and it seems to us that a theory satisfying all the previous requirements cannot reasonably exist. A crucial remark here is that the situation becomes more easy to handle if the categories $Mod(X)$, $QCoh(X)$, $Coh(X)$, and $Vect(X)$ are replaced by their derived analogs $D(X)$, $D_{\mathrm{qcoh}}(X)$, $D^b_{\mathrm{coh}}(X)$, and $D_{\mathrm{parf}}(X)$. The categorical sheaf theory we look for should then rather be a *derived categorical sheaf theory*, and is expected to satisfy the following conditions.

- To any scheme X is associated a *triangulated-2-category* $Dg(X)$, of *derived categorical sheaves on* X. Here, by *triangulated-2-category* we mean a 2-category whose categories of morphisms are endowed with triangulated structure in a way that the composition functors are bi-exacts. The 2-category $Dg(X)$ is expected to be a symmetric monoidal 2-category, in way which is compatible with the triangulated structure. Moreover, we want $Dg(X)$ to be a categorification of the derived category $D(X)$ of sheaves of $\mathcal{O}_X$-modules on X, in the sense that there is a natural triangulated equivalence between $D(X)$ and the triangulated category of endomorphisms of the unit object in $Dg(X)$.
- The 2-category $Dg(X)$ comes equipped with monoidal sub-2-categories $Dg_{\mathrm{qcoh}}(X)$, $Dg_{\mathrm{coh}}(X)$, and $Dg_{\mathrm{parf}}(X)$, which are categorifications of the derived categories $D_{\mathrm{qcoh}}(X)$, $D^b_{\mathrm{coh}}(X)$, and $D_{\mathrm{parf}}(X)$, of quasi-coherent complexes, bounded coherent sheaves, and perfect complexes. The monoidal 2-category $Dg_{\mathrm{parf}}(X)$ is moreover expected to be *rigid* (i.e., every object is dualizable).
- For a morphism $f : X \longrightarrow Y$ of schemes, there is a 2-adjunction

$$f^* : Dg(Y) \longrightarrow Dg(X) \qquad Dg(Y) \longleftarrow Dg(X) : f_*.$$

  The 2-functors f^* and f_* are supposed to preserve the sub-2-categories $Dg_{\mathrm{qcoh}}(X)$, $Dg_{\mathrm{coh}}(X)$, and $Dg_{\mathrm{parf}}(X)$, under some finiteness conditions on f.
- There exists a notion of short exact sequence in $Dg(X)$, which can be used in order to define a Grothendieck group $K_0^{(2)}(X) := K_0(Dg_{\mathrm{parf}}(X))$ (or more generally a ring spectrum $K(Dg_{\mathrm{parf}}(X))$). This Grothendieck group is called the *secondary K-theory of* X and is expected to possess the usual functorialities in X (at least pull-backs and push-forwards along proper and smooth morphisms).
- There exists a Chern character

$$K_0^{(2)}(X) \longrightarrow H^{(2)}(X),$$

  for some *secondary cohomology group* $H^{(2)}(X)$. This Chern character is expected to be functorial for pull-backs and to satisfy some version of the Grothendieck–Riemann–Roch formula for push-forwards.

The purpose of these notes is to give some ideas on how to define such triangulated-2-categories $Dg(X)$, the secondary cohomology $H^{(2)}(X)$, and finally the Chern character. In order to do this,we will follow closely one possible interpretation of the usual Chern character for vector bundles as being a kind of function on the loop space.

1.3 The Chern Character and the Loop Space

The Chern character we will construct for categorical sheaves is based on the following interpretation of the usual Chern character. Assume that X is a smooth complex algebraic manifold (or more generally a complex algebraic stack) and that V is a vector bundle on X. Let $\gamma : S^1 \longrightarrow X$ be a *loop* in X. We do not want to specify want we mean by a loop here, and the notion of loop we will use in the sequel is a rather unconventional one (see 3.1). Whatever γ truly is, we will think of it as a loop in X, at least intuitively. We consider the pull-back $\gamma^*(V)$, which is a vector bundle on S^1. Because of the notion of loops we use, this vector bundle is in fact locally constant on S^1, and thus is completely determined by a monodromy operator m_γ on the fiber $V_{\gamma(0)}$. The trace of m_γ is a complex number, and as γ varies in LX the loop space of X (again the notion of loop space we use is unconventional) we obtain a function $Ch(V)$ on LX. This function can be seen to be S^1-equivariant and thus provides an element

$$Ch(V) \in \mathcal{O}(LX)^{S^1}.$$

Our claim is that, if the objects S^1 and LX are defined correctly, then there is a natural identification

$$\mathcal{O}(LX)^{S^1} \simeq H^{ev}_{DR}(X),$$

and that $Ch(V)$ is the usual Chern character with values in the algebraic de Rham cohomology of X. The conclusion is that $Ch(V)$ can be seen as a S^1-equivariant function on LX.

One enlightening example is when X is BG, the quotient stack of a finite group G. The our loop space LBG is the quotient stack $[G/G]$, for the action of G on itself by conjugation. The space of functions on LBG can therefore be identified with $\mathbb{C}(G)$, the space of class functions on G. A vector bundle V on BG is nothing else than a linear representation of G, and the function $Ch(V)$ constructed above is the class function sending $g \in G$ to $Tr(g : V \to V)$. Therefore, the description of the Chern character above gives back the usual morphism $R(G) \longrightarrow \mathbb{C}(G)$ sending a linear representation to its class function.

Our construction of the Chern character for a categorical sheaf follows the same ideas. The interesting feature of the above interpretation of the Chern character is that it can be generalized to any setting for which traces of endomorphisms make sense. As we already mentioned, $Dg_{\mathrm{parf}}(X)$ is expected to be a rigid monoidal 2-category, and thus any endomorphism of an object possesses a trace which is itself an object in $D_{\mathrm{parf}}(X) \simeq End(1)$. Therefore, if we start with a categorical sheaf on X and do the same construction as above, we get a sheaf (rather than a function) on LX, or more precisely an object in $D_{\mathrm{parf}}(LX)$. This sheaf is moreover invariant under the action of S^1 and therefore is an object in $D^{S^1}_{\mathrm{parf}}(LX)$, the perfect

S^1-equivariant derived category of LX. This sheaf has itself a Chern character which is an element in $H^{S^1}_{DR}(LX)$, the S^1-equivariant de Rham cohomology of LX. This element is by definition the Chern character of our categorical sheaf. The Chern character should then expected to be a map

$$Ch : K_0^{(2)}(X) \longrightarrow H^{S^1}_{DR}(LX).$$

1.4 Plan of the Paper

The main purpose of this paper is to make precise all the terms of this construction. For this we will start by the definitions of the 2-categories $Dg(X)$, $Dg_{\text{qcoh}}(X)$, and $Dg_{\text{parf}}(X)$, but we do not try to define $Dg_{\text{coh}}(X)$ as the notion of coherence in this categorical setting seems unclear at the moment. The objects in $Dg(X)$ will be certain sheaves of dg-categories on X and our approach to the notion of categorical sheaves heavily relies on the homotopy theory of dg-categories recently studied in [Ta, To2]. In a second part we will recall briefly some ideas of derived algebraic geometry and of derived schemes (and stacks) as introduced in [HAG-II, Lu2]. The loop space LX of a scheme X will then be defined as the derived mapping stack from $S^1 = B\mathbb{Z}$ to X. We will argue that the ring of S^1-invariant functions on LX can be naturally identified with $H^{ev}_{dR}(X)$, the even de Rham cohomology of X, when k has characteristic zero. We will also briefly explain how this can be used in order to interpret the Chern character as we have sketched above. Finally, in a last part we will present the construction of our Chern character for categorical sheaves. One crucial point in this construction is to define an S^1-equivariant sheaf on the loop space LX. The construction of the sheaf itself is easy but the fact that it is S^1-equivariant is a delicate question which we leave open in the present work (see 4.1 and 5.1). Hopefully a detailed proof of the existence of this S^1-equivariant sheaf will appear in a future work (see TV1).

2 Categorification of Homological Algebra and dg-Categories

In this section we present our triangulated-2-categories $Dg(X)$ of derived categorical sheaves on some scheme X. We will start by an overview of a rather standard way to categorify the theory of modules over some base commutative ring using linear categories. As we will see the notion of 2-vector spaces appear naturally in this setting as the dualizable objects, exactly in the same way that the dualizable modules are the projective modules of finite rank. After arguing that this notion of 2-vector space is too rigid a notion to allow for push-forwards, we will consider dg-categories instead and show that they can be used in order to categorify homological algebra in a similar way as linear categories categorify linear algebra. By analogy with the case of modules and linear categories we will consider dualizable objects as categorified versions of perfect complexes and notice that these are precisely the smooth and proper dg-categories studied in [Ko-So, To-Va]. We will finally define the 2-categories $Dg(X)$, $Dg_{\mathrm{qcoh}}(X)$, and $Dg_{\mathrm{parf}}(X)$ for a general scheme X by some gluing procedure.

Let k be a commutative base ring. We let $Mod(k)$ be the category of k-modules, considered as a symmetric monoidal category for the tensor product of modules. Recall that an object $M \in Mod(k)$ is said to be dualizable if the natural morphism

$$M \otimes M^{\vee} \longrightarrow \underline{Hom}(M, M)$$

is an isomorphism (here $\underline{Hom}$ denotes the k-module of k-linear morphisms, and $M^{\vee} := \underline{Hom}(M, k)$ is the dual module). It is easy to see that M is dualizable if and only if it is projective and of finite type over k.

A rather standard way to categorify the category $Mod(k)$ is to consider k-linear categories and Morita morphisms. We let $Cat(k)$ be the 2-category whose objects are small k-linear categories. The category of morphisms between two k-linear categories A and B in $Cat(k)$ is defined to be the category of all $A \otimes_k B^{op}$-modules (the composition is obtained by the usual tensor product of bi-modules). The tensor product of linear categories endow $Cat(k)$ with a structure of a symmetric monoidal 2-category for which k, the k-linear category freely generated by one object, is the unit. We have $End_{Cat(k)}(k) \simeq Mod(k)$, showing that $Cat(k)$ is a categorification of $Mod(k)$. To obtain a categorification of $Mod^{pft}(k)$, the category of projective k-modules of finite type, we consider the sub-2-category of $Cat(k)$ with the same objects but for which the category of morphisms from A to B is the full sub-category of the category of $A \otimes_k B^{op}$-modules whose objects are bi-modules M such that for any $a \in A$ the B^{op}-module $M(a, -)$ is projective of finite type (i.e., a retract of a finite sum of representable B^{op}-modules). We let $Cat^c(k) \subset Cat(k)$ be this sub-2-category, which is again a symmetric monoidal 2-category for tensor product of linear categories. By definition we have $End_{Cat^c(k)}(k) \simeq Mod^{pft}(k)$. However, the tensor category $Mod^{pft}(k)$ is a rigid tensor category in the sense that every object is dualizable, but not every object in $Cat^c(k)$ is dualizable. We therefore consider $Cat^{sat}(k)$ the full sub-2-category of dualizable objects in $Cat^c(k)$. Then, $Cat^{sat}(k)$ is a rigid monoidal 2-category which is a categorification of $Mod^{pft}(k)$. It can be checked that a linear category A is in $Cat^{sat}(k)$ if and only if it is equivalent in $Cat(k)$ to an associative k-algebra B (as usual considered as a linear category with a unique object) satisfying the following two conditions.

1. The k-module B is projective and of finite type over k.
2. For any associative k-algebra A, a $B \otimes_k A$-module M is projective of finite type if and only if it is so as a A-module.

These conditions are also equivalent to the following two conditions.

1. The k-module B is projective and of finite type over k.

2. The $B \otimes_k B^{op}$-module B is projective.

When k is a field, an object in $Cat^{sat}(k)$ is nothing else than a finite dimensional k-algebra B which is universally semi-simple (i.e., such that $B \otimes_k k'$ is semi-simple for any field extension $k \to k'$). In general, an object in $Cat^{sat}(k)$ is a flat family of universally semi-simple finite dimensional algebras over $Spec\, k$. In particular, if k is an algebraically closed field any object in $Cat^{sat}(k)$ is equivalent to k^n, or in other words is a 2-vector space of finite dimension in the sense of Kapranov–Voevodsky (see for instance [Ba-Du-Ro]). For a general commutative ring k, the 2-category $Cat^{sat}(k)$ is a reasonable generalization of the notion of 2-vector spaces and can be called the *2-category of 2-vector bundles on $Spec\, k$*.

One major problem with this notion of 2-vector bundles is the lack of push-forwards in general. For instance, let X be a smooth and proper algebraic variety over some algebraically closed field k. We can consider $\underline{Vect}$, the *trivial 2-vector bundle of rank 1 over X*, which is the stack in categories sending a Zariski open $U \subset X$ to the linear category $Vect(U)$ of vector bundles over U. The push-forward of this trivial 2-vector bundle along the structure morphism $X \longrightarrow Spec\, k$ is the k-linear category of global sections of $\underline{Vect}$, or in other words the k-linear category $Vect(X)$ of vector bundles on X. This is an object in $Cat(k)$, but is definitely not in $Cat^{sat}(k)$. The linear category $Vect(X)$ is big enough to convince anyone that it cannot be *finite dimensional* in any reasonable sense. This shows that the global sections of a 2-vector bundle on a smooth and proper variety is in general not a 2-vector bundle over the base field, and that in general it is hopeless to expect a good theory of proper push-forwards in this setting.

A major observation in this work is that considering a categorification of $D(k)$ instead of $Mod(k)$, which is what we call a *categorification of homological algebra*, solves the problem mentioned above concerning push-forwards. Recall that a dg-category (over some base commutative ring k) is a category enriched over the category of complexes of k-modules (see [Ta]). For a dg-category T we can define its category of T-dg-modules as well as its derived category $D(T)$ by formally inverting quasi-isomorphisms between dg-modules (see [To2]). For two dg-categories T_1 and T_2 we can form their tensor product $T_1 \otimes_k T_2$, as well as their derived tensor product $T_1 \otimes^{\mathbb{L}}_k T_2$ (see [To2]). We now define a 2-catgeory $Dg(k)$ whose objects are dg-categories and whose category of morphisms from T_1 to T_2 is $D(T_1 \otimes^{\mathbb{L}}_k T_2^{op})$. The composition of morphisms is defined using the derived tensor product

$$- \otimes^{\mathbb{L}}_{T_2} - : D(T_1 \otimes^{\mathbb{L}}_k T_2^{op}) \times D(T_2 \otimes^{\mathbb{L}}_k T_3^{op}) \longrightarrow D(T_1 \otimes^{\mathbb{L}}_k T_3^{op}).$$

Finally, the derived tensor product of dg-categories endows $Dg(k)$ with a structure of a symmetric monoidal 2-category.

The symmetric monoidal 2-category $Dg(k)$ is a categorification of the derived category $D(k)$ as we have by definition

$$\underline{End}_{Dg(k)}(1) \simeq D(k).$$

To obtain a categorification of $D_{\text{parf}}(k)$, the perfect derived category, we consider the sub-2-category $Dg^c(k)$ having the same of objects as $Dg(k)$ itself but for which the category of morphisms from T_1 to T_2 in $Dg^c(k)$ is the full sub-category of $D(T_1 \otimes_k^{\mathbb{L}} T_2^{op})$ of bi-dg-modules F such that for all $t \in T_1$ the object $F(t,-) \in D(T^{op})$ is compact (in the sense of triangulated categories, see [Ne]). The symmetric monoidal structure on $Dg(k)$ restricts to a symmetric monoidal structure on $Dg^c(k)$, and we have

$$\underline{End}_{Dg^c(k)}(1) \simeq D_{\text{parf}}(k),$$

as an object of $D(k)$ is compact if and only if it is a perfect complex. Finally, the symmetric monoidal 2-category $Dg^c(k)$ is not rigid and we thus consider $Dg^{sat}(k)$, the full sub-2-category consisting of rigid objects in $Dg^c(k)$. By construction, $Dg^{sat}(k)$ is a rigid symmetric monoidal 2-category and we have

$$\underline{End}_{Dg^{sat}(k)}(1) \simeq D_{\text{parf}}(k).$$

The 2-category $Dg^{sat}(k)$ will be our categorification of $D_{\text{parf}}(k)$ and its objects should be thought as *perfect derived categorical sheaves on the scheme* $Spec\ k$.

It is possible to show that an object T of $Dg^c(k)$ belongs to $Dg^{sat}(k)$ if and only if it is equivalent to an associative dg-algebra B, considered as usual as a dg-category with a unique object, satisfying the following two conditions.

1. The underlying complex of k-modules of B is perfect.
2. The object $B \in D(B \otimes_k^{\mathbb{L}} B^{op})$ is compact.

In other words, a dg-category T belongs to $Dg^{sat}(k)$ if and only it is Morita equivalent to a smooth (condition (2) above) and proper (condition (1) above) dg-algebra B. Such dg-categories are also often called *saturated* (see [Ko-So, To-Va]).

As $Dg^{sat}(k)$ is a rigid symmetric monoidal 2-category we can define, for any object T a trace morphism

$$Tr : \underline{End}_{Dg^{sat}(k)}(T) \longrightarrow \underline{End}_{Dg^{sat}(k)}(1) \simeq D_{\text{parf}}(k).$$

Now, the category $\underline{End}_{Dg^{sat}(k)}(T)$ can be naturally identified with $D(T \otimes_k^{\mathbb{L}} T^{op})_c$, the full sub-category of $D(T \otimes_k^{\mathbb{L}} T^{op})$ of compact objects (here we use that T is saturated and the results of [To-Va, Sect. 2.2]). The trace morphism is then a functor

$$Tr : D(T \otimes_k^{\mathbb{L}} T^{op})_c \longrightarrow D_{\text{parf}}(k)$$

which can be seen to be isomorphic to the functor sending a bi-dg-module M to its Hochschild complex $HH(T,M) \in D_{\text{parf}}(k)$. In particular, the *rank* of an object $T \in Dg^{sat}(k)$, which by definition is the trace of its identity, is its Hochschild complex $HH(T) \in D_{\text{parf}}(k)$.

To finish this section we present the global versions of the 2-categories $Dg(k)$ and $Dg^{sat}(k)$ over some base scheme X. We let $ZarAff(X)$ be the small site of affine Zariski open sub-schemes of X. We start to define a category $dg - cat(X)$ consisting of the following data

1. For any $Spec\ A = U \subset X$ in $ZarAff(X)$, a dg-category T_U over A.
2. For any $Spec\ B = V \subset Spec\ A = U \subset X$ morphism in $ZarAff(X)$ a morphism of dg-categories over A

$$r_{U,V} : T_U \longrightarrow T_V.$$

These data should moreover satisfy the equation $r_{V,W} \circ r_{U,V} = r_{U,W}$ for any inclusion of affine opens $W \subset V \subset U \subset X$. The morphisms in $dg - cat(X)$ are defined in an obvious way as families of dg-functors commuting with the $r_{U,V}$'s.

For $T \in dg - cat(X)$ we define a category $Mod(T)$ of T-dg-modules in the following way. Its objects consist of the following data

1. For any $Spec\ A = U \subset X$ in $ZarAff(X)$, a T_U-dg-module M_U.
2. For any $Spec\ B = V \subset Spec\ A = U \subset X$ morphism in $ZarAff(X)$ a morphism of T_U-dg-modules

$$m_{U,V} : M_U \longrightarrow r^*_{U,V}(M_V).$$

These data should moreover satisfy the usual cocycle equation for $r^*_{U,V}(m_{V,W}) \circ m_{U,V} = m_{U,W}$. Morphisms in $Mod(T)$ are simply defined as families of morphisms of dg-modules commuting with the $m_{U,V}$'s. Such a morphism $f : M \longrightarrow M'$ in $Mod(T)$ is a quasi-isomorphism if it is a stalkwise quasi-isomorphism (note that M and M' are complexes of presheaves of $\mathcal{O}_X$-modules). We denote by $D(T)$ the category obtained from $Mod(T)$ by formally inverting these quasi-isomorphisms.

We now define a 2-category $Dg(X)$ whose objects are the objects of $dg-cat(X)$, and whose category of morphisms from T_1 to T_2 is $D(T_1 \otimes^{\mathbb{L}}_{\mathcal{O}_X} T_2^{op})$ (we pass on the technical point of defining this derived tensor product over $\mathcal{O}_X$, one possibility being to endow $dg - cat(X)$ with a model category structure and to use a cofibrant replacement). The compositions of morphisms in $Dg(X)$ is given by the usual derived tensor product. The derived tensor product endows $Dg(X)$ with a structure of a symmetric monoidal 2-category and we have by construction

$$\underline{End}_{Dg(X)}(1) \simeq D(X),$$

where $D(X)$ is the (unbounded) derived category of all $\mathcal{O}_X$-modules on X.

Definition 2.1. An object $T \in Dg(X)$ is *quasi-coherent* if for any inclusion of affine open subschemes

$$Spec\ B = V \subset Spec\ A = U \subset X$$

the induced morphism

$$T_U \otimes_A B \longrightarrow T_V$$

is a Morita equivalence of dg-categories.

2. A morphism $T_1 \longrightarrow T_2$, corresponding to an object $M \in D(T_1 \otimes^{\mathbb{L}}_{\mathcal{O}_X} T_2^{op})$, is called *quasi-coherent* if its underlying complex of $\mathcal{O}_X$-modules is with quasi-coherent cohomology sheaves.

The sub-2-category of $Dg(X)$ consisting of quasi-coherent objects and quasi-coherent morphisms is denoted by $Dg_{\mathrm{qcoh}}(X)$. It is called the 2-*category of quasi-coherent derived categorical sheaves on* X.

Let T_1 and T_2 be two objects in $Dg_{\mathrm{qcoh}}(X)$. We consider the full sub-category of $D(T_1 \otimes^{\mathbb{L}}_{\mathcal{O}_X} T_2^{op})$ consisting of objects M such that for any Zariski open $Spec\ A = U \subset X$ and any object $x \in (T_1)_U$, the induced dg-module $M(a,-) \in D((T_2^{op})_U)$ is compact. This defines a sub-2-category of $Dg_{\mathrm{qcoh}}(X)$, denoted by $Dg^c_{\mathrm{qcoh}}(X)$ and will be called the sub-2-category of *compact morphisms*. The symmetric monoidal structure on $Dg(X)$ restricts to a symmetric monoidal structure on $Dg_{\mathrm{qcoh}}(X)$ and on $Dg^c_{\mathrm{qcoh}}(X)$.

Definition 2.2. The 2-*category of perfect derived categorical sheaves* is the full sub-2-catgeory of $Dg^c_{\mathrm{qcoh}}(X)$ consisting of dualizable objects. It is denoted by $Dg_{\mathrm{parf}}(X)$.

By construction, $Dg_{\mathrm{parf}}(X)$ is a symmetric monoidal 2-category with

$$\underline{End}_{Dg_{\mathrm{parf}}(X)}(1) \simeq D_{\mathrm{parf}}(X).$$

It is possible to show that an object $T \in D_{\mathrm{qcoh}}(X)$ belongs to $Dg_{\mathrm{parf}}(X)$ if and only if for any affine Zariski open subscheme $Spec\ A = U \subset X$, the dg-category T_U is saturated (i.e., belongs to $Dg^{sat}(A)$).

For a morphism of schemes $f : X \longrightarrow Y$ it is possible to define a 2-adjunction

$$f^* : Dg(Y) \longrightarrow Dg(X) \qquad Dg(Y) \longleftarrow Dg(X) : f_*.$$

Moreover, f^* preserves quasi-coherent objects, quasi-coherent morphisms, as well as the sub-2-categories of compact morphisms and perfect objects. When the morphism f is quasi-compact and quasi-separated, we think that it is possible to prove that f_* preserves quasi-coherent objects and quasi-coherent morphisms as well as the sub-2-category of compact morphisms. We also guess that f_* will preserve perfect objects when f is smooth and proper, but this would require a precise investigation. As a typical example, the direct image of the unit $1 \in Dg_{\mathrm{parf}}(X)$ by f is the presheaf of dg-categories sending $Spec\ A = U \subset Y$ to the dg-category $L_{\mathrm{parf}}(X \times_Y U)$ of perfect complexes over the scheme $f^{-1}(U) \simeq X \times_Y U$. When f is smooth and proper it is known that the dg-category $L_{\mathrm{parf}}(X \times_Y U)$ is in fact saturated (see [To2, Sect. 8.3]). This shows that $f_*(1)$ is a perfect derived sheaf on Y and provides an evidence that f_* preserves perfect object. These functoriality statements will be considered in more details in a future work.

3 Loop Spaces in Derived Algebraic Geometry

In this section we present a version of the loop space of a scheme (or more generally of an algebraic stack) based on derived algebraic geometry. For us the circle S^1 is defined to be the quotient stack $B\mathbb{Z}$, where $\mathbb{Z}$ is considered as a constant sheaf of groups. For any scheme X, the mapping stack $\mathbf{Map}(S^1, X)$ is then equivalent to X, as the coarse moduli space of S^1 is simply a point. In other words, with this definition of the circle there are no interesting loops on a scheme X. However, we will explain in the sequel that there exists an interesting *derived mapping stack* $\mathbb{R}\mathbf{Map}(S^1, X)$, which is now a derived scheme and which is non-trivial. This derived mapping stack will be our loop space. In this section we recall briefly the notions from derived algebraic geometry needed in order to define the object $\mathbb{R}\mathbf{Map}(S^1, X)$. We will also explain the relation between the cohomology of $\mathbb{R}\mathbf{Map}(S^1, X)$ and cyclic homology of the scheme X.

Let k be a base commutative ring and denote by $\mathbf{Sch}_k$, resp. $\mathbf{St}_k$, the category of schemes over k and the model category of stacks over k ([HAG-II, 2.1.1] or [To1, Sect. 2, 3]), for the étale topology. We recall that the homotopy category $\mathrm{Ho}(\mathbf{St}_k)$ contains as full sub-categories the category of sheaves of sets on $\mathbf{Sch}_k$ as well as the 1-truncation of the 2-category of stacks in groupoids (in the sense of [La-Mo]). In particular, the homotopy category of stacks $\mathrm{Ho}(\mathbf{St}_k)$ contains the category of schemes and of Artin stacks as full sub-categories. In what follows we will always consider these two categories as embedded in $\mathrm{Ho}(\mathbf{St}_k)$. Finally, recall that the category $\mathrm{Ho}(\mathbf{St}_k)$ possesses internal Hom's, that will be denoted by $\mathbf{Map}$.

As explained in [HAG-II, Chap. 2.2] (see also [To1, Sect. 4] for an overview), there is also a model category $\mathbf{dSt}_k$ of derived stacks over k for the strong étale topology. The derived affine objects are simplicial k-algebras, and the model category of simplicial k-algebras will be denoted by $\mathbf{salg}_k$. The opposite (model) category is denoted by $\mathbf{dAff}_k$. Derived stacks can be identified with objects in the homotopy category $\mathrm{Ho}(\mathbf{dSt}_k)$ which in turn can be identified with the full subcategory of the homotopy category of simplicial presheaves on $\mathbf{dAff}_k$ whose objects are weak equivalences' preserving simplicial presheaves F having strong étale descent i.e., such that, for any étale homotopy hypercover $U_\bullet \to X$ in $\mathbf{dAff}_k$ ([HAG-I, Definition 3.2.3, 4.4.1]), the canonical map

$$F(X) \longrightarrow \mathrm{holim} F(U_\bullet)$$

is a weak equivalence of simplicial sets. The derived Yoneda functor induces a fully faithful functor on homotopy categories

$$\mathbb{R}\mathrm{Spec} : \mathrm{Ho}(\mathbf{dAff}_k) \hookrightarrow \mathrm{Ho}(\mathbf{dSt}_k) \ : A \mapsto (\mathbb{R}\mathrm{Spec}(A) \ : B \mapsto \mathrm{Map}_{\mathbf{salg}_k}(A, B)),$$

where $\mathrm{Map}_{\mathbf{salg}_k}$ denotes the mapping spaces of the model category $\mathbf{salg}_k$ (therefore $\mathrm{Map}_{\mathbf{salg}_k}(A, B) \simeq \underline{Hom}(Q(A), B)$, where $\underline{Hom}$ denotes the natural simplicial Hom's of $\mathbf{salg}_k$ and $Q(A)$ is a cofibrant model for A). Those derived stacks belonging to the essential image of $\mathbb{R}\mathrm{Spec}$ will be called *affine* derived stacks.

The category $\mathrm{Ho}(\mathbf{dSt}_k)$ of derived stacks has a lot of important properties. First of all, being the homotopy category of a model category, it has derived colimits and limits (denoted as hocolim and holim). In particular, given any pair of maps $F \to S$ and $G \to S$ between derived stacks, there is a derived fiber product stack $\mathrm{holim}(F \to S \leftarrow G) \equiv F \times^h_S G$. As our base ring k is not assumed to be a field, the direct product in the model category $\mathbf{dSt}_k$ is not exact and should also be derived. The derived direct product of two derived stacks F and G will be denoted by $F \times^h G$. This derived product is the categorical product in the homotopy category $\mathrm{Ho}(\mathbf{dSt}_k)$. The category $\mathrm{Ho}(\mathbf{dSt}_k)$ also admits internal Hom's, i.e., for any pair of derived stacks F and G there is a derived mapping stack denoted as

$$\mathbb{R}\mathbf{Map}(F, G)$$

with the property that

$$[F, \mathbb{R}\mathbf{Map}(G, H)] \simeq [F \times^h G, H],$$

functorially in F, G, and H.

The inclusion functor j of commutative k-algebras into $\mathbf{salg}_k$ (as constant simplicial algebras) induces a pair (i, t_0) of (left,right) adjoint functors

$$t_0 := j^* : \mathrm{Ho}(\mathbf{dSt}_k) \to \mathrm{Ho}(\mathbf{St}_k) \qquad i := \mathbb{L}j_! : \mathrm{Ho}(\mathbf{St}_k) \to \mathrm{Ho}(\mathbf{dSt}_k).$$

It can be proved that i is fully faithful. In particular we can, and will, view any stack as a derived stack (we will most of the time omit to mention the functor i and consider $\mathrm{Ho}(\mathbf{St}_k)$ as embedded in $\mathrm{Ho}(\mathbf{dSt}_k)$). The truncation functor t_0 acts on affine derived stacks as $t_0(\mathbb{R}\mathrm{Spec}(A)) = \mathrm{Spec}(\pi_0 A)$. It is important to note that the inclusion functor i does not preserve derived internal hom's nor derived fibered products. This is a crucial point in derived algebraic geometry: derived tangent spaces and derived fiber products of usual schemes or stacks are really derived objects. The derived tangent space of an Artin stack viewed as a derived stack via i is the dual of its cotangent complex while the derived fiber product of, say, two affine schemes viewed as two derived stacks is given by the derived tensor product of the corresponding commutative algebras

$$i(\mathrm{Spec}\, S) \times^h_{i(\mathrm{Spec}\, R)} i(\mathrm{Spec}\, T) \simeq \mathbb{R}\mathrm{Spec}\,(S \otimes^{\mathbb{L}}_R T).$$

Both for stacks and derived stacks there is a notion of being *geometric* ([HAGII, 1.3, 2.2.3]), depending, among other things, on the choice of a notion of smooth morphism between the affine pieces. For morphisms of commutative k-algebras this is the usual notion of smooth morphism, while in the derived case, a morphism $A \to B$ of simplicial k-algebras is said to be strongly smooth if the induced map $\pi_0 A \to \pi_0 B$ is a smooth morphism of commutative rings, and $\pi_* A \otimes_{\pi_0 A} \pi_0 B \simeq \pi_* B$. The notion of geometric stack is strictly related to the notion of Artin stack ([HAG-II, Proposition 2.1.2.1]). Any geometric derived stack has a cotangent complex ([HAG-II, Corollary 2.2.3.3]). Moreover, both functors t_0 and i preserve geometricity.

Let $\mathrm{B}\mathbb{Z}$ be the classifying stack of the constant group scheme $\mathbb{Z}$. We view $\mathrm{B}\mathbb{Z}$ as an object of $\mathbf{St}_k$, i.e., as the stack associated to the constant simplicial presheaf

$$\mathrm{B}\mathbb{Z} : \mathbf{alg}_k \to \mathbf{SSets}\,, R \mapsto \mathrm{B}\mathbb{Z},$$

where by abuse of notations, we have also denoted as $\mathrm{B}\mathbb{Z}$ the classifying simplicial set, i.e., the nerve of the (discrete) group $\mathbb{Z}$. Such a nerve is naturally a pointed simplicial set, and we call 0 that point.

Definition 3.1. Let X be a derived stack over k. The *derived loop stack* of X is the derived stack

$$\mathrm{L}X := \mathbb{R}\mathbf{Map}(\mathrm{B}\mathbb{Z}, X).$$

We will be mostly interested in the case where X is a scheme or an algebraic (underived) stack. Taking into account the homotopy equivalence

$$\mathrm{B}\mathbb{Z} \simeq S^1 \simeq * \coprod_{*\coprod *}^{h} *\,,$$

we see that we have

$$\mathrm{L}X \simeq X \times^h_{X\times^h X} X,$$

where X maps to $X \times^h X$ diagonally and the homotopy fiber product is taken in $\mathbf{dSt}_k$. Evaluation at $0 \in \mathrm{B}\mathbb{Z}$ yields a canonical map of derived stacks

$$p : \mathrm{L}X \longrightarrow X.$$

On the other hand, since the limit maps canonically to the homotopy limit, we get a canonical morphism of derived stacks $X \to \mathrm{L}X$, a section of p, describing X as the "constant loops" in $\mathrm{L}X$.

If X is an affine scheme over k, $X = \operatorname{Spec} A$ with A a commutative k-algebra, we get that

$$\mathrm{L}X \simeq \mathbb{R}\mathrm{Spec}(A \otimes^{\mathbb{L}}_{A\otimes^{\mathbb{L}} A} A),$$

where the derived tensor product is taken in the model category $\mathbf{salg}_k$. One way to rephrase this is by saying that "functions" on $\mathrm{L}\operatorname{Spec} A$ *are* Hochschild homology classes of A with values in A itself. Precisely, we have

$$\mathcal{O}(\mathrm{L}X) := \mathbb{R}\underline{Hom}(\mathrm{L}X, \mathbb{A}^1) \simeq \mathrm{HH}(A, A),$$

where $\mathrm{HH}(A, A)$ is the simplicial set obtained from the complex of Hochschild homology of A by the Dold–Kan correspondence, and $\mathbb{R}\underline{Hom}$ denotes the natural enrichment of $\mathrm{Ho}(\mathbf{dSt}_k)$ into $\mathrm{Ho}(SSet)$. When X is a general scheme then $\mathcal{O}(\mathrm{L}X)$ can be identified with the Hochschild homology complex of X, and we have

$$\pi_i(\mathcal{O}(\mathrm{L}X)) \simeq HH_i(X).$$

In particular, when X is a smooth and k is of characteristic zero, the Hochschild–Kostant–Rosenberg theorem implies that

$$\pi_0(\mathcal{O}(\mathrm{L}X)) \simeq \oplus_i H^i(X, \Omega^i_{X/k}).$$

The stack $S^1 = \mathrm{B}\mathbb{Z}$ is a group stack, and it acts naturally on $\mathrm{L}X$ for any derived stack X by "rotating the loops." More precisely, there is a model category $\mathbf{dSt}^{S^1}_{/k}$, or S^1-equivariant stacks, and $\mathrm{L}X$ is naturally an object in the homotopy category $\mathrm{Ho}(\mathbf{dSt}^{S^1}_{/k})$. This way, the simplicial algebra of functions $\mathcal{O}(\mathrm{L}X)$ is naturally an S^1-equivariant simplicial algebra, and thus can also be considered as an S^1-equivariant complex or in other words as an object in $D^{S^1}(k)$, the S^1-equivariant derived category of k. The category $D^{S^1}(k)$ is also naturally equivalent to $D(k[\epsilon])$, the derived category of the dg-algebra $k[\epsilon]$ freely generated by an element ϵ of degree -1 and with $\epsilon^2 = 0$. The derived category $D^{S^1}(k)$ is thus naturally equivalent to the derived category of mixed complexes (see [Lo]) (multiplication by ϵ providing the second differential). When k is of characteristic zero and $X = \mathrm{Spec}\ A$ one can show (see TV2) that

$$\pi_i(\mathcal{O}(\mathrm{L}X)^{hS^1}) \simeq H^{ev}_{dR}(A),$$

where K^{hS^1} denotes the simplicial set of homotopy fixed points of an S^1-equivariant simplicial set K. In other words, there is a natural identification between S^1-invariant functions on $\mathrm{L}X$ and even de Rham cohomology of X. This statement of course can be generalized to the case of a scheme X.

Proposition 3.2. *For a scheme X we have*

$$\pi_0(\mathcal{O}(\mathrm{L}X)^{hS^1}) \simeq H^{ev}_{dR}(X),$$

In the first version of this paper, we conjectured a similar statement, for k of any characteristic and with even de Rham cohomology replaced by negative cyclic homology. At the moment, however, we are only able to prove the weaker results above (see TV2).

The even part of de Rham cohomology of X can be identified with S^1-equivariant functions on the derived loop space $\mathrm{L}X$. This fact can also be generalized to the case where X is a smooth Deligne–Mumford stack over k (again assumed to be of characteristic zero), but the right hand side should rather be replaced by the (even part of) de Rham orbifold cohomology of X, which is the de Rham cohomology of the inertia stack $IX \simeq t_0(\mathrm{L}X)$.

To finish this part, we would like to mention that the construction of the Chern character for vector bundles we suggested in Sect. 1 can now be made precise, and through the identification of Proposition 3.2, this Chern character coincides with the usual one. We start with a vector bundle V on X and we consider its pull-back $p^*(V)$ on $\mathrm{L}X$, which is a vector bundle on the derived scheme $\mathrm{L}X$. This vector bundle $p^*(V)$ comes naturally equipped with an automorphism u. This follows by considering the evaluation morphism $\pi : S^1 \times \mathrm{L}X \longrightarrow X$, and the vector bundle $\pi^*(V)$. As $S^1 = B\mathbb{Z}$, a vector bundle on $S^1 \times \mathrm{L}X$ consists precisely of a vector bundle on $\mathrm{L}X$ together with an action of $\mathbb{Z}$, or in other words together with an automorphism. We can then consider the trace of u, which is an element in $\pi_0(\mathcal{O}(\mathrm{L}X)) \simeq HH_0(X)$. A difficult issue here is to argue that this function $Tr(u)$ has a natural refinement to an S^1-invariant function $Tr(u) \in \pi_0(\mathcal{O}(\mathrm{L}X)^{hS^1}) \simeq H^{ev}_{dR}(X/k)$, which is the Chern character of V. The S^1-invariance of $Tr(u)$ will be studied in a future work, and we refer to our last section below, for some comments about how this would follow from the general theory of rigid tensor ∞-categories.

4 Construction of the Chern Character

We are now ready to sketch the construction of our Chern character for a derived categorical sheaf. This construction simply follows the lines we have just sketched for vector bundles. We will meet the same difficult issue of the existence of an S^1-invariant refinement of the trace, and we will leave this question as an conjecture. However, in the next section we will explain how this would follow from a very general fact about rigid monoidal ∞-categories.

Let $T \in Dg_{\mathrm{parf}}(X)$ be a perfect derived categorical sheaf on some scheme X (or more generally on somealgebraicc stack X). We consider the natural morphism $p : \mathrm{L}X \longrightarrow X$ and we consider $p^*(T)$, which is a perfect derived categorical sheaf on $\mathrm{L}X$. We have not defined the notions of categorical sheaves on derived schemes or derived stacks but this is rather straightforward. As in the case of vector bundles explained in the last section, the object $p^*(T)$ comes naturally equipped with an autoequivalence u. This again follows from the fact that a derived categorical sheaf on $S^1 \times \mathrm{L}X$ is the same thing as a derived categorical sheaf on $\mathrm{L}X$ together with an autoequivalence. We consider the trace of u in order to get a perfect complex on the derived loop space

$$Tr(u) \in \underline{End}_{Dg_{\mathrm{parf}}(\mathrm{L}X)}(1) = D_{\mathrm{parf}}(\mathrm{L}X).$$

The main technical difficulty here is to show that $Tr(u)$ possesses a natural lift as an S^1-equivariant complex on $\mathrm{L}X$. We leave this as a conjecture.

Conjecture 4.1. The complex $Tr(u)$ has a natural lift

$$Tr^{S^1}(u) \in D^{S^1}_{\mathrm{parf}}(\mathrm{L}X),$$

where $D^{S^1}_{\mathrm{parf}}(\mathrm{L}X)$ is the S^1-equivariant perfect derived category of $\mathrm{L}X$.

The above conjecture is not very precise as the claim is not that a lift simply exists, but rather than there exists a natural one. One of the difficulty in the conjecture above is that it seems difficult to characterize the required lift by some specific properties. We will see however that the conjecture can be reduced to a general conjecture about rigid monoidal ∞-categories.

Assuming Conjecture 4.1, we have $Tr^{S^1}(u)$ and we now consider its class in the Grothendieck group of the triangulated category $D^{S^1}_{\mathrm{parf}}(\mathrm{L}X)$. This is our definition of the categorical Chern character of T.

Definition 4.2. The categorical Chern character of T is

$$Ch^{cat}(T) := [Tr^{S^1}(u)] \in K_0^{S^1}(\mathrm{L}X) := K_0(D^{S^1}_{\mathrm{parf}}(\mathrm{L}X)).$$

The categorical Chern character $Ch^{cat}(T)$ can be itself refined into a *cohomological Chern character* by using now the S^1-equivariant Chern character map for S^1equivariantt perfect complexes on $\mathrm{L}X$. We skip some technical details here but the final result is an element

$$Ch^{coh}(T) := Ch^{S^1}(Ch^{cat}(T)) \in \pi_0(\mathcal{O}(\mathrm{L}^{(2)}X)^{h(S^1\times S^1)}),$$

where $\mathrm{L}^{(2)}X := \mathbb{R}\mathbf{Map}(S^1 \times S^1, X)$ is now the derived double loop space of X. The space $\pi_0(\mathcal{O}(\mathrm{L}^{(2)}X)^{h(S^1\times S^1)})$ can reasonably be called the *secondary negative cyclic homology of* X and should be thought (and actually is) the S^1-equivariant negative cyclic homology of $\mathrm{L}X$. We therefore have

$$Ch^{coh}(T) \in HC_0^{-,S^1}(\mathrm{L}X).$$

Definition 4.3. The cohomological Chern character of T is

$$Ch^{coh}(T) := Ch^{S^1}(Ch^{cat}(T)) \in HC_0^{-,S^1}(\mathrm{L}X) := \pi_0(\mathcal{O}(\mathrm{L}^{(2)}X)^{hS^1\times S^1})$$

defined above.

Obviously, it is furthermore expected that the constructions $T \mapsto Ch^{cat}(T)$ and $T \mapsto Ch^{coh}(T)$ satisfy standard properties such as additivity, multiplicativity and functoriality with respect to pull-backs. The most general version of our Chern character map should be a morphism of commutative ring spectra

$$Ch^{cat} : Kg(X) \longrightarrow K^{S^1}(\mathrm{L}X),$$

where $Kg(X)$ is a ring spectrum constructed using a certain Waldhausen category of perfect derived categorical sheaves on X and $K^{S^1}(\mathrm{L}X)$ is the K-theory spectrum of S^1-equivariant perfect complexes on $\mathrm{L}X$. This aspect of the Chern character will be investigated in more details in a future work.

5 Final Comments

On S^1-equivariant trace maps. Our Conjecture 4.1 can be clarified using the language of higher categories. Recall that a $(1,\infty)$-category is an ∞-category in which all n-morphisms are invertible (up to higher morphisms) as soon as $n > 1$. There exist several well behaved models for the theory of $(1,\infty)$-categories, such as simplicially enriched categories, quasi-categories, Segal categories, and Rezk's spaces. We refer to [Ber1] for an overview of these various notions. What we will say below can be done in any of these theories, but, to fix ideas, we will work with $\mathbb{S}$-categories (i.e., simplicially enriched categories).

We will be using $Ho(\mathbb{S}-Cat)$ the homotopy category of $\mathbb{S}$-categories, which is the category obtained from the category of $\mathbb{S}$-categories and $\mathbb{S}$-functors by inverting the (Dwyer–Kan) equivalences (a mixture between weak equivalences of simplicial sets and categorical equivalences). An important property of $Ho(\mathbb{S}-Cat)$ is that it is cartesian closed (see [To2] for the corresponding statement for dg-categories whose proof is similar). In particular, for two $\mathbb{S}$-categories C and C' we can construct an $\mathbb{S}$-category $\mathbb{R}\underline{Hom}(C,C')$ with the property that

$$[C'', \mathbb{R}\underline{Hom}(C,C')] \simeq [C'' \times C, C'],$$

where $[-,-]$ denote the Hom sets of $Ho(\mathbb{S}-Cat)$. Any $\mathbb{S}$-category C gives rise to a genuine category $[C]$ with the same objects and whose sets of morphisms are the connected components of the simplicial sets of morphisms of C.

We let Γ be the category of pointed finite sets and pointed maps. The finite set $\{0,\dots,n\}$ pointed at 0 will be denoted by n^+. Now, a *symmetric monoidal $\mathbb{S}$-category M* is a functor

$$M : \Gamma \longrightarrow \mathbb{S}-Cat$$

such that for any $n \geq 0$ the so-called *Segal morphism*

$$M(n^+) \longrightarrow M(1^+)^n,$$

induced by the various projections $n^+ \to 1^+$ sending $i \in \{1,\dots,n\}$ to 1 and everything else to 0, is an equivalence of $\mathbb{S}$-categories. The full sub-category of the homotopy category of functors $Ho(\mathbb{S}-Cat^{\Gamma})$ consisting of symmetric monoidal $\mathbb{S}$-categories will be denoted by $Ho(\mathbb{S}-Cat^{\otimes})$. As the category $Ho(\mathbb{S}-Cat)$ is a model for the homotopy category of $(1,\infty)$-categories, the category $Ho(\mathbb{S}-Cat^{\otimes})$ is a model for the homotopy category of symmetric monoidal $(1,\infty)$-categories. For $M \in Ho(\mathbb{S}-Cat^{\otimes})$ we will again use M to denote its underlying $\mathbb{S}$-category $M(1^+)$. The $\mathbb{S}$-category $M(1^+)$ has a natural structure of a commutative monoid in $Ho(\mathbb{S}-Cat)$. This monoid structure will be denoted by $\otimes$.

We say that a symmetric monoidal $\mathbb{S}$-category M is *rigid* if for any object $x \in M$ there is anobjectc $x^{\vee} \in M$ and a morphism $1 \to x \otimes x^{\vee}$ such that for any pair of objects $y,z \in M$, the induced morphism of simplicial sets

$$M(y \otimes x, z) \longrightarrow M(y \otimes x \otimes x^{\vee}, z \otimes x^{\vee}) \longrightarrow M(y, z \otimes x^{\vee})$$

is an equivalence. In particular, the identity of $x^{\vee}$ provides a trace morphism $x \otimes x^{\vee} \to 1$ ($y = x^{\vee}$, $z = 1$). Therefore, for any rigid symmetric monoidal $\mathbb{S}$-category M and an object $x \in M$ we can define a trace morphism

$$Tr_x : M(x, x) \simeq M(1, x \otimes x^{\vee}) \longrightarrow M(1, 1).$$

Let M be a fixed rigid symmetric monoidal $\mathbb{S}$-category and $S^1 = B\mathbb{Z}$ be the groupoid with a unique object with $\mathbb{Z}$ as automorphism group. The category S^1 is an abelian group object in categories and therefore can be considered as a group object in $\mathbb{S}$-categories. The $\mathbb{S}$-category of functors $\mathbb{R}\underline{Hom}(S^1, M)$ is denoted by $M(S^1)$, and is equipped with a natural action of S^1. We consider the sub-$\mathbb{S}$-category of invertible (up to homotopy) morphisms in $M(S^1)$ whose classifying space is an S^1-equivariant simplicial set. We denote this simplicial set by LM ("L" stands for "loops"). It is possible to collect all the trace morphisms Tr_x defined above into a unique morphism of simplicial sets (well defined in $Ho(SSet)$)

$$Tr : LM \longrightarrow M(1, 1).$$

Note that the connected components of LM are in one to one correspondence with the set of equivalences classes of pairs (x, u), consisting of an object x in M and an autoequivalence u of x. The morphism Tr is such that $Tr(x, u) = Tr_x(u) \in \pi_0(M(1, 1))$. We are in fact convinced that the trace map Tr can be made equivariant for the action of S^1 on LM, functorially in M. To make a precise conjecture, we consider $\mathbb{S} - Cat^{rig}$ the category of all rigid symmetric monoidal $\mathbb{S}$-categories (note that $\mathbb{S} - Cat^{rig}$ is not a homotopy category, it is simply a full sub-category of $\mathbb{S} - Cat^{\Gamma}$). We have two functors

$$\mathbb{S} - Cat^{rig} \longrightarrow S^1 - SSet,$$

to the category of S^1-equivariant simplicial sets. The first one sends M to LM together with its natural action of S^1. The second one sends M to $M(1, 1)$ with the trivial S^1-action. These two functors are considered as objects in $Ho(Fun(\mathbb{S} - Cat^{rig}, S^1 - SSet))$, the homotopy category of functors. Let us denote these two objects by $L : M \mapsto LM$ and $E : M \mapsto M(1, 1)$.

Conjecture 5.1. There exists a morphism in $Ho(Fun(\mathbb{S} - Cat^{rig}, S^1 - SSet))$

$$Tr : L \longrightarrow E,$$

in such a way that for any rigid symmetric monoidal $\mathbb{S}$-category T the induced morphism of simplicial sets

$$Tr : LM \longrightarrow M(1,1)$$

is the trace map described above.

It can be shown that Conjecture 5.1 implies Conjecture 4.1. In fact the tensor 2-categories $Dg_{\text{parf}}(X)$ are the 2-truncation of natural rigid symmetric monoidal $(2,\infty)$-categories, which can also be considered as $(1,\infty)$-categories by only considering invertible higher morphisms. An application of the above conjecture to these rigid symmetric monoidal $(1,\infty)$-categories give a solution to Conjecture 4.1, but this will be explained in more detailed in a future work. To finish this part on rigid $(1,\infty)$-categories let us mention that a recent work of Lurie and Hopkins on universal properties of $(1,\infty)$-categories of 1-bordisms seems to solve Conjecture 5.1 ([Lu3]). We think we have another solution to the part of Conjecture 5.1 concerned with the rigid symmetric monoidal $(1,\infty)$-category of saturated dg-categories, which is surely enough to imply Conjecture 4.1. This again will be explained in a future work.

The Chern character as part of a 1-TFT over X. The recent work of Lurie and Hopkins (see [Lu3]) mentioned above also allows us to fit our Chern character into the framework of 1-dimensional topological field theories. Indeed, let X be a scheme and $T \in Dg_{\text{parf}}(X)$ be a perfect derived categorical sheaf on X. It is represented by a morphism of derived stacks

$$T : X \longrightarrow Dg_{\text{parf}},$$

where Dg_{parf} is considered here as a symmetric monoidal $(1,\infty)$-category as explained above. The object Dg_{parf} is thus a *derived stack in rigid symmetric monoidal* $(1,\infty)$*-categories*, and the results of [Lu3] imply that the morphism T induces a well defined morphism of derived stacks in rigid symmetric monoidal $(1,\infty)$-categories

$$1 - Bord(X) \longrightarrow Dg_{\text{parf}}.$$

Here $1 - Bord(X)$ is the categorical object in derived stacks of relative 1-bordisms over X (see [Lu3] for details). In terms of functor of points it is defined by sending a simplicial commutative algebra A to the $(1,\infty)$-category $1 - Bord(X(A))$ of relative 1-bordisms over the space $X(A)$. It is not hard to see that $1 - Bord(X)$ is itself a derived algebraic stack, and thus a *rigid symmetric monoidal categorical object in algebraic derived stacks*. The morphism

$$1 - Bord(X) \longrightarrow Dg_{\text{parf}}$$

is therefore a relative *1-dimensional TQFT over X and takes its values in Dg_{parf}*. Our Chern character can be easily reconstructed from this 1-dimensional field theory by considering its restriction to the space of endomorphisms of the unit object $\emptyset \to X$. Indeed, this space contains as a component the *moduli space of oriented circles over X*, which is nothing else than $[LX/S^1]$ (the quotient stack of LX by the action of S^1). From the 1-dimensional field theory above we thus extract a morphism of derived stacks

$$[LX/S^1] \longrightarrow End_{Dg_{\mathrm{parf}}}(1) \simeq D_{\mathrm{parf}},$$

whose datum is equivalent to the datum of an object in $D^{S^1}_{\mathrm{parf}}(LX)$. This object turns out to be isomorphic to our $Ch^{cat}(T)$.

Relations with variations of Hodge structures (see TV1). The derived loop space $\mathrm{L}X$ and the S^1-equivariant derived category $D^{S^1}_{\mathrm{parf}}(\mathrm{L}X)$ have already been studied in [Be-Na]. In this work the category $D^{S^1}_{\mathrm{parf}}(\mathrm{L}X)$ is identified with a certain derived category of modules over the Rees algebra of differential operators on X (when, say, X is smooth over k of characteristic zero). We do not claim to fully understand this identification but it seems clear that objects in $D^{S^1}_{\mathrm{parf}}(\mathrm{L}X)$ can be identified with some kind of filtered complexes of D-modules on X. Indeed, when $X = Spec\ A$ is smooth and affine over k of characteristic zero, the category $D^{S^1}_{\mathrm{parf}}(\mathrm{L}X)$ can be identified with the derived category of dg-modules over the dg-algebra $\Omega^*_A[\partial]$, generated by the graded algebra of differential forms (here Ω^i_A sits in degree $-i$) together with an extra element ∂ of degree -1 acting as the de Rham differential on Ω^*_A. This dg-algebra is itself Kozsul dual to the Rees algebra $\mathcal{R}_X$ of differential operators on X defined as follows. The dg-algebra $\mathcal{R}_X$ is generated by $\oplus_i \mathcal{D}^{\leq i}_X[-2i]$ together with an element u of degree 2 acting by the natural inclusions $\mathcal{D}^{\leq i}_X \subset \mathcal{D}^{\leq i+1}_X$ (here $\mathcal{D}^{\leq i}_X$ denotes as usual the ring of differential operators of degree less than i). We thus have an equivalence $D^{S^1}_{\mathrm{parf}}(LX) \simeq D_{\mathrm{parf}}(\mathcal{R}_X)$. Now, apart from its unusual grading, $\mathcal{R}_X$ is essentially the Rees algebra associated to the filtered ring $\mathcal{D}_X$, and thus $D_{\mathrm{parf}}(\mathcal{R}_X)$ is essentially the derived category of filtered D-modules (which are perfect over X, and with this unusual grading).

Using this identification our categorical Chern character $Ch^{cat}(T)$ probably encodes the data of the negative cyclic complex $HC^-(T)$ of T over X together with its *Gauss–Manin connection and Hodge filtration*. In other words, $Ch^{cat}(T)$ seems to be nothing more than the variation of Hodge structures induced by the family of dg-categories T over X. As far as we know the construction of such a structure of variations of Hodge structures on the family of complexes of cyclic homology associated to a family of saturated dg-categories is a new result (see however [Ge] for the construction of a Gauss–Manin connection on cyclic homology). We also think it is a remarkably nice fact that variations of Hodge structures appear naturally out of the construction of our Chern character for categorical sheaves.

It is certainly possible to describe the cohomological Chern character of 4.3 using this point of view of Hodge structure. Indeed, $HC_0^{-,S^1}(\mathrm{L}X)$ is close to be the S^1-equivariant de Rham cohomology of $\mathrm{L}X$, and using a localization formula it is probably possible to relate $HC_0^{-,S^1}(\mathrm{L}X)$ with $HP_0(X)[[t]][t^{-1}]$, where $HP_0(X)$ is periodic cyclic homology of X and t is a formalparameterr. We expect at least a morphism

$$HC_0^{-,S^1}(\mathrm{L}X) \longrightarrow HP_0(X)[[t]][t^{-1}].$$

The image of $Ch^{coh}(T)$ under this map should then be closely related to the *Hodge polynomial of* T, that is $\sum_p Ch(Gr^p HC^-(T))t^p$, where $Gr^p HC^-(T)$ is the p-th graded piece of the Hodge filtration of Hochschild homology and Ch is the usual Chern character for sheaves on X.

Back to elliptic cohomology ? In the Introduction we mentioned that our motivation for thinking about categorical sheaf theory originated from elliptic cohomology. However, our choice to work in the context of algebraic geometry drove us rather far from elliptic cohomology and it is at the moment unclear whether our work on the Chern character can really bring any new insight on elliptic cohomology. About this we would like to make the following remark. Since what we have been considering are categorified version of algebraic vector bundles, it seems rather clear that what we have done so far might have some relations with what could be called *algebraic elliptic cohomology* (by analogy with the distinction between algebraic and topological K-theory). However, the work of Walker shows that algebraic K-theory determines completely topological K-theory (see [Wa]), and that it is possible to recover topological K-theory from algebraic K-theory by an explicit construction. Such a striking fact suggests the possibility that understanding enough about the algebraic version of elliptic cohomology could also provide some new insights on usual elliptic cohomology. We leave this vague idea to future investigations.

In a similar vein, as observed by the referee, the analogy with the chromatic picture, hinted at in the Introduction, would suggest the existence of some kind of action of the second Morava stabilizer group $\mathbb{G}_2$ on the ring spectrum $Kg(X)$ of Sect. 4 (suitably p-completed at some prime p), and also the possibility that $Ch^{cat} : Kg(X) \longrightarrow K^{S^1}(\mathrm{L}X)$ could detect some kind of v_2-periodicity (i.e., some evidence of chromatic type 2) in $K^{S^1}(\mathrm{L}X)$. Unfortunately, at the moment, we are not able to state these suggestions more precisely, let alone giving an answer.

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Voevodsky's Lectures on Motivic Cohomology 2000/2001

Pierre Deligne

Abstract These lectures cover several important topics of motivic homotopy theory which have not been covered elsewhere. These topics include the definition of equivariant motivic homotopy categories, the definition and basic properties of solid and ind-solid sheaves and the proof of the basic properties of the operations of twisted powers and group quotients relative to the $\mathbf{A}^1$-equivalences between ind-solid sheaves.

1 Introduction

The lectures which provided the source for these notes covered several different topics which are related to each other but which do not in any reasonable sense form a coherent whole. As a result, this text is a collection of four parts which refer to each other but otherwise are independent.

In the first part we introduce the motivic homotopy category and connect it with the motivic cohomology theory discussed in [7]. The exposition is a little unusual because we wanted to avoid any references to model structures and still prove the main theorem 2. We were able to do it modulo 6 where we had to refer to the next part.

The second part is about we the motivic homotopy category of G-schemes where G is a finite flat group scheme with respect to an equivariant analog of the Nisnevich topology. Our main result is a description of the class of $\mathbf{A}^1$-equivalences (formerly called $\mathbf{A}^1$-weak equivalences) given in Theorem 4 (also in Theorem 5). For the trivial group G we get a new description of the $\mathbf{A}^1$-equivalences in the non equivariant setting.

In the third part we define a class of sheaves on G-schemes which we call solid sheaves. It contains all representable sheaves and quotients of representable sheaves by subsheaves corresponding to open subschemes. In particular the Thom

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spaces of vector bundles are solid sheaves. The key property of solid sheaves can be expressed by saying that any right exact functor which takes open embeddings to monomorphisms is left exact on solid sheaves. A more precise statement is Theorem 6.

In the fourth part we study two functors. One is the extension to pointed sheaves of the functor from G-schemes to schemes which takes X to X/G. The other one is extension to pointed sheaves of the functor which takes X to X^W where W is a finite flat G-scheme. We show that both functors take solid sheaves to solid sheaves and preserve local and $\mathbf{A}^1$-equivalences between termwise (ind-)solid sheaves.

The material of all the parts of these notes but the first one was originally developed with one particular goal in mind – to extend non-additive functors, such as the symmetric product, from schemes to the motivic homotopy category. More precisely, we were interested in functors given by

$$T : X \mapsto (X^W \times E)/G$$

where G is a finite flat group scheme, W is a finite flat G-scheme and E any G-scheme of finite type. The equivariant motivic homotopy category was introduced to represent T as a composition

$$X \mapsto X^W \mapsto X^W \times E \mapsto (X^W \times E)/G$$

and solid sheaves as a natural class of sheaves on which the derived functor $\mathbf{L}T$ coincides with T.

In the present form these notes are the result of an interactive process which involved all listeners of the lectures. A very special role was played by Pierre Deligne. The text as it is now was completely written by him. He also cleared up a lot of messy parts and simplified the arguments in several important places.

Princeton, NJ, 2001.
Vladimir Voevodsky

2 Motivic Cohomology and Motivic Homotopy Category

We will recall first some of last year results (see [7]).

2.1 Last Year

1.1 We work over a field k which sometimes will have to be assumed to be perfect. The schemes over k we consider will usually be assumes separated and smooth of finite type over k. We note Sm/k their category. Three Grothendieck topologies on Sm/k will be useful: Zariski, Nisnevich and etale. For each of these topologies a sheaf on (Sm/k) amounts to the data for X smooth over k of a sheaf F_X on the small site X_{Zar} (resp. X_{Nis}, X_{et}) of the open subsets U of X (resp. of $U \to X$ etale), with F_X functorial in X: a map $f : X \to Y$ induces $f^* : f^* F_Y \to F_X$.

1.2 The definition of the motivic cohomology groups of X smooth over k has the following form:

(a) One defines for each $q \in \mathbf{Z}$ a complex of presheaves of abelian groups $\mathbf{Z}(q)$ on Sm/k. It is in fact a complex of sheaves for the etale topology, hence a fortiori for the Nisnevich and Zariski topology. For any abelian group A the same applies to $A(q) := A \otimes \mathbf{Z}(q)$.

(b) The motivic cohomology groups of X with coefficients in A are the hypercohomology groups of the $A(q)$, in the Nisnevich topology:

$$H^{p,q}(X, A) := \mathbf{H}^p(X_{Nis}, A(q))$$

For $A = \mathbf{Z}$ we will write simply $H^{p,q}(X)$.

Motivic cohomology has the following properties:

1. The complex $\mathbf{Z}(q)$ is zero for $q < 0$. For any q it lives in cohomological degree $\leq q$. As a complex of Nisnevich sheaves it is quasi-isomorphic to $\mathbf{Z}$ for $q = 0$ and to $\mathbf{G}_m[-1]$ for $q = 1$.
2. $H^{p,p}(Spec(k)) = K_p^M(k)$ for any $p \geq 0$.
3. For any X in Sm/k one has

$$H^{p,q}(X) = CH^q(X, 2q - p)$$

where $CH^q(X, 2q - p)$ is the $(2q - p)$-th higher Chow group of cycles of codimension q.

4. In the etale topology, for n prime to the characteristic of k, the complex $\mathbf{Z}/n(q)$ is quasi-isomorphic to $\mu_n^{\otimes q}$, giving for the etale analog of $H^{p,q}$ the formula

$$H_{et}^{p,q}(X, \mathbf{Z}/n) := \mathbf{H}^p(X_{et}, \mathbf{Z}/n(q)) = H^p(X_{et}, \mu_n^{\otimes q}).$$

1.3 The category $SmCor(k)$ is the category with objects separated schemes smooth of finite type over k, for which a morphism $Z : X \to Y$ is a cycle $Z = \sum n_i Z_i$ on $X \times Y$ each of whose irreducible components Z_i is finite over X and projects onto a connected component of X. A morphism Z can be thought of as a finitely valued map from X to Y. For $x \in X$, with residue field $k(x)$, it defines a zero cycle $Z(x)$ on $Y_{k(x)}$, and the assumption made on Z implies that the degree of this 0-cycle is locally constant on X.

A morphism of schemes $f : X \to Y$ defines a morphism in $SmCor(k)$: the graph of f. This graph construction defines a faithful functor from Sm/k to the additive category $SmCor(k)$.

A *presheaf with transfers* is a contravariant additive functor from the category $SmCor(k)$ to the category of abelian groups. The embedding of Sm/k in

$SmCor(k)$ allows us to view a presheaf with transfers as a presheaf on Sm/k endowed with an extra structure. A sheaf with transfers (for a given topology on Sm/k, usually the Nisnevich topology) is a presheaf with transfers which, as a presheaf on Sm/k, is a sheaf. The Nisnevich and the etale topologies have the virtue that if F is a presheaf with transfers, the associated sheaf $a(F)$ carries a structure of a sheaf with transfers. This structure is uniquely determined by $F \to a(F)$ being a morphism of presheaves with transfers. For any sheaf with transfers G, one has

$$Hom(a(F), G) \xrightarrow{\sim} Hom(F, G)$$

(Hom of presheaves with transfers). All of this fails for the Zariski topology.

The complexes $\mathbf{Z}(q)$ (or $A(q)$) start life as complexes of sheaves with transfers.

1.4 A presheaf F on Sm/k is called *homotopy invariant* if $F(X) = F(X \times \mathbf{A}^1)$. As the point 0 of $\mathbf{A}^1$ defines a section of the projection of $X \times \mathbf{A}^1$ to X, for any presheaf of abelian groups F, $F(X)$ is naturally a direct factor of $F(X \times \mathbf{A}^1)$; it follows that the condition "homotopy invariant" is stable by kernels, cokernels and extensions of presheaves. The following construction is a derived version of the left adjoint to the inclusion

$$(homotopy \quad invariant \quad presheaves) \subset (all \quad presheaves)$$

(a) For S a finite set, let $A(S)$ be the affine space freely spanned (in the sense of barycentric calculus) by S. Over $\mathbf{C}$, or $\mathbf{R}$, $A(S)$ contains the standard topological simplex spanned by S. The schemes $\Delta^m := A(\{0, \ldots, n\})$ form a cosimplicial scheme.

(b) For F a presheaf, $C_\bullet(F)$ (the "singular complex of F") is the simplicial presheaf $C_n(F) : X \mapsto F(X \times \Delta^n)$.

Arguments imitated from topology show that for F a presheaf of abelian groups, the cohomology presheaves of the complex $C_*(F)$, obtained from $C_\bullet(F)$ by taking alternating sum of the face maps, are homotopy invariant. If F has transfers so do the $C_n(F)$ and hence the $H_n C_*(F)$. A basic theorem proved last year is:

Theorem 1. *Let F be a homotopy invariant presheaf with transfers over a perfect field with the associated Nisnevich sheaf $a_{Nis}(F)$. Then the presheaves with transfers*

$$X \mapsto H^i(X_{Nis}, a_{Nis}(F))$$

are homotopy invariant as well.

The particular case of this theorem for $i = 0$ claims the homotopy invariance of the sheaf with transfers $a_{Nis}(F)$.

Last year, the equivalence of a number of definitions of $\mathbf{Z}(q)$ was proven. Equivalence means: a construction of an isomorphism in a suitable derived category, implying an isomorphism for the corresponding motivic cohomology groups. For our present purpose the most convenient definition is as follows.

Let $\mathbf{Z}_{tr}(X)$ be the sheaf with transfers represented by X (on the category $SmCor(k)$). We set

$$K_q = \begin{cases} 0 & \text{for } q < 0 \\ \mathbf{Z}_{tr}(\mathbf{A}^q)/\mathbf{Z}_{tr}(\mathbf{A}^q - \{0\}) & \text{for } q \geq 0 \end{cases}$$

and $\mathbf{Z}(q) = C_*(K_q)[-q]$.

2.2 Motivic Homotopy Category

The motivic homotopy category $Ho_{\mathbf{A}^1,\bullet}(S)$ (pointed $\mathbf{A}^1$-homotopy category of S), for S a finite dimensional noetherian scheme, will be the category deduced from a category of simplicial sheaves by two successive localizations.[1]

One starts with the category Sm/S of schemes smooth over S, with the Nisnevich topology, and the category of pointed simplicial sheaves on Sm/S. For any site $\mathcal{S}$ (for instance $(Sm/S)_{Nis}$), there is a notion of *local equivalence* of (pointed) simplicial sheaves. It proceeds as follows:

(a) A sheaf G defines a simplicial sheaf G_* with all $G_n = G$ and all simplicial maps the identities. The functor $G \mapsto G_*$ has a left adjoint $F \mapsto \pi_0(F)$:

$$Hom(F_*, G_*) = Hom(\pi_0(F_*), G)$$

The sheaf $\pi_0(F_*)$ can be described as the equalizer of $F_1 \rightrightarrows F_0$, as well as the sheaf associated to the presheaf

$$U \mapsto \pi_0(|F_*(U)|)$$

The same holds in the pointed context. We will often write simply G for G_*.

(b) If F_* is a simplicial sheaf, and u a section of F_0 over U, one also disposes of sheaves $\pi_i(F_*, u)$ over U: the sheaves associated to the presheaves

$$V/U \mapsto \pi(|F_*(V)|, u).$$

(c) A morphism $F_* \to G_*$ is a local equivalence, if it induces an isomorphism on π_0 as well as, for any local section u of F_0, an isomorphism on all π_i. This applies also to pointed simplicial sheaves: one just forgets the marked point.

One defines $Ho_\bullet(Sm/S)$ as the category derived from the category of pointed simplicial sheaves on $(Sm/S)_{Nis}$ by formally inverting local equivalences. Until made more concrete, this definition could lead to set-theoretic difficulties, which we leave the reader to solve in its preferred way.

[1] In the Appendix we have assembled the properties of "localization" to be used in this talk and in the next.

For G a pointed sheaf on Sm/S, Proposition 7 applies to G_* and to the localization by local equivalences: one has

$$Hom_{Ho_\bullet}(F_*, G_*) = Hom(F_*, G_*) = Hom(\pi_0(F_*), G) \tag{0.1}$$

Definition 1. An object X of $Ho_\bullet(Sm/S)$ is called $\mathbf{A}^1$-local if for any simplicial sheaf Y, one has

$$Hom_{Ho_\bullet}(Y, X) \xrightarrow{\sim} Hom_{Ho_\bullet}(Y \times \mathbf{A}^1 / * \times \mathbf{A}^1, X)$$

At the right hand side, $/ * \times \mathbf{A}^1$ means that in the product, $* \times \mathbf{A}^1$ is contracted to a point, the new base point.

Proposition 1. *For G a pointed sheaf on Sm/S, the simplicial sheaf G_* is $\mathbf{A}^1$-local if and only if G is homotopy invariant.*

Proof. We have $\pi_0(Y \times \mathbf{A}^1) = \pi_0(Y) \times \mathbf{A}^1$, so that by (0.1) "$\mathbf{A}^1$-local" means that for any pointed sheaf Y, one has

$$Hom(Y, G) = Hom(Y \times \mathbf{A}^1 / * \times \mathbf{A}^1, G)$$

A morphism $Y \to G$ can be viewed as the data, for each $y \in Y(U)$, of $f(y) \in G(U)$, functorial in U and marked point going to marked point. A morphism $g : Y \times \mathbf{A}^1 \to G$ can similarly be described as data for $y \in Y(U)$ of $g(y) \in G(U \times \mathbf{A}^1)$. Homotopy invariance hence implies $\mathbf{A}^1$-locality. The converse is checked by taking for Y the disjoint sum of a representable sheaf and the base point. $\square$

Definition 2. (1) A morphism $f : Y_1 \to Y_2$ in $Ho_\bullet(Sm/S)$ is an $\mathbf{A}^1$-equivalence if for any $\mathbf{A}^1$-local X, one has in $Ho_\bullet(Sm/S)$

$$Hom(Y_2, X) \xrightarrow{\sim} Hom(Y_1, X)$$

(2) The category $Ho_{\mathbf{A}^1,\bullet}(Sm/S)$ is deduced from $Ho_\bullet(Sm/S)$ by formally inverting $\mathbf{A}^1$-equivalences.

Remark 1. If a morphism in $Ho_\bullet(Sm/S)$ becomes an isomorphism in $Ho_{\bullet,\mathbf{A}^1}$ (Sm/S) it is an $\mathbf{A}^1$-equivalence. Indeed, if X in $Ho_\bullet(Sm/S)$ is $\mathbf{A}^1$-local, an application of Proposition 7 shows that for any Y,

$$Hom_{Ho_\bullet}(Y, X) \to Hom_{Ho_{\bullet,\mathbf{A}^1}}(Y, X)$$

is bijective. If $f : Y_1 \to Y_2$ in $Ho_\bullet(Sm/S)$ has an image in $Ho_{\bullet,\mathbf{A}^1}(Sm/S)$ which is an isomorphism, it follows that for any $\mathbf{A}^1$-local X, one has

$$Hom_{Ho_\bullet}(Y_2, X) \xrightarrow{\sim} Hom_{Ho_\bullet}(Y_1, X).$$

Such an f is an $\mathbf{A}^1$-equivalence.

Example 1. Arguments similar to those given before show that if G is a homotopy invariant pointed sheaf, then for any simplicial pointed sheaf F_*, one has

$$Hom_{Ho_{\mathbf{A}^1,\bullet}}(Sm/S)(F_*, G) = Hom(F_*, G) = Hom(\pi_0(F_*), G)$$

in particular, if U is smooth over S and if U_+ is the disjoint union of U and of a base point,

$$Hom_{Ho_{\mathbf{A}^1,\bullet}}(U_+, G) = G(U)$$

2.3 Derived Categories Vs. Homotopy Categories

For any topos T, which for us will be the category of sheaves on some site S, the pointed homotopy category $Ho_\bullet(S)$ as well as the derived category $D(S)$ are obtained by localization. For the derived category, one starts with the category of complexes of abelian sheaves. The subcategory of complexes living in homological degree ≥ 0 is naturally equivalent, by the Dold Puppe construction, to the category of simplicial sheaves of abelian groups. The equivalence is

$$N : (simplicial\ F_*) \mapsto complex(\bigcap_{i \neq 0} ker(\partial_i), \partial_0)$$

We will write K for the inverse equivalence. For S a point, and π an abelian group, $|K(\pi[n])|$ is indeed the Eilenberg–Maclane space $K(\pi, n)$. For a complex C not assumed to live in homological degree ≥ 0, we define

$$K(C) := K(\tau_{\geq 0} C)$$

where $\tau_{\geq n} C$ is the subcomplex in C of the form

$$\dots C_{n+2} \stackrel{d_{n+1}}{\to} C_{n+1} \to ker(d_n) \to 0$$

Note that $C \mapsto \tau_{\geq 0} C$ is right adjoint to the inclusion functor

$$\text{(complexes in homological degree} \geq 0) \hookrightarrow \text{(all complexes)}$$

so that (N, K) form a pair of adjoint functors:

$$\text{(simplicial abelian sheaves)} \rightleftarrows \text{(complexes of sheaves)}$$

Theorem 2. *Assume that $S = Spec(k)$ with k perfect. Then, for F a presheaf with transfers, and U_+ as above, and $p \geq 0$*

$$Hom_{Ho_{\mathbf{A}^1,\bullet}}(U_+, K(F[p])) = \mathbf{H}^p(U_{Nis}, C_*(F))$$

In this theorem $K(F[p])$ is the simplicial sheaf of abelian groups whose normalized chain complex is F in homological degree p.

To prove the theorem we establish the chain of equalities,

$$\begin{aligned}\mathbf{H}^p(U_{Nis}, C_*(F)) = Hom_{Ho_\bullet}(U_+, K(C_*(F)[p])) = \\ = Hom_{Ho_{\bullet,\mathbf{A}^1}}(U_+, K(C_*(F)[p])) = Hom_{Ho_{\bullet,\mathbf{A}^1}}(U_+, K(F[p])),\end{aligned} \tag{1.1}$$

the first equality is proved right before Proposition 4, the second right after Proposition 5 and the last one follows from Lemma 1.

Let $forget$ be the forgetting functor from abelian sheaves to sheaves of sets. Its left adjoint is $F \mapsto \mathbf{Z}[F]$: the sheaf associated to the presheaf

$$U \mapsto \text{(abelian group freely generated by } F(U))$$

In the pointed context , the adjoint is

$$(F, *) \mapsto \tilde{\mathbf{Z}}(F) : \mathbf{Z}(F)/\mathbf{Z}(*)$$

We have the same adjunction for (pointed) simplicial objects.

Proposition 2. *On a site with enough points (and presumably always), one has*

(1) The functor $F_* \mapsto N\mathbf{Z}(F_*)$ from pointed simplicial sheaves to complexes of abelian groups transforms local equivalences into quasi-isomorphisms
(2) The right adjoint $C \mapsto forget(K(C))$ transforms quasi-isomorphisms to local equivalences.

The assumption "enough points" applies to Sm/k with the Nisnevich topology: for any U in Sm/k and any point x of U, $F \mapsto F(Spec(\mathcal{O}^h_{U,x}))$ is a point, and they form a conservative system.

Proof. Local equivalence (resp. quasi-isomorphism) can be checked point by point, and the two functors considered commute with passage to the fiber at a point. This reduces our proposition to the case when S is just a point, i.e. to usual homotopy theory. In that case, (1) boils down to the fact that a weak equivalence induces an isomorphism on reduced homology, a theorem of Whitehead, and (1.1) reduces to the fact: for a complex C, $\pi_i(K(C))$, computed using any base point, is $H_i(C)$. The $\pi_i(K(C))$ are easy to compute because $K(C)$ has the Kan property. □

Applying Proposition 8, we deduce from Proposition 2 the following.

Proposition 3. *Under the same assumptions as in Proposition 2, for F_* a pointed simplicial sheaf and C a complex of abelian sheaves, one has*

$$Hom_{Ho_\bullet}(F_*, K(C)) = Hom_D(N\tilde{\mathbf{Z}}(F_*), C) \tag{3.1}$$

Let F be a sheaf and F_+ be deduced from F by adjunction of a base point. We also write F and F_+ for the corresponding "constant" simplicial sheaf. One has

$$N\tilde{\mathbf{Z}}(F_+) = N\mathbf{Z}(F) = (\mathbf{Z}(F) \text{ in degree zero})$$

For the pointed simplicial sheaf F_+, the group $Hom_D(\mathbf{Z}(F), C)$ which now occurs at the right-hand side of (3.1) can be interpreted as hypercohomology of C "over F viewed as a space", i.e. in the topos of sheaves over F. For F defined by an object U of the site S, this is the same as hypercohomology of the site S/U. As we do not want to assume C bounded below (in cohomological numbering), checking this requires a little care.

For a complex of sheaves K over a site S, not necessarily bounded below, $\mathbf{H}^0(S, K)$ can be defined as the Hom group in the derived category $Hom_D(\mathbf{Z}, K)$. For F in a topos T and the topos T/F: "F viewed as a space", besides the morphism of toposes $(T/F) \xrightarrow{j} T$, i.e. the adjoint pair (j^*, j_*), we have for abelian sheaves an adjoint pair $(j_!, j^*)$, with $j_!$ and j^* both exact. By Proposition 8, $(j_!, j^*)$ induce an adjoint pair for the corresponding derived categories. As $j_!\mathbf{Z} = \mathbf{Z}[F]$, we get

$$Hom_D(\mathbf{Z}(F), C) = \mathbf{H}^0(T/F, j^*(C)) \tag{3.2}$$

hence

$$Hom_{Ho_\bullet}(F_+, K(C)) = \mathbf{H}^0(T/F, j^*(C)) \tag{3.3}$$

Let us consider the particular case of Sm/k with the Nisnevich topology. For any complex of sheaves, (3.3) gives for U smooth over k

$$Hom_{Ho_\bullet}(U_+, K(C)) = \mathbf{H}^0(U_{big-Nis}, C) \tag{3.4}$$

Here, $U_{big-Nis}$ is the site $(Sm/S)/U$ with the Nisnevich topology. It has however the same hypercohomology as the small Nisnevich site U_{Nis}. Indeed, one has a morphism $\epsilon : U_{big,Nis} \to U_{Nis}$ and the functors ϵ^* and ϵ_* are exact. One again applies Proposition 8. If we apply (3.4) to a translate (shift) of C, we get

$$Hom_{Ho_\bullet}(U_+, K(C[p])) = \mathbf{H}^p(U_{Nis}, C) \tag{3.5}$$

Applying (3.5) to $C_*(F)$ we get the first equality in (1.1).

Proposition 4. *Let C be a complex of abelian sheaves on Sm/k. The following conditions are equivalent:*

1. *$K(C)$ is $\mathbf{A}^1$-local.*
2. *For $i \leq 0$, the functor $U \mapsto \mathbf{H}^i(U, C)$ is homotopy invariant.*
3. *For any complex L in cohomological degree ≤ 0, one has in the derived category*

$$Hom(L \otimes \mathbf{Z}(\mathbf{A}^1), C) \xrightarrow{\sim} Hom(L, C).$$

Proof. By Proposition 3, condition (1) can be rewritten: for any pointed simplicial sheaf F_*.

$$Hom_D(N\tilde{\mathbf{Z}}(F_*), C) = Hom_D(N\tilde{\mathbf{Z}}(F_* \times \mathbf{A}^1 / * \times \mathbf{A}^1))$$

The operation $F_* \mapsto F_* \times \mathbf{A}^1 / * \times \mathbf{A}^1$ is better written as a smash product $F_* \wedge \mathbf{A}^1_+$ with $\mathbf{A}^1_+$. For pointed sets E and F, $\tilde{\mathbf{Z}}(E \wedge F) = \tilde{\mathbf{Z}}(E) \otimes \tilde{\mathbf{Z}}(F)$. It follows that

$$\tilde{\mathbf{Z}}(F_* \times \mathbf{A}^1 / * \times \mathbf{A}^1) = \tilde{\mathbf{Z}}(F_* \wedge \mathbf{A}^1_+) = \tilde{\mathbf{Z}}(F_*) \otimes \tilde{\mathbf{Z}}(\mathbf{A}^1_+) = \tilde{\mathbf{Z}}(F_*) \otimes \mathbf{Z}(\mathbf{A}^1)$$

(isomorphisms of simplicial sheaves), hence

$$N\tilde{\mathbf{Z}}(F_* \times \mathbf{A}^1 / * \times \mathbf{A}^1) = N\tilde{\mathbf{Z}}(F_*) \otimes \mathbf{Z}(\mathbf{A}^1)$$

It follows that (1) is the particular case of (3) for L of the form $\tilde{\mathbf{Z}}(F_*)$. Similarly, (2) is the particular case of (3) for L of the form $\mathbf{Z}(U)[i]$, with $i \geq 0$.

The suspension $\Sigma^i F_*$ of a simplicial pointed sheaf F_* is its smash product with the simplicial sphere S^i (the i-simplex modulo its boundary). It follows that

$$\tilde{\mathbf{Z}}(\Sigma^i F_*) = \tilde{\mathbf{Z}}(F_*) \otimes \tilde{\mathbf{Z}}(S^i)$$

(isomorphism of simplicial sheaves), and by Eilenberg–Zilber, the normalized complex $N\mathbf{Z}(\Sigma^i F_*)$ is homotopic to the tensor product of the normalized complexes of $\tilde{\mathbf{Z}}(F_*)$ and $\tilde{\mathbf{Z}}(S^i)$. The latter is simply $\mathbf{Z}[i]$:

$$N\tilde{\mathbf{Z}}(\Sigma^i F_*) \cong \tilde{\mathbf{Z}}(F_*)[i]$$

This is just a high-brown way of telling that the reduced homology of a suspension is just a shift of the reduced homology of the space one started with.

Applying this to $F_* = U_+$, one obtains that (1)$\Rightarrow$(3). Indeed, $\tilde{\mathbf{Z}}(\Sigma^i U_+)$ is homotopic to $\tilde{\mathbf{Z}}(U_+)[i] = \mathbf{Z}(U)[i]$.

We now prove that (2)$\Rightarrow$(1). For L a complex, let (*) be the statement that the conclusion of (3) holds for all $L[i]$, $i \geq 0$. The assumption (2) is that (*) holds for L reduced to $\mathbf{Z}(U)$ in degree 0, and we will conclude that it holds for all L in cohomological degree ≤ 0 by "devissage":

(a) The case of a sum of $\mathbf{Z}(U)$, in degree zero, follows from Corollary 10.
(b) Suppose that L is bounded, is in cohomological degree ≤ 0 and that (*) holds for all L^n.

The functors

$$\begin{aligned} h' &: L \mapsto Hom^n(L, C) \\ h'' &: L \mapsto Hom^n(L \otimes \mathbf{Z}(\mathbf{A}^1), C) \end{aligned} \tag{3.6}$$

are contravariant cohomological functors, hence give rise to convergent spectral sequences

$$E_1^{pq} = h^q(L^{-p}) \Rightarrow h^{p+q}(L).$$

One has a morphism of spectral sequences

$$E(\text{for } h') \to E(\text{for } h'')$$

which is an isomorphism for $q \leq 0$, and both E^{pq} vanish for $p < 0$ or p large. It follows that $h'^n(L) \xrightarrow{\sim} h''^n(L)$ for $n \leq 0$, i.e. that L satisfies (*).

The same argument can be expressed as an induction on the number of i such that $L^i \neq 0$. If n is the largest (with $n \leq 0$), the induction assumption applies to $\sigma^{<n}L$, even to $(\sigma^{<n}L)[-1]$, and one concludes by the long exact sequence defined by

$$0 \to L^n[-n] \to L \to \sigma^{<n}L \to 0$$

(c) Expressing L as the inductive limit of the $\sigma^{\geq -n}L$ and using Proposition 11, one sees that we need not assume that L is bounded.
(d) If $L' \to L''$ is a quasi-isomorphism, $L' \otimes \mathbf{Z}(\mathbf{A}^1) \to L'' \otimes \mathbf{Z}(\mathbf{A}^1)$ is one too (flatness of $\mathbf{Z}(\mathbf{A}^1)$), and (*) holds for L' if and only if it holds for L''.
(e) Any abelian sheaf L is a quotient of a direct sum of sheaves $\mathbf{Z}(U)$. For instance, the sum over (U, s), $s \in \Gamma(U, L)$, of $\mathbf{Z}(U)$ mapping to L by s. It follows that L admits a resolution L^* by such sheaves. By (a) and (c), L^* satisfies (*). It follows from (d) that L satisfies (*) and then by (c) that any complex in degree ≤ 0 satisfies (*). □

2.4 Application to Presheaves with Transfers

Let F be a presheaf with transfers. A formal argument [7] shows that the presheaves with transfers $H^iC_*(F)$ are homotopy invariant. By the basic result (Theorem 1) recalled in the first lecture, it follows that for any U, one has

$$\mathbf{H}^*(U, C_*(F)) = \mathbf{H}^*(U \times \mathbf{A}^1, C_*(F)) \tag{4.1}$$

Indeed, as U and $U \times \mathbf{A}^1$ are of finite cohomological dimension, both sides are abutment of a convergent spectral sequence

$$E_2^{pq} = H^p(U, H^qC_*(F)) \Rightarrow \mathbf{H}^{p+q}(U, C_*(F))$$

and the same for $U \times \mathbf{A}^1$. By Theorem 1 applied to $H^qC_*(F)$,

$$H^p(X, H^qC_*(F)) := H^p(X, aH^qC_*(F))$$

is the same for $X = U$ or for $X = U \times \mathbf{A}^1$. Applying (3.5), we conclude from Proposition 4((2)⇒(1)) that

Proposition 5. *For k perfect, if F is a presheaf with transfers, for all p, $K(C_*(F)[p])$ is $\mathbf{A}^1$-local.*

Combining Proposition 5 with Proposition 7 we get the second equality in (1.1).

2.5 End of the Proof of Theorem 2

For any pointed simplicial sheaf $G_\bullet$, $C_\bullet(G_\bullet)$ is a pointed bisimplicial sheaf of which one can take the diagonal $\Delta C_\bullet(G_\bullet)$. For any pointed sheaf G, one has a natural map $G \to C_\bullet(G)$, and for a pointed simplicial sheaf $G_\bullet$, those maps for the G_n induce

$$a : G_\bullet \to \Delta C_\bullet(G_\bullet)$$

Proposition 6. *The morphism $a : G_\bullet \to \Delta C_\bullet(G_\bullet)$ is an $\mathbf{A}^1$-equivalence.*

Proof. We deduce Proposition 6 from Proposition 20.

The two maps $0, 1 : F_\bullet \to F_\bullet \wedge \mathbf{A}^1_+$ are equalized by $F_\bullet \wedge \mathbf{A}^1_+ \to F_\bullet$, hence become equal in the $\mathbf{A}^1$-homotopy category. If two maps of pointed simplicial sheaves $F_\bullet \rightrightarrows G_\bullet$ factor as $F_\bullet \rightrightarrows F_\bullet \wedge \mathbf{A}^1_+ \to G_\bullet$, they also become equal. By the adjunction of $\wedge \mathbf{A}^1_+$ and of $C_1(-) = \underline{Hom}(\mathbf{A}^1_+, -)$, such a factorization can be rewritten as

$$F_\bullet \to C_1(G_\bullet) \to G_\bullet$$

Particular case: the maps $C_1(G_\bullet) \to G_\bullet$, become equal in the homotopy category. Evaluated on X, these maps are the restriction maps $0^*, 1^* : G_\bullet(X \times \mathbf{A}^1) \to G_\bullet(X)$.

The affine space $\mathbf{A}^n$ is homotopic to a point in the sense that $H : \mathbf{A}^1 \times \mathbf{A}^n \to \mathbf{A}^n : (t, x) \mapsto tx$ interpolates between the identity map (for $t = 1$) and the constant map 0 (for $t = 0$). The map H induces

$$G_\bullet(S \times \mathbf{A}^n) \to G_\bullet(S \times \mathbf{A}^n \times \mathbf{A}^1)$$

and, composing with 0, 1 in $\mathbf{A}^1$, we obtain that

$$G_\bullet(S \times \mathbf{A}^n) \rightrightarrows G_\bullet(S \times \mathbf{A}^n)$$

the identity map, and the map induced by $0 : \mathbf{A}^n \to \mathbf{A}^n$, are equal in the $\mathbf{A}^1$-homotopy category. The map of simplicial sheaves $G_\bullet \to C_n G_\bullet$ is hence an $\mathbf{A}^1$-equivalence. It has as inverse in the $\mathbf{A}^1$-homotopy category the map induced by

$0 : Spec(k) \to \mathbf{A}^n$ and one applies Remark 1. We now apply Proposition 20 to the bisimplicial sheaves

$$G_{pq} := G_p$$
$$H_{pq} := C_q G_p : S \mapsto G_p(S \times \Delta^q)$$

and to the natural map $G_{pq} \to H_{pq}$. For fixed q, this is just $G(S) \to G(S \times \mathbf{A}^q)$, and Proposition 20 gives Proposition 6. □

To prove the last equality in (1.1), it suffices to show that:

Lemma 1. *For any abelian sheaf F, $F[p] \to C_*(F)[p]$ induces an $\mathbf{A}^1$-equivalence from $K(F[p])$ to $K(C_*(F)[p])$.*

Proof. For G a monoid (with unit), the pointed simplicial set $B_\bullet G$ is given by

$$B_n G = \left\{ \begin{array}{c} \text{functors from the ordered set } (0, \ldots, n) \text{ viewed as a category} \\ \text{to } G \text{ viewed as a category with one object} \end{array} \right\}$$

This construction can be sheafified, and can be applied termwise to a simplicial sheaf of monoids, leading to a pointed bisimplicial sheaf of which one can take the diagonal

$$BG_\bullet := \Delta B_\bullet(G_\bullet)$$

This construction commutes with the construction $G_\bullet \to \Delta C_\bullet(G_\bullet)$. Indeed, $B_n G_p$ is naturally isomorphic to G_p^n, the operation C_m commutes with products, and $B(\Delta C_\bullet(G_\bullet))$ and $\Delta C_\bullet(BG_\bullet)$ are both diagonals of the trisimplicial pointed sheaf $C_\bullet B_\bullet G_\bullet$.

For abelian simplicial sheaves, the operation B gives again abelian simplicial sheaves, hence can be iterated, and $\Delta C_\bullet$ commutes with B^n.

Via Dold–Puppe construction, B corresponds, up to homotopy, to the shift [1] of complexes:

$$NBG_\bullet \cong (NG_\bullet)[1].$$

This can be viewed as an application of the Eilenberg–Zilber Theorem (see [9, Theorem 8.5.1]): one has

$$NBG_\bullet \cong BG_* \cong Tot B_* G_* \quad \text{(Eilenberg–Zilber)},$$

and for each G_q, the normalization of $B_\bullet G_q$ is just $G_q[1]$, so that the double complexes $B_* G_*$ and $H_{pq} := G_q$ for $p = 1$, 0 otherwise, have homotopic Tot.

If $G_\bullet$ is an abelian simplicial sheaf, applying Proposition 6 to $B^p G_\bullet$, we obtain that

$$B^p G_\bullet \to \Delta C_\bullet B^p G_\bullet = B^p \Delta C_\bullet G_\bullet \tag{6.1}$$

is an $\mathbf{A}^1$-equivalence. The functor K transforms chain homotopy equivalences into simplicial equivalences. For any simplicial abelian group $L_\bullet$ (to be $G_\bullet$ or $\Delta C_\bullet G_\bullet$), we hence have a simplicial homotopy equivalence

$$B^p L_\bullet = KNB^p L_\bullet \cong K((NL_\bullet)[p])$$

Simplicial homotopy equivalences being $\mathbf{A}^1$-equivalences, we conclude that (6.1) induces an $\mathbf{A}^1$-equivalence

$$K((NG_\bullet)[p]) \to K(N(\Delta C_\bullet G_\bullet)[p])$$

□

2.6 Appendix: Localization

Let C be a category and S be a set of morphisms of C. The localized category $C[S^{-1}]$ is deduced from C by "formally inverting all $s \in S$". With this definition, it is clear that one has a natural functor $loc : C \to C[S^{-1}]$, bijective on the set of objects, and that for any category D,

$$F \mapsto F \circ loc : Hom(C[S^{-1}], D) \to Hom(C, D)$$

is a bijection from $Hom(C[S^{-1}], D)$ to the set of functors from C to D transforming morphisms in S into isomorphisms.

If one remembers that the categories form a 2-category, and if one agree with the principle that one should not try to define a category more precisely than up to equivalence (unique up to unique isomorphism), the universal property of $C[S^{-1}]$ given above is doubly unsatisfactory. The easily checked and useful universal property is the following: $F \mapsto F \circ loc$ is an equivalence from the category $Hom(C[S^{-1}], D)$ to the full subcategory of $Hom(C, D)$ consisting of the functors F which map S to isomorphisms.

Proposition 7. *If Y in C is such that the functor*

$$h_Y : C^{op} \to Sets : X \mapsto Hom_C(X, Y)$$

transforms maps in S into bijections, then

$$Hom_C(X, Y) \xrightarrow{\sim} Hom_{C[S^{-1}]}(X, Y)$$

Proof. By Yoneda construction $Y \mapsto h_Y$, C embeds into the category $C^\wedge$ of contravariant functors from C to *Sets*, while $C[S^{-1}]$ embeds into $C[S^{-1}]^\wedge$, identified by (a) with the full subcategory of $C^\wedge$ consisting of F transforming S into bijections. For Y in C, with image $\bar{Y}$ in $C[S^{-1}]$, and for any F in $C(S^{-1})^\wedge \subset C^\wedge$, one has in $C^\wedge$

$$Hom(h_Y, F) = Hom(h_{\bar{Y}}, F).$$

Indeed, by (a) and Yoneda lemma for C and $C[S^{-1}]$ both sides are $F(Y)$. This means that $h_{\bar{Y}}$ is the solution of the universal problem of mapping h_Y into an object of $C[S^{-1}]^\wedge \subset C^\wedge$. In particular, for Y as in (b), i.e. in $C[S^{-1}]^\wedge$, $h_{\bar{Y}}$ coincides with h_Y, as claimed by (b). □

Proposition 8. *Let (L, R) be a pair of adjoint functors between categories C and D. Let S and T be sets of morphisms in C and D. Assume that F maps S to T and that G maps T to S. Then the functors $\bar{L}$, $\bar{R}$ between $C[S^{-1}]$ and $D[T^{-1}]$ induced by L and R again form an adjoint pair.*

Proof. The functors $\bar{L}$ and $\bar{R}$ induced by F and G are characterized by commutative diagrams

$$\begin{array}{ccc} C & \xrightarrow{L} & D \\ \downarrow & & \downarrow \\ C[S^{-1}] & \xrightarrow{\bar{L}} & D[T^{-1}] \end{array} \qquad \begin{array}{ccc} D & \xrightarrow{R} & C \\ \downarrow & & \downarrow \\ D[T^{-1}] & \xrightarrow{\bar{R}} & C[S^{-1}] \end{array}$$

Adjunction can be expressed by the data of $\epsilon : Id \to RL$ and $\eta : LR \to Id$ such that the compositions

$$R \to RLR \to R$$

$$L \to LRL \to L$$

are the identity automorphisms of R and L respectively (see, e.g. [6]).

By the universal property of localization, ϵ induces a morphism $\bar{\epsilon} \to \bar{R}\bar{L}$, indeed, to define such a morphism amounts to defining a morphism $loc \to \bar{R}\bar{L}loc$, and $\bar{R}\bar{L}loc = locRL$. Similarly, η induces $\bar{\eta} : \bar{L}\bar{R} \to Id$. The morphism $\bar{L} \to \bar{L}\bar{R}\bar{L} \to \bar{L}$ is induced by $L \to LRL \to L$, similarly for $\bar{R} \to \bar{R}\bar{L}\bar{R} \to \bar{R}$, and the proposition follows. □

Proposition 9. *Suppose that:*

1. *The localization $C[S^{-1}]$ gives rise to a right calculus of fractions.*
2. *Coproducts exist in C, and S is stable by coproducts.*

Then, a coproduct in C is also a coproduct in $C[S^{-1}]$.

For the definition of "gives rise to a right calculus of fractions" see [10]. It implies that for X in C, the category of $s : X' \to X$ with s in S is filtering, and that

$$Hom_{C[S^{-1}]}(X, Y) = colim_{s:X' \to X} Hom_C(X', Y)$$

Proof. For X in C, let (S/X) be the filtering category of morphisms $X' \to X$ in S. For X the coproduct of X_α, $\alpha \in A$, one has a functor "coproduct":

$$\prod (S/X_\alpha) \to (S/X)$$

It is cofinal: for $s : X' \to X$ in S, one can construct a diagram

$$\begin{tikzcd} X'_\alpha \arrow[r] \arrow[d] & X_\alpha \arrow[d] \\ X' \arrow[r] & X \end{tikzcd}$$

with $s_\alpha : X'_\alpha \to X$ in S, and $\coprod s_\alpha$ dominates s. For any Y, it follows that

$$Hom_{C[S^{-1}]}(X, Y) = colim_{(S/X)} Hom_C(X', Y) =$$

$$= colim_{\prod(S/X_\alpha)} Hom(\coprod X'_\alpha, Y) = colim_{\prod(S/X_\alpha)} \prod Hom(X'_\alpha, Y) =$$

$$= \prod colim_{S/X_\alpha} Hom(X'_\alpha, Y) = \prod Hom_{C[S^{-1}]}(X_\alpha, Y),$$

meaning that X is also the coproduct of the X_α in $C[S^{-1}]$. □

Corollary 10. *Suppose that in the abelian category A arbitrary direct sums exist and are exact. Then, arbitrary direct sums exist in the derived category $D(A)$, and the localization functor*

$$C(A) \to D(A)$$

commutes with direct sums.

Proof. The functor $C(A) \to D(A)$ factors through the category $K(A)$ of complexes and maps up to homotopy. Direct sums in $C(A)$ are also direct sums in $K(A)$. Indeed,

$$Hom_{K(A)}(\oplus K_\alpha, L) = H^0 Hom^\bullet(\oplus K_\alpha, L) =$$

$$= H^0 \prod Hom^\bullet(K_\alpha, L) = \prod H^0 Hom^\bullet(K_\alpha, L),$$

as $\prod$ is exact for abelian groups. Exactness of $\oplus$ in A ensures that a direct sum of quasi-isomorphisms is again a quasi-isomorphism, and Proposition 9 applies to $K(A)$ and the set S of quasi-isomorphisms, proving the corollary. □

If A_i, $i \geq 0$ is an inductive system of objects of A, the colimit of A_i is the cokernel

$$\oplus A_i \xrightarrow{d} \oplus A_i \to colim\, A_i \to 0$$

of the difference of the identity map and of the sum of the $A_i \to A_{i+1}$. If taking the inductive limit of a sequence is an exact functor, the map d is injective: it is the colimit of the

$$\oplus_{i=0}^{n} A_i \to \oplus_{i=0}^{n+1} A_i$$

each of which is injective, as its graded for the filtration by the $\oplus_{i \geq p} A_i$ is the identity inclusion.

Under the assumptions of Corollary 10 if a complex K is the colimit of an inductive sequence $K_{(i)}$, and if the sequence

$$0 \to \oplus K_{(i)} \xrightarrow{d} \oplus K_{(i)} \to K \to 0 \tag{10.1}$$

is exact, then for any L, the long exact sequence of cohomology reads

$$\to Hom(K, L) \to \prod Hom(K_{(i)}, L) \xrightarrow{d} \prod Hom(K_{(i)}, L) \to$$

The kernel of d is simply the projective limit of the $Hom(K_{(i)}, L)$. The cokernel is lim^1. One concludes.

Proposition 11. *Suppose that in A countable direct sums exist and are exact. If the complex K is the colimit of the $K_{(i)}$, and if the sequence (10.1) is exact, for instance if either:*

1. *In A inductive limits of sequences are exact.*
2. *In each degree n, each $K^n_{(i)} \to K^n_{(i+1)}$, is the inclusion of a direct factor.*

then, one has a short exact sequence

$$0 \to lim^1 Hom(K_{(i)}, L[-1]) \to Hom(K, L) \to lim Hom(K_{(i)}, L) \to 0$$

Proof. It remains to check that condition (2) implies the exactness of (10.1). This is to be seen degree by degree. By assumption, the $A_i := K^n_{(i)}$, have decompositions compatible with the transition maps $A_i = \oplus^i_{j=0} B_i$. A corresponding decomposition of (10.1) in direct sum follows, and we are reduced to check exactness of the particular case $(10.1)_{(A,n)}$ of (10.1) when $B_i = 0$ for $i \neq n$, i.e. when A_i is a fixed A from $i \geq n$, and is 0 otherwise. Let $(10.1)_0$ be the sequence $(10.1)_{\mathbf{Z},n}$ in Ab for $A = \mathbf{Z}$. It is a split exact sequence of free abelian groups. Because direct sums exist, $L \otimes A$, for L a free abelian group is defined and functorial in L. It is a sum of copies of A, indexed by a basis of L, and is characterized by

$$Hom(L \otimes A, B) = Hom(L, Hom(A, B))$$

(functorial in B). The sequence $(10.1)_{A,n}$ is $(10.1)_0 \otimes A$ and, $(10.1)_0$ being split exact, it splits and in particular is exact. □

The truncation $\sigma_{\leq n} K = \sigma^{\geq -n} K$ of a complex K is the subcomplex which coincides with K in homological degree $\leq n$ and is 0 in homological degree $> n$. For any complex K, one has

$$K = colim \sigma_{\leq n} K$$

and this colimit satisfies condition (2) of Proposition 11. It follows that

Corollary 12. *Under the assumptions of Corollary 10, for any K and L, one has a short exact sequence*

$$0 \to lim^1 Hom(\sigma_{\leq n} K, L[-1]) \to Hom(K, L) \to lim Hom(\sigma_{\leq n} K, L) \to 0$$

3 $\mathbf{A^1}$-Equivalences of Simplicial Sheaves on G-Schemes

3.1 Sheaves on a Site of G-Schemes

We fix a base scheme S, supposed to be separated noetherian and of finite dimension; fiber product $X \times_S Y$ will be written simply as $X \times Y$. We also fix a group scheme G over S, supposed to be finite and flat. We note h_X the representable sheaf defined by X.

Let QP/G be the category of schemes quasi-projective over S, given with an action of G. Any X in QP/G admits an open covering (U_i) by affine open subschemes which are G-stable. This makes it possible to define a reasonable quotient X/G in the category of schemes over S (rather than in the larger category of algebraic spaces). For each U_i, U_i/G is defined as the spectrum of the equalizer

$$\mathcal{O}(U_i) \rightrightarrows \mathcal{O}(U_i \times G),$$

and X/G is obtained by gluing the U_i/G. It is a categorical quotient, i.e. the coequalizer of $G \times X \rightrightarrows X$. The map $X \to X/G$ is finite, open, and the topological space $|X/G|$ is the coequalizer of the map of topological spaces $|G \times X| \rightrightarrows |X|$. One can show that X/G is again quasi-projective. Remark 2 below shows that this fact, while convenient for the exposition, is irrelevant.

One defines on QP/G a pretopology [3, II.1.3] by taking as coverings the family of etale maps $Y_i \to X$ with the following property: X admits a filtration by closed equivariant subschemes $\emptyset = X_n \subset \cdots \subset X_1 \subset X_0 = X$ such that for each j, some map $Y_i \to X$ has a section over $X_j - X_{j+1}$. The *Nisnevich topology* on QP/G is the topology generated by this pretopology. The category QP/G with the Nisnevich topology is the *Nisnevich site* $(QP/G)_{Nis}$.

Remark 1. The corresponding topos is not the classifying topos of [2, IV.2.5]. A morphism $X \to Y$ can become a Nisnevich covering after forgetting the action of G, and not be a Nisnevich covering. Example: $S = Spec(k)$, $G = \mathbf{Z}/2$, $X = S$, $Y = S \coprod S$ and G permutes two copies of S in Y.

Remark 2. Let $(affine/G)_{Nis}$ be the site defined as above, with "quasi-projective" replaced by "affine". It is equivalent to $(QP/G)_{Nis}$, in the sense that restriction to $(affine/G)_{Nis}$ is an equivalence from the category of sheaves on $(QP/G)_{Nis}$ to the category of sheaves on $(affine/G)_{Nis}$.

Remark 3. If G is the trivial group e, the definition given above recovers the usual Nisnevich topology. For $G = e$, the condition usually considered: "every point x of X is the image of a point with the same residue field of some Y_i", is indeed equivalent to the condition imposed above. This is checked by noetherian induction: if a generic point ξ of X can be lifted to Y_i, some open neighborhood $U \subset X$ of ξ can be lifted to Y_i, and one applies the induction hypothesis to $X_1 = (X - U)_{red}$.

We write $(QP)_{Nis}$ for the category of quasi-projective schemes over S, with the Nisnevich topology.

Lemma 2. *If $\mathcal{U} : Y_i \to X$ $(i \in I)$ is a covering of X in $(QP/G)_{Nis}$, there is a covering $\mathcal{V}$ of X/G in $(QP)_{Nis}$ whose pull-back to X is finer than $\mathcal{U}$.*

Proof. Fix a filtration $\emptyset = X_n \subset \cdots \subset X_0 = X$ as in the definition of the Nisnevich topology. We write q for the quotient map $X \to X/G$. For x in X/G, $(q^{-1}(x))_{red}$ is in some $X_j - X_{j+1}$, by equivariance of the X_j, and one of the maps $Y_i \to X$ has an equivariant section s over $X_j - X_{j+1}$. Let $(X/G)^h_x$ be the henselization of X/G at x. The map q being finite, the pull-back of $(X/G)^h_x$ to X is the coproduct of the X^h_y for $q(y) = x$. The map from Y_i to X being etale, the section s, restricted to $(q^{-1}(y))_{red}$, extends uniquely to a section (automatically equivariant) of Y_i over $\coprod_{q(y)=x} X^h_y$. Writing $(X/G)^h_x$ as the limit of etale neighborhoods of x, one finds that x has an etale neighborhood $V(x)$ such that Y_i has an equivariant section over $X \times_{X/G} V(x)$. The $V(x)$ form the required covering $\mathcal{V}$. □

We define the G-local henselian schemes to be the schemes Y obtained in the following way. For X in (QP/G), y a point of X/G, and $(X/G)^h_y$ the henselization of X/G at y, take the fiber product $Y := X \times_{X/G} (X/G)^h_y$. As X is finite over X/G, this fiber product is a finite disjoint union of local henselian schemes, and G-local henselian schemes are simply the G-equivariant finite disjoint unions of Y of local henselian schemes, for which Y/G is local.

Proposition 13. *If Y is G-local henselian, the functor $X \mapsto Hom(Y, X)$ is a point of the site $(QP/G)_{Nis}$, i.e. it defines a morphism of the punctual site $(Sets)$ to $(QP/G)_{Nis}$. If $Y = X \times_{X/G} (X/G)^h_y$, the corresponding fiber functor is $F \mapsto colim F(X \times_{X/G} V)$, the colimit being taken over the etale neighborhoods of y in X/G. The collection of fiber functors so obtained is conservative.*

Proof. The functor $X \mapsto Hom(Y, X)$ commutes with finite limits. It follows from Lemma 2 that it transforms coverings into surjective families of maps, hence is a morphism of sites $(Sets) \to (QP/G)_{Nis}$.

To check that the resulting set of fiber functors is conservative, it suffices to check that a family of etale $f_i : U_i \to X$ is a covering if for any G-local henselian Y,

$$\coprod Hom(Y, U_i) \to Hom(Y, X)$$

is onto. The proof, parallel to that of Lemma 2 is left to the reader. □

3.2 The Brown–Gersten Closed Model Structure on Simplicial Sheaves on G-Schemes

We recall that a commutative square of simplicial sets (or pointed simplicial sets)

$$\begin{array}{ccc} K & \longrightarrow & L \\ \downarrow & & \downarrow \\ M & \longrightarrow & N \end{array} \tag{13.1}$$

is *homotopy cartesian* (or a *homotopy pull-back square*) if, when L is replaced by L' weakly equivalent to it and mapping to N by (Kan) fibration: $L \xrightarrow{\cong} L' \to N$, the map from K to $L' \times_N M$ is a weak equivalence.

Definition 3. A simplicial presheaf $F_\bullet$ on $(QP/G)_{Nis}$ is *flasque* if $F(\emptyset)$ is contractible and if for any (upper) distinguished square:

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{j} & X \end{array}$$

(p – etale, j open embedding, $B = p^{-1}(A)$ and $Y - B \cong X - A$), the square

$$\begin{array}{ccc} F(X) & \longrightarrow & F(Y) \\ \downarrow & & \downarrow{\scriptstyle p} \\ F(A) & \xrightarrow{j} & F(B) \end{array}$$

is homotopy cartesian.

Theorem 3. *Let $f : F_\bullet \to F'_\bullet$ be a morphism of flasque simplicial presheaves. If the induced morphism of simplicial sheaves $aF_\bullet \to aF'_\bullet$ is a local equivalence, then, for any U in QP/G, $F_\bullet(U) \to F'_\bullet(U)$ is a weak equivalence.*

Proof. For a G-scheme X let X_{Nis} be the small Nisnevich site of X and for a presheaf F on (QP/G) let $F_{|X}$ be the restriction of F to X_{Nis}. Our assumption that $aF_\bullet \to aF'_\bullet$ is a local equivalence implies that $aF_{\bullet,|U} \to aF'_{\bullet,|U}$ is a local equivalence. The map $U \to U/G$ defines a morphism of sites $p : U_{Nis} \to (U/G)_{Nis}$ and Lemma 2 implies that the direct image functor p_* commutes with the associated sheaf functor and takes local equivalences to local equivalences. Therefore the morphism $ap_*(F_{\bullet,|U}) \to ap_*(F'_{\bullet,|U})$ is a local equivalence. The presheaves $p_*(F_{\bullet,|U})$ and $p_*(F'_{\bullet,|U})$ are flasque on $(U/G)_{Nis}$ and by [8, Lemma 3.1.18] we conclude that

$$F_\bullet(U) = p_*(F_{\bullet,|U})(U/G) \to p_*(F'_{\bullet,|U})(U/G) = F'_\bullet(U)$$

is a weak equivalence. $\square$

In [1], Brown and Gersten define a simplicial closed model structure on the category of pointed simplicial sheaves on a Noetherian topological space of finite dimension. As in Jardine [11], the equivalences are the local equivalences . The homotopy category is hence the same as Joyal's, but the model structure is different: less cofibrations, more fibrations.

The arguments of [1] work as well in the Nisnevich topology, for the big as well as for the small Nisnevich site, or for $(QP/G)_{Nis}$, once Theorem 3 is available.

We review the basic definitions, working in $(QP/G)_{Nis}$. Let $\Lambda^{n,k}$ be the sub-simplicial set of $\partial\Delta^n$, union of all faces but the k-th face. For $n = 0$, $\Lambda^{0,0} = \emptyset$. One takes as *generating trivial cofibrations* the maps of the form (J):

(J_a) $(\Lambda^{n,k} \times h_X)_+ \to (\Delta^n \times h_X)_+$
(J_b) for $U \to X$ an open embedding,

$$(\Delta^n \times h_U \coprod_{\Lambda^{n,k}\times h_U} \Lambda^{n,k} \times h_X)_+ \to (\Delta^n \times h_X)_+$$

One then defines the *fibrations* to be the morphisms p having the right lifting property with respect to generating trivial cofibrations (see, e.g. [5]), the *(weak) equivalences* to be the local equivalences, the trivial fibrations to be fibrations which are also (weak) equivalences, and the *cofibrations* to be the morphisms having the left lifting property with respect to trivial fibrations.

Following [1] and using Theorem 3, one proves that the trivial fibrations can be equivalently described as morphisms having the right lifting property with respect to the following class of morphisms (I):

(I_a) $(\partial\Delta^n \times h_X)_+ \subset (\Delta^n \times h_X)_+$
(I_b) for $U \to X$ open embedding,

$$(\Delta^n \times h_U \coprod_{\partial\Delta^n\times h_U} \partial\Delta^n \times h_X)_+ \to (\Delta^n \times h_X)_+$$

The maps of the form (I) are called *generating cofibrations*.

For X and Y pointed simplicial sheaves, one defines a pointed simplicial set $S(X, Y)$ by

$$S(X, Y)_n = Hom(X \wedge (\Delta^n)_+, Y)$$

Following [1], one sees that the classes of cofibrations, (weak) equivalences, fibrations, and S are a simplicial closed model structure in the sense of []. This has the following consequences.

Corollary 14. *If X is cofibrant and Y fibrant, for any pointed simplicial set K, one has in the relevant homotopy categories*

$$Hom_{Ho}(X \wedge K, Y) = Hom_{Ho}(K, S(X, Y))$$

In particular, taking $k = (\Delta^0)_+$ one gets

$$Hom_{Ho}(X, Y) = \pi_0 S(X, Y)$$

Corollary 15. *If $X \to Y$ is a cofibration and Z a cofibrant object, then $X \wedge Z \to Y \wedge Z$ is a cofibration.*

Corollary 16. *If X is cofibrant and Y is fibrant, then for any Z*

$$Hom_{Ho}(Z, \underline{Hom}(X, Y)) = Hom_{Ho}(Z \wedge X, Y) \tag{16.1}$$

In (16.1), $\underline{Hom}(X, Y)$ is the pointed simplicial sheaf with components the sheaves of homomorphisms from $X \wedge (\Delta^n)_+$ to Y.

We now apply this framework to prove the following criterion for $\mathbf{A}^1$-locality.

Proposition 17. *Let F be a pointed simplicial sheaf on (QP/G). If, as a simplicial presheaf, F is flasque, then F is $\mathbf{A}^1$-local if and only if, for any U in (QP/G),*

$$F(U) \to F(U \times \mathbf{A}^1)$$

is a weak equivalence.

We recall that $\mathbf{A}^1$-*local* means that for any Y one has the following in the homotopy category

$$Hom_{Ho}(Y, F) = Hom_{Ho}(Y \wedge (h_{\mathbf{A}^1})_+, F) \tag{17.1}$$

Lemma 3. *A fibrant pointed simplicial sheaf is flasque.*

Proof. The right lifting property of $F \to *$ relative the morphisms (J_b) means that for $U \subset X$ an open embedding, the morphism $F(X) \to F(U)$ is a Kan fibration. As F is a sheaf, an upper distinguished square

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow \\ A & \longrightarrow & X \end{array}$$

gives rise to a Cartesian square

$$\begin{array}{ccc} F(X) & \longrightarrow & F(Y) \\ \downarrow & & \downarrow \\ F(A) & \longrightarrow & F(B) \end{array}$$

As $F(Y) \to F(B)$ is a Kan fibration, this square is also homotopy Cartesian. □

Lemma 4. *Proposition 17 holds of the assumption "F is flasque" is replaced by the assumption "F is fibrant".*

Proof. "Only if" (I_a) for $n = 0$ says that for any U, $(h_U)_+$ is cofibrant. By Corollary 14, for any pointed simplicial set K, one has

$$Hom_{Ho}((h_U)_+ \wedge K, F) = Hom_{Ho}(K, S((h_U)_+, F))$$

and $S((h_U)_+, F)$ is just $F(U)$. If in (17.1) we take $Y = K \wedge (h_U)_+$, so that $Y \wedge (h_{\mathbf{A}^1})_+ = K \wedge (h_{U \times \mathbf{A}^1})_+$ we get

$$Hom_{Ho}(K, F(U \times \mathbf{A}^1)) = Hom_{Ho}(K, F(U))$$

That this holds for any K means that $F(U) \to F(U \times \mathbf{A}^1)$ becomes an isomorphism in the homotopy category , hence is a weak equivalence.
"If" We apply Corollary 16. As $(h_{\mathbf{A}^1})_+$ is cofibrant and F fibrant,

$$Hom_{Ho}(Y \wedge (h_{\mathbf{A}^1})_+, F) = Hom_{Ho}(Y, \underline{Hom}((h_{\mathbf{A}^1})_+, F))$$

and it suffice to show that

$$F \to \underline{Hom}((h_{\mathbf{A}^1})_+, F)$$

is a local equivalence. This $\underline{Hom}$ is a simplicial sheaf $U \mapsto F(U \times \mathbf{A}^1)$ and the claim follows. □

Proof. We can now finish the proof of Proposition 17. Let $F \to F'$ be a fibrant replacement of F. As F and F' are flasque, $F(U) \to F'(U)$ is a weak equivalence for any U. That all $F(U) \to F(U \times \mathbf{A}^1)$ be weak equivalences is hence equivalent to all $F'(U) \to F'(U \times \mathbf{A}^1)$ be weak equivalences, while F is $\mathbf{A}^1$-local if and only if F' is. □

3.3 $\bar{\Delta}$-Closed Classes

The proof of the main theorem of this section will be postponed.

Definition 4. A class S of morphisms of pointed simplicial sheaves is Δ-closed if:

1. (Simplicial) homotopy equivalences are in S.
2. If two of f, g and fg are in S then so is the third.
3. S is stable by finite coproducts.
4. If $F_{**} \to G_{**}$ is a morphism of pointed bisimplicial sheaves, and if all $F_{*p} \to G_{*p}$ are in S, so is the diagonal $\Delta(F) \to \Delta(G)$.

Definition 5. The class S is $\bar{\Delta}$-closed if, in addition, it is stable by arbitrary coproducts and colimits of sequences $(F_* \to G_*)_n$ with the property that, degree by degree, $(F_k)_n \to (F_k)_{n+1}$ (resp. $(G_k)_n \to (G_k)_{n+1}$) is isomorphic to an embedding $A \subset A \coprod B$ of pointed sheaves.

Theorem 4. *The class of $\mathbf{A}^1$-equivalences is the $\bar{\Delta}$-closure of the union of the classes of:*

1. *Local equivalences*
2. *Morphisms $(U \times \mathbf{A}^1)_+ \to U_+$ for U in Sm/k*

In particular, the class of $\mathbf{A}^1$-equivalences is $\bar{\Delta}$-closed.

3.4 The Class of $\mathbf{A}^1$-Equivalences Is $\bar{\Delta}$-Closed

The properties 4(1), 4(2), 4(3) are clear. The last property is proved in Proposition 20.

Lemma 5. *Let A be a pointed simplicial set and X a pointed simplicial sheaf. If X is fibrant and $\mathbf{A}^1$-local, then X^A is $\mathbf{A}^1$-local.*

Proof. Because X is fibrant, for any Y, one has in the homotopy category

$$Hom_{Ho}(Y, X^A) = Hom_{Ho}(A \wedge Y, X) \tag{17.1}$$

Applying this to Y and $Y \wedge (\mathbf{A}^1_+)$ and using

$$(A \wedge Y) \wedge (\mathbf{A}^1_+) = A \wedge (Y \wedge (\mathbf{A}^1_+))$$

one deduces from the $\mathbf{A}^1$-locality of X that of X^A. □

Lemma 6. *Let $f : K \to L$ be a morphism of pointed simplicial sheaves and A be a pointed simplicial set. If f is an $\mathbf{A}^1$-equivalence, then so is $f \wedge A : X \wedge A \to Y \wedge A$.*

Proof. One has to check that for any $\mathbf{A}^1$-local X one has in the homotopy category

$$Hom_{Ho}(A \wedge L, X) = Hom_{Ho}(A \wedge K, X)$$

Replacing X by a fibrant replacement, one may assume X fibrant. Applying (17.1) one is reduced to Lemma 5. □

Lemma 7. *Let $f : K \to L$ be a morphism of pointed simplicial sheaves. If K and L are cofibrant, then f is a $\mathbf{A}^1$-equivalence if and only if for any fibrant $\mathbf{A}^1$-local X, the morphism of simplicial sets*

$$S(L, X) \to S(K, X)$$

is a weak equivalence.

Proof. "If" Taking π_0 one deduces from the assumptions that

$$Hom_{Ho}(L, X) \overset{\rightarrow}{\rightarrow} Hom_{Ho}(K, X)$$

"Only if" The assumptions imply that $S(K, X)$ and $S(L, X)$ are fibrant. For any pointed simplicial set A one has

$$Hom_{Ho}(A, S(K, X)) = Hom_{Ho}(K \wedge A, X)$$

and similarly for L and one applies Lemma 6. □

Proposition 18. *The coproduct of a family of $\mathbf{A}^1$-equivalences*

$$f_\alpha : X_\alpha \to Y_\alpha$$

is an $\mathbf{A}^1$-equivalence.

Proof. There are commutative diagrams

$$\begin{array}{ccccc}
* & \longrightarrow & X'_\alpha & \xrightarrow{f'_\alpha} & Y'_\alpha \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & X_\alpha & \xrightarrow{f_\alpha} & Y_\alpha
\end{array}$$

where morphisms on the first line are cofibrations, and where the vertical maps are local equivalences, and similarly for Y. Replacing X_α (resp. Y_α) by X'_α (resp. Y'_α) we may and shall assume that the X_α and Y_α are cofibrant. The coproducts $\coprod X_\alpha$, $\coprod Y_\alpha$ are then cofibrant too. One has

$$S(\coprod X_\alpha, X) = \prod S(X_\alpha, X)$$

and similarly for the Y_α, and one applies Lemma 7, and the fact that a product of a family of weak equivalences of fibrant pointed simplicial sets is a weak equivalence. □

Proposition 19. *The colimit*

$$f : colim F_n \to colim G_n$$

of an inductive sequence of $\mathbf{A}^1$*-equivalences* $f_n : F_n \to F_n$ *is again an* $\mathbf{A}^1$*-equivalence.*

Proof. One inductively constructs an inductive sequence of commutative squares

$$\begin{tikzcd} F'_n \arrow[r, "f'_n"] \arrow[d] & G'_n \arrow[d] \\ F_n \arrow[r, "f_n"] & G_n \end{tikzcd}$$

in which the vertical maps are local equivalences, the F'_n and G'_n are cofibrant and the transition maps $F'_n \to F'_{n+1}$, $G'_n \to G'_{n+1}$ are cofibrations. A colimit of local equivalences being a local equivalence, it is sufficient to prove the proposition for the sequence (f'_n). We hence may and shall assume that $* \to F_1 \to \ldots \to F_n \to$ is a sequence of cofibrations and similarly for the $* \to G_1 \to \ldots \to G_n \to$. The colimits F and G of those sequences are then cofibrant.

If X is fibrant and $\mathbf{A}^1$-local, $S(G, X) \to S(F, X)$ is the limit of the sequence of weak equivalences

$$S(G_n, X) \to S(F_n, X)$$

In the sequences $S(G_n, X)$ and $S(F_n, X)$ the transition maps are fibrations of fibrant objects. It follows that the limit is again a weak equivalence: the π_i of the limit map onto the limit of π_i, with fibers $(lim^1 \pi_{i+1})$-torsors. It remains to apply Lemma 7. □

Proposition 20. *Let* $F_{**} \to G_{**}$ *be a morphism of pointed bisimplicial sheaves. If all* $F_{p*} \to G_{p*}$ *are* $\mathbf{A}^1$*-equivalences, so is* $\Delta(F) \to \Delta(G)$.

To prove Proposition 20 we will functorially attach to F_{**} an inductive sequence of pointed simplicial sheaves $F^{(n)}$, whose colimit maps to $\Delta(F)$ by a local equivalence. We will then inductively prove that $F^{(n)} \to G^{(n)}$ is an $\mathbf{A}^1$-equivalence, and apply Proposition 19. We begin with preliminaries to the construction of the $F^{(n)}$.

Proposition 21. *Let* Δ_{inj} *be the category of finite ordered sets* $\Delta^n = (0, \ldots, n)$ *and increasing injective maps. For any category* C *with finite coproducts,* *the forgetting functor*

$$\omega : \Delta^{op} C \to \Delta^{op}_{inj} C$$

has a left adjoint ω'*: "formally adding degenerate simplicies":* $(\omega' X)_n$ *is the coproduct, over all* p *and all increasing surjective maps* $s : \Delta^n \to \Delta^p$*, of copies of* X_p

$$(\omega' X)_n = \coprod_s X_p$$

We define the wrapping functor $Wr : \Delta^{op} C \to \Delta^{op} C$ *as the composite* $Wr := \omega' \omega$*. For* C *the category of sets or of pointed sets one has the following.*

Lemma 8. *The adjunction map* $a : Wr(X) \to X$ *is a weak equivalence.*

Proof. We will prove it for C the category of sets. The pointed case is similar. The *fundamental groupoid* of X is the category with set of objects X_0, in which all maps are isomorphisms, and universal for the property that:

(1) $\sigma \in X_1$ defines a morphism $f(\sigma) : \partial_1(\sigma) \to \partial_0(\sigma)$.
(2) For $\tau \in X_2$, $f(\partial_1 \tau) = f(\partial_0 \tau) f(\partial_2 \tau)$.

One has $X_0 = Wr(X)_0$. To handle π_0 and π_1 it suffice to show that a induces an isomorphism of fundamental groupoids. For any X and any $p \in X_0$, $f(s_0(p))$ is the identity of p. This results from (2) applied to $s_0 s_0(p)$ which gives

$$f(s_0(p)) = f(s_0(p)) f(s_0(p))$$

As generators of the fundamental groupoid, it hence suffices to take non degenerate $\sigma \in X_1$. For relations, it then suffices to take those coming from non degenerate $\tau \in X_2$: the degenerate τ give nothing new.

If we apply this to $Wr(X)$, we find as set of generators X_1, and relations indexed by X_2, the same relations as for X.

The functor Wr commutes with passage to connected components and to passage to a covering. To handle higher π_i, this reduces us to the case where X (and hence $Wr(X)$) is connected and simply connected. In this case it suffices to check that a induces an isomorphism in homology. It does because one has a commutative diagram

$$\begin{tikzcd} C_*(X) \arrow[r, "\sim"] \arrow[dr] & C_*(Wr(X))/degeneracies \arrow[d] \\ & C_*(X)/degeneracies \end{tikzcd}$$

in which the first arrow is an isomorphism, the second the effect of a on homology, and the composite is a homotopy equivalence. □

Proposition 22. *For* X *a pointed simplicial sheaf, let* $sk_n(X)$ *be the n-th skeleton of* X*, i.e. simplicial subsheaf of* X *for which* $(sk_n(X))_p$ *is the union of the images of the degeneracies* $X_q \to X_p$ *for* $q \le n$*. One has push-out squares*

$$\begin{array}{ccc} X_{n+1} \wedge (\partial \Delta^{n+1})_+ & \longrightarrow & sk_n(Wr(X)) \\ \downarrow & & \downarrow \\ X_{n+1} \wedge (\Delta^{n+1})_+ & \longrightarrow & sk_{n+1}(Wr(X)) \end{array} \tag{22.1}$$

Let now F be bisimplicial. Each $F_{n,\bullet}$ is simplicial, and they form a simplicial system of pointed simplicial sheaves. Let us apply Wr and sk_n to the first variable, i.e. to the simplicial sheaf $F_{\bullet,m}$ for each fixed m. We again have diagrams (22.1) and, taking the diagonal Δ, one obtains push-out squares:

$$\begin{array}{ccc} F_{n+1} \wedge (\partial \Delta^{n+1})_+ & \longrightarrow & \Delta(sk_n(Wr(F))) \\ \downarrow & & \downarrow \\ F_{n+1} \wedge (\Delta^{n+1})_+ & \longrightarrow & \Delta(sk_{n+1}(Wr(F))) \end{array} \tag{22.2}$$

where F_n now stands for the pointed simplicial sheaf $F_{n,\bullet}$. This way the simplicial sheaf $\Delta(Wr(F))$, which by Lemma 8 maps to $\Delta(F)$ by a local equivalence, appears as an inductive limit of (22.2).

Proof of Proposition 20: With the notations of 22 it suffice to show that the

$$\Delta\, sk_n\, Wr(F) \to \Delta\, sk_n\, Wr(G)$$

(with $sk_n\, Wr$ applied in the first variable) are $\mathbf{A}^1$-equivalences. We prove it by induction on n.

For $n = 0$, $\Delta\, sk_0\, Wr(F) = F_0$, and $F_0 \to G_0$ is assumed to be an $\mathbf{A}^1$-equivalence. From n to $n + 1$, we have a morphism of push out squares

$$(22.2) \text{ for } F \to (22.2) \text{ for } G$$

As $F_{n+1} \to G_{n+1}$ is an $\mathbf{A}^1$-equivalence, by Lemma 6, so are its smash product with $(\partial\Delta^{n+1})_+$ and $(\Delta^{n+1})_+$. It remain to apply the

Lemma 9. *Suppose given a morphism of push out squares*

$$\begin{array}{ccc} \mathbf{1} & \longrightarrow & \mathbf{2} \\ \downarrow & & \downarrow \\ \mathbf{3} & \longrightarrow & \mathbf{4} \end{array} \quad \longrightarrow \quad \begin{array}{ccc} \mathbf{1}' & \longrightarrow & \mathbf{2}' \\ \downarrow & & \downarrow \\ \mathbf{3}' & \longrightarrow & \mathbf{4}' \end{array}$$

which is an $\mathbf{A}^1$-equivalence in positions $\mathbf{1}, \mathbf{2}$ and $\mathbf{3}$. If in each square the first vertical map is injective, then the morphism of squares is an $\mathbf{A}^1$-equivalence in position $\mathbf{4}$ as well.

Proof. Replacing the push out squares by push out squares of local equivalent objects, we may and shall assume that all objects considered are cofibrant, and that the vertical maps are cofibrant.

If X is fibrant applying $S(-, X)$ to each of the squares we get a morphism of cartesian squares, of pointed simplicial sets in which the vertical maps are cofibrations:

$$\begin{array}{ccccccc} S(\mathbf{4}, X) & \longrightarrow & S(\mathbf{3}, X) & & S(\mathbf{4}', X) & \longrightarrow & S(\mathbf{3}', X) \\ \downarrow & & \downarrow & \longleftarrow & \downarrow & & \downarrow \\ S(\mathbf{2}, X) & \longrightarrow & S(\mathbf{1}, X) & & S(\mathbf{2}', X) & \longrightarrow & S(\mathbf{1}', X) \end{array}$$

If X is in addition $\mathbf{A}^1$-local, it is a weak equivalence in positions $\mathbf{1}, \mathbf{2}$ and $\mathbf{3}$, hence also in position $\mathbf{4}$. By Lemma 7, this proves Lemma 9, finishing the proof of Proposition 20 as well as of the claim that the class of $\mathbf{A}^1$-equivalences is $\bar{\Delta}$-closed. □

3.5 The Class of $\mathbf{A}^1$*-Equivalences as a* $\bar{\Delta}$*-Closure*

In this section we finish the proof of Theorem 4.

Proposition 23. *The homotopy push-out of a diagram*

$$Q \quad : \quad \begin{array}{ccc} K & \longrightarrow & L \\ \downarrow & & \\ M & & \end{array} \tag{23.1}$$

is the push-out K_Q *of*

$$Q \quad : \quad \begin{array}{ccc} K \vee K & \longrightarrow & M \vee L \\ \downarrow & & \\ K \wedge (\Delta^1)_+ & & \end{array} \tag{23.2}$$

where the vertical map $K \wedge (\Delta^0 \coprod \Delta^0)_+ \to K \wedge (\Delta^1)_+$ *is induced by* $\partial_1, \partial_0 : \Delta^0 \to \Delta^1$ *mapping* Δ^0 *to* 0 *(resp.* 1*) in* Δ^1*. In the case of simplicial sets,* $|K_Q|$ *maps to* $|\Delta^1| = [0, 1]$ *with fibers* $|M|$ *above* 0*,* $|L|$ *above* 1*, and* $|K|$ *above* $(0, 1)$*.*

The homotopy push-out K_Q is the diagonal of the bisimplicial object with columns $M \vee K^{\vee n} \vee L$ obtained by formally adding degeneracies to

$$K \overset{\rightarrow}{\rightarrow} M \vee L$$

in $\Delta^{op}_{inj}\Delta^{op}(Sh_\bullet)$ (cf. 21) [∂_0 maps K to L, ∂_1 maps K to M]. If $f : Q \to Q'$ is a morphism of diagrams (23.1), the induced morphism from K_Q to $K_{Q'}$ is hence in the closure of the three components of f for the operation of finite coproduct and diagonal (4(3), (4)).

A commutative square

$$\begin{tikzcd} K \arrow[r] \arrow[d] & L \arrow[d] \\ M \arrow[r] & N \end{tikzcd} \tag{23.3}$$

induces a morphism $K_Q \to N$.

Example 1. Let $f : K \to L$ be a morphism. The homotopy push-out of

$$\begin{tikzcd} K \arrow[r] \arrow[d, "id"'] & L \\ K & \end{tikzcd}$$

is the cylinder $cyl(f)$ of f. The morphisms

$$L \to cyl(f) \to L$$

are homotopy equivalences. To check that the composite $cyl(f) \to L \to cyl(f)$ is homotopic to the identity, one observes that $cyl(f)$ is the push-out of

$$\begin{tikzcd} K \arrow[r] \arrow[d] & L \\ K \wedge (\Delta^1)_+ & \end{tikzcd}$$

(the vertical map induced by $\partial_0 : \Delta^0 \to \Delta^1$ mapping Δ^0 to 1) and that the composite $cyl(f) \to cyl(f)$ is induced by $\Delta^1 \to \Delta^0 \to \Delta^1$, homotopic to the identity by a homotopy fixing 1.

Similar arguments would show that the homotopy push out $cyl'(f)$ of

$$\begin{tikzcd} K \arrow[r, "id"] \arrow[d] & K \\ M & \end{tikzcd}$$

is homotopic to L by $L \to cyl'(L) \to L$.

Example 2. In any category with finite coproducts, a *coprojection* is a map isomorphic to the natural map $A \to A \coprod B$ for some A and B. If in a push-out square of pointed simplicial sheaves

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ \downarrow & & \downarrow \\ M & \longrightarrow & N \end{array} \tag{23.4}$$

the morphism f is a coprojection: $L = K \vee A$, the square (23.4) is the coproduct of the squares

$$\begin{array}{ccc} K & \xrightarrow{id} & K \\ \downarrow & & \downarrow \\ M & \xrightarrow{id} & M \end{array} \quad \text{and} \quad \begin{array}{ccc} * & \longrightarrow & A \\ \downarrow & & \downarrow \\ * & \longrightarrow & A \end{array} \tag{23.5}$$

and the resulting morphism $K_Q \to N$ is a homotopy equivalence, being the coproduct of the homotopy equivalences of Example 1 resulting from the two squares (23.5). The same conclusion applies if $K \to M$ is a coprojection.

A morphism of pointed simplicial sheaves $K \to L$ is a termwise coprojection if each $K_n \to L_n$ is a coprojection of pointed sheaves. Example: for any diagram (23.1), the morphisms $L, M \to K_Q$ are termwise coprojections. For any morphism $f : K \to L$, this applies in particular to $K, L \to cyl(f)$.

Proposition 24. *If in a cocartesian square (23.3) either $K \to L$ or $K \to M$ is a termwise coprojection, then the resulting morphism from K_Q to N is in the Δ-closure of the empty set of morphisms.*

Proof. For each n, we have a cocartesian square of pointed sheaves

$$Q_n \quad : \quad \begin{array}{ccc} K_n & \longrightarrow & L_n \\ \downarrow & & \downarrow \\ M_n & \longrightarrow & N_n \end{array}$$

Let us view it as a cocartesian square of pointed simplicial sheaves. By Example 2, it gives rise to a homotopy equivalence $K_{Q_n} \to N_n$. One concludes by observing that $K_Q \to N$ is the diagonal of this simplicial system of morphisms. □

Corollary 25. *If in a cocartesian square (23.3):*

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ M & \xrightarrow{f'} & N \end{array}$$

f or g is a termwise coprojection, then:

1. *f' is in the Δ-closure of $\{f\}$.*
2. *g' is in the Δ-closure of $\{g\}$.*

Proof of (1): The morphism of cocartesian squares

$$\begin{tikzcd} Q' : & K \arrow[r, "id"] \arrow[d] & K \arrow[d] & \longrightarrow & Q : & K \arrow[r] \arrow[d] & L \arrow[d] \\ & M \arrow[r, "id"] & M & & & M \arrow[r] & N \end{tikzcd}$$

defines a commutative square

$$\begin{tikzcd} K_{Q'} \arrow[r] \arrow[d] & K_Q \arrow[d] \\ M \arrow[r] & N \end{tikzcd}$$

in which the vertical maps are in the Δ-closure of the empty set by Proposition 24, while the first horizontal map is in the Δ-closure of f by 8.

Proof of (2): One similarly uses

$$\begin{tikzcd} Q' : & K \arrow[r] \arrow[d, "id"'] & L \arrow[d, "id"] & \longrightarrow & Q : & K \arrow[r] \arrow[d] & L \arrow[d] \\ & K \arrow[r, "i"] & L & & & M \arrow[r] & N. \end{tikzcd}$$

Proposition 26. *A pointed simplicial sheaf $F_\bullet$ is* reliably compact *if it coincides with its n-skeleton for some n and each F_i is compact in the sense that the functor $Hom(F_i, -)$ commutes with filtering colimits. This property is stable by $F_\bullet \to F_\bullet \wedge K$ for K a finite pointed simplicial set (finite number of non degenerate simplices) and implies that $F_\bullet$ is compact.*

Construction 27. Let E and N be classes of morphisms such that:

(a) Sources and targets are reliably compact.
(b) Each f in N is a termwise coprojection.

We will construct a functor Ex from pointed simplicial sheaves to pointed simplicial sheaves and a morphism $Id \to Ex$ such that:

(i) For any F, $F \to Ex(F)$ is in the $\bar{\Delta}$-closure of E
(ii) If $f : K \to L$ is in E, the morphism

$$S(L, Ex(X)) \to S(K, Ex(X)) \tag{27.1}$$

is a weak equivalence.

(iii) If $f : K \to L$ is in N, the morphism (27.1) is a Kan fibration.

Let us factorize $f : K \to L$ as $K \to cyl(f) \to L$. As the second map is a homotopy equivalence, the first is in the Δ-closure of E. In the corresponding factorization of (27.1):

$$S(L, Ex(F)) \to S(cyl(f), Ex(F)) \to S(K, Ex(F))$$

the first map is a homotopy equivalence. To obtain (ii), it hence suffices that $S(cyl(f), Ex(F)) \to S(K, Ex(F))$ be a weak equivalence.

Replacing each $f : K \to L$ in E by the corresponding $K \to cyl(f)$, this reduces us to the case where

(c) each f in E is a termwise coprojection,

and we will construct in this case a functor Ex such that

(iii)* for f in E, (27.1) is a trivial fibration.

The conditions (ii), (iii)* are lifting properties:

for f in E, in squares:

$$\begin{tikzcd} \partial\Delta^n_+ \arrow[r] \arrow[d] & S(L, Ex(F)) \arrow[d] \\ \Delta^n_+ \arrow[r] & S(K, Ex(F)) \end{tikzcd}$$

for f in N, in squares:

$$\begin{tikzcd} (\Lambda^n_k)_+ \arrow[r] \arrow[d] & S(L, Ex(F)) \arrow[d] \\ \Delta^n_+ \arrow[r] & S(K, Ex(F)) \end{tikzcd}$$

In the first case, the data are morphisms $\Delta^n_+ \wedge K \to Ex(F)$ and $\partial\Delta^n_+ \wedge L \to Ex(F)$ agreeing on $\partial\Delta^n_+ \wedge K$, i.e. a morphism

$$(\Delta^n_+ \wedge K) \coprod_{\partial\Delta^n_+ \wedge K} (\partial\Delta^n_+ \wedge L) \to Ex(F)$$

and we want it to extend to $\Delta^n_+ \wedge L$. Similarly in the second case, with $\partial\Delta^n$ replaced by Λ^n_k:

for f in E:

$$\begin{tikzcd} (\Delta^n_+ \wedge K) \coprod_{\partial\Delta^n_+ \wedge K} (\partial\Delta^n_+ \wedge L) \arrow[r] \arrow[d] & Ex(F) \arrow[d] \\ \Delta^n_+ \wedge L \arrow[r] & * \end{tikzcd} \tag{27.2}$$

for f in N:

$$\begin{array}{ccc} (\Delta^n_+ \wedge K) \coprod_{(\Lambda^n_k)_+ \wedge K} ((\Lambda^n_k)_+ \wedge L) & \longrightarrow & Ex(F) \\ \big\downarrow & & \big\downarrow \\ \Delta^n_+ \wedge L & \longrightarrow & * \end{array} \tag{27.3}$$

The left vertical maps are termwise coprojections, and their sources are compact. One now uses the standard trick of defining $Ex(F)$ as the inductive limit of the iterates of functors $F \to T(F)$, where $T(F)$ is deduced from F by push out, simultaneously for all

$$(\Delta^n_+ \wedge K) \coprod_{\partial\Delta^n_+ \wedge K} (\partial\Delta^n_+ \wedge L) \to Ex(F) \quad (f : K \to L \text{ in } E)$$

and

$$(\Delta^n_+ \wedge K) \coprod_{(\Lambda^n_k)_+ \wedge K} ((\Lambda^n_k)_+ \wedge L) \to Ex(F) \quad (f : K \to L \text{ in } N)$$

The push out is by

$$\bigvee (\text{sources}) \to \bigvee (\Delta^n_+ \wedge L)$$

a morphism which is a termwise coprojection. By 26, to check that the resulting $F \to Ex(F)$ is in the $\bar{\Delta}$-closure of E, it suffices to check that the left vertical morphism in (27.2) (resp. (27.3)) is in the Δ-closure of E (resp. of the empty set).

For (27.2), this is the map marked **3** in

$$\begin{array}{ccccc} \partial\Delta^n_+ \wedge K & \xrightarrow{\mathbf{1}} & \partial\Delta^n_+ \wedge L & & \\ \big\downarrow & & \big\downarrow & & \\ \Delta^n_+ \wedge K & \xrightarrow{\mathbf{2}} & \dots & \xrightarrow{\mathbf{3}} & \Delta^n_+ \wedge L \end{array}$$

The morphisms **1** and **3** $\circ$ **2** are in the Δ-closure of E. So is **2** by 26 and one applies the 2 out of 3 property.

For (27.3), the diagram is

$$\begin{array}{ccccc} (\Lambda^n_k)_+ \wedge K & \longrightarrow & (\Lambda^n_k)_+ \wedge L & & \\ \big\downarrow{\scriptstyle \mathbf{1}} & & \big\downarrow{\scriptstyle \mathbf{2}} & & \\ \Delta^n_+ \wedge K & \longrightarrow & \dots & \xrightarrow{\mathbf{3}} & \Delta^n_+ \wedge L \end{array}$$

with **1** and **3** $\circ$ **2** in the Δ-closure of the empty set. Indeed, Λ_k^n and Δ^n are both contractible.

Remark 4. Let P be a property of pointed simplicial schemes stable by coproduct, and suppose that:

(a) For $f : K \to L$ in E, the K_n and L_n have property P.
(b) For $f : K \to L$ in N, f is in degree n isomorphic to the natural map $K_n \to K_n \vee A$ for some A having property P.

The functor Ex constructed in Construction 27 is then such that for any K, each morphism $K_n \to Ex(K)_n$ is isomorphic to some $K_n \to K_n \vee A$ where A has property P. In particular, if the K_n have property P, so have the $Ex(K)_n$.

Proposition 28. *(Proof of Theorem 4)* We apply Construction 11, on the site QP/G, taking for E and N the following classes.

E: For any X in the site, the morphism

$$(X \times \mathbf{A}^1)_+ \to X_+ \tag{28.1}$$

and for any upper distinguished square

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow \\ A & \longrightarrow & X, \end{array} \tag{28.2}$$

the morphism

$$(K_Q)_+ \to X_+ \tag{28.3}$$

N: For any X in the site,

$$(\emptyset)_+ \to X_+ \tag{28.4}$$

If a pointed simplicial sheaf G is of the form $Ex(F)$, that (28.4) is in N ensures that each $G(X)$ is Kan. That (28.3) is in E ensures that for each upper distinguished square (28.2), the morphism

$$G(X) \to S((K_Q)_+, G)$$

is a weak equivalence. As each $G(Y)$ is Kan, $S((K_Q)_+, G)$ is the homotopy fiber product of $G(A)$ over $G(B)$, and

$$\begin{array}{ccc} G(X) & \longrightarrow & G(A) \\ \downarrow & & \downarrow \\ G(Y) & \longrightarrow & G(B) \end{array}$$

is homotopy cartesian: G is flasque.

Further, as (28.1) is in E, for each X,

$$G(X) \to G(X \times \mathbf{A}^1)$$

is a weak equivalence: by Theorem 3, G is $\mathbf{A}^1$-local.

Suppose now that $f : F \to G$ is a $\mathbf{A}^1$-equivalence. In the commutative diagram

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & & \downarrow \\ Ex(F) & \longrightarrow & Ex(G) \end{array}$$

the vertical maps are in the $\bar{\Delta}$-closure of the morphisms (28.1) and (28.2), the later being local equivalences. In particular, they are $\mathbf{A}^1$-equivalences and $Ex(F) \to Ex(G)$ is an $\mathbf{A}^1$-equivalence between $\mathbf{A}^1$-local objects , hence is a local equivalence. It follows that f is in the required $\bar{\Delta}$-closure, proving Theorem 4.

The functor Ex used introduced in 28 can also be used to prove the following lemma.

Lemma 10. *If $F^{(i)} \to F^{(j)}$ is a filtering system of $\mathbf{A}^1$-equivalences, then $F^{(n)} \to colim_i F^{(i)}$ is again an $\mathbf{A}^1$-equivalence.*

Proof. Consider the square:

$$\begin{array}{ccc} F^{(n)} & \longrightarrow & colim_i F^{(i)} \\ \downarrow & & \downarrow \\ Ex(F^{(n)}) & \longrightarrow & Ex(colim_i F^{(i)}). \end{array}$$

Since the functor Ex commutes with filtering colimits, the bottom arrow is a filtering colimit of local equivalences, hence a local equivalence. The vertical maps are $\mathbf{A}^1$-equivalences, hence the top map is an $\mathbf{A}^1$-equivalence. □

3.6 One More Characterization of Equivalences

Denote by $[QP/G]_+$ the full subcategory in the category of pointed sheaves on QP/G generated by all coproducts of sheaves of the form $(h_X)_+$.

Theorem 5. *The class of local equivalences (resp. $\mathbf{A}^1$-equivalences) in Δ^{op} $[QP/G]_+$ is the smallest class W which contains morphisms $(K_Q)_+ \to X_+$ for Q upper distinguished and has the following properties:*

1. *Simplicial homotopy equivalences (resp. and $\mathbf{A}^1$-homotopy equivalences) are in W.*

2. *If two of f, g and fg are in W then so is the third.*
3. *If $F^{(i)} \to F^{(j)}$ is a filtering system of termwise coprojections in W, then $F^{(n)} \to colim_i F^{(i)}$ is again in W.*
4. *If $F_{**} \to F'_{**}$ is a morphism of bisimplicial objects, and if all $F_{*p} \to F'_{*p}$ are in W, so is the diagonal $\Delta(F) \to \Delta(F')$.*

The proof is given in 32.

Lemma 11. *If the morphism $f : F \to G$ is such that, for each U, $F(U) \to G(U)$ is a weak equivalence, and if the F_n and G_n are all of the form $(\coprod h_{U_i})_+$, then f is in the $\bar{\Delta}$-closure of the empty set.*

The proof will use the following construction.

Construction 29. Let C be a category, and let C_0 be a set of objects of C, such that any isomorphism class has a representative in C_0. Let i_* be the functor which to a presheaf of pointed sets on C attaches the family of pointed sets $(F(U))_{U \in C_0}$. It has a left adjoint i^*:

$$\text{family } (A_U)_{U \in C_0} \mapsto \bigvee_U ((h_U)_+ \wedge A_U) =$$

$$= (\text{disjoint sum over the } U \in C_0 \text{ and } (A_U - *) \text{ of } h_U)_+$$

If C_0 is viewed as a category whose only morphisms are identities, the natural functor

$$i : C_0 \to C$$

defines a morphism of sites $C \to C_0$, both endowed with the trivial topology (any presheaf a sheaf), and i_*, i^* are the corresponding direct and inverse image of pointed sheaves.

By a general story valid for any pair of adjoint functors, for any pointed presheaf F on C, the $(i^* i_*)^{n+1}(F)$ form a pointed simplicial presheaf $R(F)$ augmented to F:

$$a : R(F) \to F$$

Further:

(a) If F is of the form $i^* A$, i.e. of the form $(\coprod h_{U_i})_+$, a is a homotopy equivalence.
(b) For any F, $i^*(a)$ is a homotopy equivalence: for each U in C, $R(F)(U) \to F(U)$ is a homotopy equivalence.

For a simplicial presheaf F we define

$$R(F) = \Delta(\text{simplicial system of } R(F_p))$$

Proposition 30. *(Proof of Lemma 11)* Let us say that X in QP/G is connected if it is not empty and is not a disjoint union: G should act transitively on the set of connected components of X. Let $C \subset QP/G$ be the full subcategory of connected objects. A sheaf F on QP/G is determined by its restriction to C. Indeed, $F(\coprod X_i) = \prod F(X_i)$. To apply Construction 29, we will use this remark to identify the category of sheaves on QP/G to a full subcategory of the category of presheaves on C. For any C_0 as in Construction 29, the functor i^* takes values in sheaves, that is in the restriction of sheaves to C. Indeed, for X connected, $(\coprod h_{U_i})_+(X)$ is the same, whether $\coprod$ and $_+$ are taken in the sheaf or in the presheaf sense.

Fix $f : F \to G$ as in Lemma 11. For each n, the assumption on F_n ensures that $R(F_n) \to F_n$ is in the Δ-closure of the empty set, and similarly for G.

For each (connected) U, the morphism of pointed simplicial sets $F(U) \to G(U)$ is a weak equivalence, hence in the $\bar{\Delta}$-closure of the empty set. It follows that $i^*i_*(F) \to i^*i_*(G)$: the $\vee$ over C_0 of the

$$(h_U)_+ \wedge F(U) \to (h_U)_+ \wedge G(U)$$

is in the $\bar{\Delta}$-closure of the empty set. Iterating one finds the same for $(i^*i_*)^n(F) \to (i^*i_*)^n(G)$, and $R(F) \to R(G)$ is in this $\bar{\Delta}$-closure too. It remains to apply the two out of three property to

$$\begin{array}{ccc} R(F) & \longrightarrow & R(G) \\ \downarrow & & \downarrow \\ F & \longrightarrow & G \end{array}$$

Lemma 12. *If $f : F \to G$ is a local equivalence and if the F_n and G_n are all of the form $(\coprod h_{U_i})_+$, then f is in the $\bar{\Delta}$-closure of the $(K_Q)_+ \to X_+$ for Q upper distinguished.*

Proof. We will use the Construction 27 for E the class of morphisms $(K_Q)_+ \to X_+$ for Q upper distinguished, and for N the class of morphisms $* \to X_+$. By Remark 4, if the F_n are of the form $(\coprod h_{U_i})_+$, so are the $Ex(F)_n$. In the commutative diagram

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & & \downarrow \\ Ex(F) & \longrightarrow & Ex(G) \end{array}$$

the vertical maps are in the required Δ-closure. They are in particular local equivalences and so is $Ex(f)$. One verifies as in 28 that $Ex(F)$ and $Ex(G)$ are flasque. By Construction 27(ii), for each U, $Ex(f)(U)$ is a weak equivalence, and it remains to apply Lemma 11 to Ex(f). □

Lemma 13. *If $f : F \to G$ is an $\mathbf{A}^1$-equivalence and if the F_n and G_n are all of the form $(\coprod h_{U_i})_+$, then f is in the $\bar{\Delta}$-closure of the $(K_Q)_+ \to X_+$ for Q upper distinguished and $(X \times \mathbf{A}^1 \to X)_+$ for $X \in QP$.*

Proof. Similar to the proof of Lemma 12. □

Proposition 31. *Since for any simplicial sheaf F the map $R(F) \to F$ is a local equivalence Lemmas 12 and 13 imply that for any local (resp. $\mathbf{A}^1$-) equivalence $f : F \to G$, the morphism $R(f)$ belongs to the $\bar{\Delta}$-closure of the $(K_Q)_+ \to X_+$ for Q upper distinguished (resp. the $(K_Q)_+ \to X_+$ for Q upper distinguished and $(X \times \mathbf{A}^1 \to X)_+$ for $X \in QP$).*

Proposition 32. *Proof of Theorem 5*: We consider only the case of $\mathbf{A}^1$-equivalences. Proposition 20 and Lemma 10 imply that $\mathbf{A}^1$-equivalences contain the class W. In view of Lemma 13 it remains to see that W is $\bar{\Delta}$-closed. The only condition to check is that it is closed under coproducts. Let

$$f_\alpha : F^{(\alpha)} \to H^{(\alpha)}, \quad \alpha \in A$$

be a family of morphisms in W. For a finite subset I in A set

$$\Phi_I = (\coprod_{\alpha \in I} H^{(\alpha)}) \coprod (\coprod_{\alpha \in A - I} F^{(\alpha)})$$

For $I \in J$ we have a morphism $\Phi_I \to \Phi_J$ and the map $\coprod_{f_\alpha}$ is isomorphic to the map

$$\Phi_\emptyset \to colim_{I \subset A} \Phi_I$$

It remains to show that $\Phi_I \to \Phi_{I \cup \{\alpha}$ is in W. This morphism is of the form $Id_H \coprod (f : F \to F')$ where f is in W. Using the fact that W is closed for diagonals we reduce to the case $H = \coprod (h_U)_+$. Using the same reasoning as above we further reduce to the case $H = (h_U)_+$.

Consider the class of f such that $Id_{(h_U)_+} \coprod f$ is in W. This class clearly contains morphisms $(K_Q)_+ \to X_+$, has the two out of three property and is closed under filtering colimits. It also contains simplicial homotopy equivalences. It contains morphisms of the form $p_+ : (X \times \mathbf{A}^1)_+ \to X_+$ because such morphisms are $\mathbf{A}^1$-homotopy equivalences.

4 Solid Sheaves

4.1 Open Morphisms and Solid Morphisms of Sheaves

We fix S and G as in Sect. 3.1, and will work in $(QP/G)_{Nis}$. The story could be repeated in $(Sm/S)_{Nis}$.

Definition 6. A morphism of sheaves $f : F \to G$ is open if it is relatively representable by open embeddings, i.e. if for any morphism $u : h_X \to G$ (that is,

$u \in G(X)$, X in QP/G), the fiber product $F \times_G h_X$ mapping to h_X is isomorphic to $h_U \to h_X$ for U a G-stable open subset of X.

In other words: f should identify F with a subsheaf of G and, for any $s \in G(X)$, there is U open in X and G-stable such that the pull-back of s with respect to $Y \to X$ is in $F(Y)$ if and only if Y maps to U.

The property "open" is stable under composition. It is also stable by pull-back: if in a cartesian square

$$\begin{array}{ccc} F' & \xrightarrow{f'} & G' \\ \downarrow & & \downarrow{\scriptstyle u} \\ F & \xrightarrow{f} & G \end{array} \tag{32.1}$$

f is open, then f' is open. This follows from transitivity of pull-backs. Conversely, if f' is open and u is an epimorphism, then f is open. Indeed,

Lemma 14. *For $F \to h_X$ a morphism, the property that F is represented by U open in X is local on X (for the Nisnevich topology).*

Proof. Suppose that the X_α cover X, and that each $F_\alpha = F \times_{h_X} h_{X_\alpha}$ is represented by $U_\alpha \subset X_\alpha$. For $Y \to X_\alpha$, $F_Y := F \times_{h_X} h_Y$ is then represented by $U_Y \subset Y$ with U_Y the inverse image of U_α. By descent for open embedding, the U_α come from some $U \subset X$, we have locally on X an isomorphism $F \xrightarrow{\sim} h_U$ and by descent for isomorphisms of sheaves one has $F \xrightarrow{\sim} h_U$. $\square$

Given a square of the form (32.1) with f' open and u an epimorphism, if s is in $G(X)$, s can locally be lifted to a section of G'. As f' is open, it follows that locally on X, $F \times_G h_X$ is represented by an open subset. Applying Lemma 14 one concludes that f is open. The same argument shows that if we have cartesian diagrams

$$\begin{array}{ccc} F'_\alpha & \xrightarrow{f'_\alpha} & G'_\alpha \\ \downarrow & & \downarrow{\scriptstyle u_\alpha} \\ F & \xrightarrow{f} & G \end{array}$$

with each f'_α open and $\coprod u_\alpha : \coprod G'_\alpha \to G$ onto, then f is open.

Proposition 33. *The property "open" is stable by push-outs.*

Proof. Suppose

$$\begin{array}{ccc} F & \xrightarrow{f} & G \\ \downarrow & & \downarrow \\ F' & \xrightarrow{f'} & G' \end{array}$$

is a cocartesian diagram, with f open. In particular f is a monomorphism, and it follows that f' is a monomorphism and that the square is cartesian as well. The morphism $F' \coprod G \to G'$ is onto. The pull-back of f' by $f' : F' \to G'$ is an isomorphism (f' being a monomorphism) hence open. The pull-back of f' by $G \to G'$ is just f, open by assumption. It follows that f' is open. □

We now fix a class C of open embeddings $U \to V$ in (QP/G). We require the following stabilities

Conditions 34. 1. If $U \to U' \to V$ are open embeddings and if $U \to V$ is in C, so is $U' \to V$.

2. If $U \to V$ is an open embedding in C, and if $f : V' \to V$ is etale, with $f^{-1}(Y) \subset Y$ for Y the complement of U in V, then $f^{-1}(U) \to V'$ is in C.

The classes C we will have to consider are the following:

1. The open embeddings $U \to V$ with V smooth.
2. The open embeddings $U \to V$ with V smooth such that the action of G is free on $V - U$. Equivalently: V is the union of U and the open subset on which the action of G is free.
3. When working in (Sm/S): all open embeddings.

Definition 7. A morphism $f : F \to G$ is C-solid if it is a composite $F = F_0 \to F_1 \to \ldots \to F_n = G$ where each $F_i \to F_{i+1}$ is deduced by push-out from some $h_U \to h_X$, $U \subset X$ in C.

A sheaf F is solid if $\emptyset \to F$ is C-solid.

In the pointed context, a pointed sheaf is (pointed) C-solid if the morphism $pt \to F$ is C-solid.

Example 1. For U open in X, let $h_{X/U}$ be the sheaf h_X, with the subsheaf h_U contracted to a point p. If $U \to X$ is in C, then $p : pt \to h_{X/U}$ is solid: it is the push-out of $h_U \to h_X$ by $h_U \to pt$. Thom spaces are of this form: starting from a vector bundle V on Y, one contracts, in the total space of this vector bundle, the complement of the zero section to a point.

The class of solid morphisms is the smallest class closed by compositions and push-outs which contains all $h_U \to h_X$ for $U \subset X$ in C. By Proposition 33 solid morphisms are open.

For $F \to G$ a monomorphism of sheaves, define G/F to be the pointed sheaf obtained by contracting F to a point: one has a cocartesian square

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & & \downarrow \\ pt & \longrightarrow & F/G \end{array}$$

By transitivity of push-out, any cocartesian diagram

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & & \downarrow \\ F' & \longrightarrow & G' \end{array}$$

induces an isomorphism $G/F \to G'/F'$.

Proposition 35. *A morphism of sheaves $f : F \to G$ is C-solid if and only if it is a composite $F = F_0 \to F_1 \to \ldots \to F_n = G$ of monomorphisms where each F_i/F_{i+1} is isomorphic to some $h_{V/U} = h_V/h_U$ for $U \subset V$ in C.*

Proof. If a morphism $F \to G$ is deduced by push-out from $U \to V$, G/F is isomorphic to $h_{V/U}$. From this, "only if" results. Conversely, if we have

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & & \downarrow \\ * & \longrightarrow & h_{V/U} \end{array} \quad , \quad \begin{array}{ccc} h_U & \longrightarrow & h_V \\ \downarrow & & \downarrow \\ * & \longrightarrow & h_{V/U} \end{array} \tag{35.1}$$

cocartesian, and if $h_V \to h_{V/U}$ lifts to G, then $F \to G$ is deduced by push-out from $h_U \to h_V$. Indeed, the diagrams (35.1) being cartesian as well as cocartesian, we have a cartesian

$$\begin{array}{ccc} h_U & \longrightarrow & h_V \\ \downarrow & & \downarrow \\ F & \longrightarrow & G \end{array}$$

If G_1 is deduced from $h_U \to h_V$ by push-out:

$$\begin{array}{ccccc} h_U & \longrightarrow & h_V & \xrightarrow{=} & h_V \\ \downarrow & & \downarrow & & \downarrow \\ F & \longrightarrow & G_1 & \longrightarrow & G \end{array}$$

then $G_1/F \cong G/F$ and it follows that $G_1 = G$.

Let us suppose only that we have $v : \tilde{V} \to V$ etale, inducing an isomorphism from $\tilde{V} - v^{-1}(U)$ to $V - U$ and a lifting of $h_{\tilde{V}} \to h_{V/U}$ to G. If $\tilde{U} := v^{-1}(U)$, $h_{\tilde{V}/\tilde{U}} \to h_{V/U}$ is an isomorphism. This is most easily checked by applying the fiber functors defined by a G-local henselian Y: a morphism $Y \to V$, if it does not map to U, lifts uniquely to a morphism to $\tilde{V}$. The assumptions made hence imply that $F \to G$ is a push-out of $h_{\tilde{U}} \to h_{\tilde{V}}$. Note that by the second stability property of C, $\tilde{U} \to \tilde{V}$ is in C.

We will reduce the proof of "if" to that case. We have to show that if a monomorphism $f : F \to G$ is such that $G/F \cong h_{V/U}$ with $U \to V$ in C, then f is C solid. The cocartesian square

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & & \downarrow \\ pt & \longrightarrow & F/G \end{array} \tag{35.2}$$

induces an epimorphism $pt \coprod G \to h_{V/U}$. The natural section of $h_{V/U}$ on V can hence locally be lifted to pt or to G: for some filtration $\emptyset = Z_n \subset \cdots \subset Z_1 \subset Z_0 = V$ of V by closed equivariant subschemes, we have etale maps $\phi_i : Y_i \to V$ with a (equivariant) section over $Z_i - Z_{i+1}$, and a lifting of $h_{Y_i} \to h_{V/U}$ to pt or to G. Note $V_i := V - Z_{i+1}$. We may:

1. Start with $V_0 = U$, taking $Y_0 = U$: here the lifting is to pt
2. Assume $V_i \neq V_{i+1}$; the succeeding liftings then cannot be to pt: they must be to G
3. Shrink Y_i, first so that it maps to V_i, next so that it induces an isomorphism from $Y_i - \phi_i(V_{i-1})$ to $Z_i - Z_{i+1}$

As $F \to G$ is a monomorphism the cocartesian (35.2) is cartesian as well. The composition

$$pt = h_{V_0/U} \to h_{V_1/U} \to \ldots \to h_{V/U}$$

gives by pull-back a factorization of $F \to G$ as

$$F \to F_1 \to \ldots \to G$$

with each

$$\begin{array}{ccc} F_i & \longrightarrow & F_{i+1} \\ \downarrow & & \downarrow \\ h_{V_i/U} & \longrightarrow & h_{V_{i+1}}/U \end{array}$$

cartesian and cocartesian, hence $F_{i+1}/F_i \cong h_{V_{i+1}}/h_{V_i}$. Further, the morphism $\phi_{i+1} : h_{Y_{i+1}} \to h_{V_{i+1}} \to h_{V/U}$ lifts to G, hence $h_{Y_{i+1}} \to h_{V_{i+1}/U}$ lifts to F_{i+1}. It follows that $F_i \to F_{i+1}$ is a push-out of $\phi_{i+1}^{-1}(V_i) \to Y_{i+1}$, which is in C, and solidity follows. □

Remark 1. Another formulation of Proposition 35 is: a morphism $F \to G$ is C-solid if and only if the pointed sheaf G/F is an iterated extension of $h_{V/U}$'s with $U \to V$ in C, in the sense that there are morphisms

$$pt = H_0 \to \ldots \to H_n = G/F$$

with each H_{i+1}/H_i of the form $h_{V/U}$.

Proposition 36. *If $f : F \to G$ is open and G is C-solid, then f is C-solid.*

Proof. In the proof we say "solid" instead of "C-solid". Let (*) be the property of a sheaf G that any open $f : F \to G$ is solid. If G is solid, G sits at the end of a chain $\emptyset = G_0 \to G_1 \to \ldots \to G_n = G$ with each $G_i \to G_{i+1}$ push out of some $h_U \to h_X$ for $U \to X$ in C. We prove by induction on i that G_i satisfies (*).

For $i = 1$, $G_1 = h_X$ is representable and $\emptyset \to X$ is in C. If $f : F \to G_1$ is open, it is of the form $h_U \to h_X$ for U open in X, hence solid by Condition 34(1). It remains to check that if in a cocartesian square

$$\begin{array}{ccc} h_U & \longrightarrow & h_X \\ \downarrow & & \downarrow \\ G' & \longrightarrow & G \end{array} \tag{36.1}$$

the sheaf G' satisfies (*), so does G. In (36.1), $h_U \to h_X$ is a monomorphism and the square (36.1) hence cartesian as well as cocartesian.

Fix $f : F \to G$ open, and take the pull-back of (36.1) by f. It is again a cartesian and cocartesian square and, f being open, it is of the form

$$\begin{array}{ccc} h_V & \longrightarrow & h_Y \\ \downarrow & & \downarrow \\ F' & \longrightarrow & F \end{array} \tag{36.2}$$

where Y is open in X and $V = U \cap Y$. The diagram

$$\begin{array}{ccccc} F' & \longrightarrow & F & \longrightarrow & h_{Y/V} \\ \downarrow & & \downarrow & & \downarrow \\ G' & \longrightarrow & G & \longrightarrow & h_{X/U} \\ \downarrow & & \downarrow & & \downarrow \\ G'/F' & \longrightarrow & G/F & \longrightarrow & h_{X/(U\cup V)} \end{array}$$

expresses G/F as an extension of $h_{X/(U\cup Y)}$ by G'/F' and one concludes by Remark 1 using (*) for G' and the fact that $U \cup Y \to X$ is in C. $\square$

Proposition 37. *The pull-back of a solid morphism f by an open morphism s is solid. In particular, if $g : F \to G$ is open and if G is solid, so is F.*

Proof. Since the pull-back of an open morphism is open, it suffices to check the proposition for f a push-out of $h_U \subset h_X$ for U open in X:

$$\begin{array}{ccc} h_U & \longrightarrow & h_X \\ \downarrow & & \downarrow \\ G' & \xrightarrow{f} & G \end{array}$$

Pulling back by g, we obtain a cocartesian square

$$\begin{tikzcd} h_{U'} \arrow[r] \arrow[d] & h_{X'} \arrow[d] \\ F' \arrow[r] & F \end{tikzcd}$$

with U' open in U and X' open in X. This shows that $F' \to F$ is solid. $\square$

Suppose now that we are given two classes C and C' of open embeddings satisfying conditions 34. We define $C \times C'$ as the smallest class stable by Condition 34 containing the

$$(U \times V') \cup (U' \times V) \subset V \times V'$$

for $U \subset V$ in C and $U' \subset V'$ in C'.

Example 2. If C is a class of all open embeddings and C' is the class of those $U' \subset V'$ for which G acts freely outside U', then $C \times C' = C'$.

Proposition 38. *If the pointed sheaves F and F' are respectively C and C'-solid, the $F \wedge F'$ is $C \times C'$-solid.*

Proof. By assumption, F is an iterated extension in the sense of Remark 1 of pointed sheaves h_{V_i/U_i} with $U_i \to V_i$ in C. Similarly for F', with $U_j' \to V_j'$ in C'. The smash product $F \wedge F'$ is then an iterated extension of the

$$h_{V_i/U_i} \wedge h_{V_j'/U_j'} = h_{V_i \times V_j'/((U_i \times V_j') \cup (V_i \times U_j'))},$$

taken for instance in the lexicographical order, hence it is $C \times C'$ solid. $\square$

Definition 8. A morphism is called ind-solid relative to C if it is a filtering colimit of C-solid morphisms.

Exercise 39. We take G to be the trivial group. A section on Y of a push-out

$$\begin{tikzcd} h_U \arrow[r] \arrow[d, "\psi"'] & h_X \arrow[d] \\ F \arrow[r] & G \end{tikzcd}$$

can be described as follows. For an open subset V of Y and a section ϕ of F on V consider on the small Nisnevich site Y_{Nis} of Y the presheaf $\Phi(V, \phi)$ which sends $a : W \to Y$ to the set of morphisms $f : W \to X$ such that $f^{-1}(U) = a^{-1}(V)$ and $\phi_{|a^{-1}(V)} = f^*(\psi)$. A section of G on Y is given by data:

1. An open subset V of Y.
2. A section ϕ of F on V.

3. A section of $i^*(a_{Nis}\Phi(V,\phi))$ on $Y - V$ where i is the closed embedding $Y - V \to Y$ and a_{Nis} denotes the associated Nisnevich sheaf.

Exercise 40. In the notations of Exercise 39, if F is a sheaf for the etale topology, so is G. For any Y, the (V,ϕ) as in (1),(2) above form a sheaf for the etale topology. It hence suffices to prove that for (V,ϕ) fixed, the datum (3) forms a sheaf for the etale topology. This is checked by using the following criterion to check if a Nisnevich sheaf is etale. For $y \in Y$, and for L a finite separable extension of k_y, let $\mathcal{O}^h_{L,y}$ be deduced by "extension of the residue field" from the henselization $\mathcal{O}^h_y$ of Y at y. The criterion is that $Spec(L) \mapsto F(Spec(\mathcal{O}^h_{L,y}))$ should be an etale sheaf on $Spec(k_y)_{et}$.

Exercise 41. It follows from Exercises 39 and 40 that if $f : F \to G$ is ind solid, and if F is etale, then G is etale. In particular, a solid sheaf, as well as a pointed solid sheaf, are etale sheaves.

Remark 2. The same formalism of open and solid morphisms holds in the site of all schemes of finite type over S with the etale topology.

4.2 A Criterion for Preservation of Local Equivalences

We work with pointed sheaves on QP/G. Our aim in this section is to prove the following result

Theorem 6. *Let Φ be a functor from pointed sheaves to pointed sets. Suppose that Φ commutes with all colimits, and that for any open embedding $U \to X$, $\Phi((h_U)_+) \to \Phi((h_X)_+)$ is a monomorphism. Then if $f : F_\bullet \to G_\bullet$ is a local equivalence and if F_n and G_n are (pointed) ind-solid, then $\Phi(f)$ is a weak equivalence.*

Suppose that

$$Q \quad : \quad \begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow \\ A & \longrightarrow & X \end{array}$$

is an upper distinguished square. Adding a base point, we obtain Q_+. The morphism $K_{Q_+} \to X_+$ is then a local equivalence. Let us check that $\Phi(K_{Q_+}) \to \Phi(X_+)$ is a weak equivalence. As Φ commutes with coproducts, this morphism is deduced from the commutative square

$$\begin{array}{ccc} \Phi((h_B)_+) & \longrightarrow & \Phi((h_Y)_+) \\ \downarrow & & \downarrow \\ \Phi((h_A)_+) & \longrightarrow & \Phi((h_X)_+) \end{array}$$

by applying the same construction (23.3). This square is cocartesian because Q is. The top horizontal line being a monomorphism, it is homotopy cocartesian, and the claim follows. As Φ commutes with colimits, this special case implies that more generally one has

Lemma 15. *If f is in the $\bar{\Delta}$-closure of the $(K_Q)_+ \to X_+$ as above, then $\Phi(f)$ is a weak equivalence.*

Proposition 42. *(Proof of Theorem 6)* For any pointed sheaf F, $R(F) \to F$ is a local equivalence. Indeed for any connected X in QP/G, $R(F)(X) \to F(X)$ is a weak equivalence by Construction 29. It follows that for $f : F \to G$ a local equivalence,

$$\begin{array}{ccc} R(F) & \longrightarrow & R(G) \\ \downarrow & & \downarrow \\ F & \longrightarrow & G \end{array}$$

is a commutative square of local equivalences. By Lemmas 15 and 12, $\Phi(R(f))$ is a weak equivalence. It remains to see that $\Phi(R(F)) \to \Phi(F)$ is a weak equivalence – and the same for G. For this it suffices to see that for a pointed ind-solid sheaf F, $\Phi(R(F)) \to \Phi(F)$ is a weak equivalence. As Φ and R commute with filtering colimits, the ind-solid reduces to solid, and by the inductive definition of solid, it suffices to prove the following lemma.

Lemma 16. *Let $U \to X$ be an open embedding. If in a cartesian square of pointed sheaves*

$$Q \quad : \quad \begin{array}{ccc} (h_U)_+ & \longrightarrow & (h_X)_+ \\ \downarrow & & \downarrow \\ F & \longrightarrow & G \end{array}$$

F is such that $\Phi R(F) \to \Phi(F)$ is a weak equivalence, the same holds for G.

Proof. Consider the cocartesian square

$$Q' \quad : \quad \begin{array}{ccc} R((h_U)_+) & \longrightarrow & R((h_X)_+) \\ \downarrow & & \downarrow \\ R(F) & \longrightarrow & R \end{array}$$

One can easily see that the top morphism is a monomorphism. It follows that Q' is point by point homotopy cocartesian, and $R \to R(G)$ is a local equivalence. The functor $i^* i_*$ of Construction 29 transforms a monomorphism into a coprojection of the form $A \to A \vee ((\coprod h_{U_i})_+)$. It follows that each R_n is of the form $(\coprod h_{U_i})_+$ and, by Lemmas 15 and 12, $\Phi(R) \to \Phi(R(G))$ is a weak equivalence. It remains to show that $\Phi(R) \to \Phi(G)$ is a weak equivalence.

Let us apply Φ to the morphism of cocartesian squares $Q' \to Q$. By Construction 29 both $R((h_U)_+) \to (h_U)_+$ and $R((h_X)_+) \to (h_X)_+$ are homotopy equivalences, and remain so by applying Φ. We assumed $\Phi R(F) \to \Phi(F)$ to be a weak equivalence. As $\Phi(Q')$ and $\Phi(Q)$ are cocartesian with a top morphism which is a monomorphism (by the assumption on Φ, for Q), it follows that $\Phi(R) \to \Phi(G)$ is a weak equivalence. Hence so is $\Phi(R(G)) \to \Phi(G)$. □

5 Two Functors

5.1 The Functor $X \mapsto X/G$

One has a natural morphism of sites

$$\eta : (QP/G)_{Nis} \to (QP)_{Nis}$$

given by the functor

$$\eta^f : QP \to QP/G : X \mapsto (X \text{ with the trivial } G\text{-action})$$

Indeed, the functor η^f commutes with finite limits and transforms covering families into covering families.

In particular the functor η^f is continuous: if F is a sheaf on $(QP/G)_{Nis}$, the presheaf

$$X \mapsto F(X \text{ with the trivial } G\text{-action})$$

is a sheaf on $(QP)_{Nis}$. The functor η^f has a left adjoint $\lambda^f : X \mapsto X/G$. As η^f is continuous, the functor λ^f is cocontinuous, and the functor η^* from sheaves on $(QP)_{Nis}$ to sheaves on $(QP/G)_{Nis}$ is

$$F \mapsto (\text{sheaf associated to the presheaf } X \mapsto F(X/G))$$

Proposition 43. *The cocontinuous functor $\lambda^f : X \mapsto X/G$ is also continuous, that is, if F is a sheaf on $(QP)_{Nis}$, the presheaf $X \mapsto F(X/G)$ on $(QP/G)_{Nis}$ is a sheaf.*

Proof. By Lemma 2 it is sufficient to test the sheaf property of $X \mapsto F(X/G)$ for a covering of X deduced by pull-back from a Nisnevich covering $V_i \to X/G$ of X/G. Passage to quotient commutes with flat base change. Taking as base X/G, this gives that

$$X \times_{X/G} V_i \to V_i$$

identifies V_i with the quotient of $X \times_{X/G} V_i$ by G. Similarly, if $V_{ij} = V_i \times_{X/G} V_j$, the quotient by G of the pull-back to X of V_{ij} is V_{ij} again. This reduces the sheaf property of $X \mapsto F(X/G)$, for the covering of X by the $X \times_{X/G} V_i$, to the sheaf property of F for the covering (V_i) of X/G. □

The functor $\lambda^f : X \mapsto X/G$ gives rise to a pair of adjoint functors (λ_*, λ^*) between the categories of presheaves on (QP/G) and (QP), with $\lambda_*(F)$ being $X \mapsto F(X/G)$. As λ^f is continuous , it induces a similar pair of adjoint functors between the categories of sheaves. This pair is

$$(\eta_\# := (\text{associated sheaf}) \circ \lambda^*, \eta^* = \lambda_*),$$

so that one has a sequence of adjunctions $(\eta_\#, \eta^*, \eta_*)$. If F on (QP/G) is representable: $F = h_X$, then $\eta_\#(F) = h_{X/G}$. In particular, $\eta_\#$ transforms the final sheaf h_S on $(QP/G)_{Nis}$, also called "point", into the final sheaf on $(QP)_{Nis}$, and $(\eta_\#, \eta^*)$ is a pair of adjoint functors in the category of pointed sheaves as well. It is clear that $\eta_\#$ takes solid sheaves to solid sheaves. We also have the following.

Proposition 44. *Let F be a pointed sheaf solid with respect to open embeddings $U \subset V$ of smooth schemes such that the action of G on V is free outside U. Then $\eta_\#(F)$ is solid with respect to open embeddings of smooth schemes.*

Proof. If V' is the open subset of V where the action of G is free, then $U \cup V' = V$ and if $U' := U \cap V'$, a push-out of $U \to V$ is also a push-out of $U' \to V'$: we gained that the action is free everywhere. The next step is applying $\eta_\#$, from pointed sheaves on (QP/G) to pointed sheaves on (QP). This functor is a left adjoint, hence respects colimits and in particular push-outs. It transforms h_U to $h_{U/G}$, and in particular, for $U = S$, the final object into the final object. To check that it respects solidity it is hence sufficient to apply:

Lemma 17. *If G acts freely on U smooth over S, then U/G is smooth.*

Proof. If G is finite etale, for instance S_n, the case which most interests us, this is clear, resulting from $U \to U/G$ being etale. In general one proceeds as follows. The assumption that G acts freely on U implies that U is a G-torsor over U/G. In particular, $U \to U/G$ is faithfully flat. As U is flat over S, this forces U/G to be flat over S. To check smoothness of U/G over S it is hence sufficient to check it geometric fiber by geometric fiber. For $\bar{s}$ a geometric point of S, smoothness of $(U/G)_{\bar{s}}$ amounts to regularity. As $U_{\bar{s}}$ is smooth over $\bar{s}$, hence regular, and $U_{\bar{s}} \to (U/G)_{\bar{s}}$ is faithfully flat, this is [4] (an application of Serre's cohomological criterion for regularity). $\square$

Proposition 45. *The functor $\eta_\#$ respects local (resp. $\mathbf{A}^1$-) equivalences between termwise ind-solid simplicial sheaves.*

Proof. Let $f : F \to F'$ be a local equivalence between termwise ind-solid simplicial sheaves on QP/G. To verify that $\eta_\#(f)$ is a local equivalence it is sufficient to check that for any X in QP and $x \in X$ the map

$$\eta_\#(F)(Spec\mathcal{O}^n_{X,x}) \to \eta_\#(F')(Spec\mathcal{O}^n_{X,x})$$

is a weak equivalence of simplicial sets. Since $\eta_\#$ is a left adjoint, the functor

$$F \mapsto \eta_\#(F)(Spec\mathcal{O}^n_{X,x}) \tag{45.1}$$

commutes with colimits. For an open embedding $U \to V$ in QP/G, $U/G \to V/G$ is again an open embedding and we can apply to (45.1) Theorem 6.

Let $f : F \to F'$ be an $\mathbf{A}^1$-equivalence. Consider the square

$$\begin{array}{ccc} R(F) & \xrightarrow{R(f)} & R(F') \\ \downarrow & & \downarrow \\ F & \xrightarrow{f} & F' \end{array}$$

By the first part of proposition $\eta_\#$ takes the vertical maps to local equivalences. Since $\eta_\#$ commutes with colimits, Lemma 13 implies that $\eta_\#(R(f))$ is in the $\bar{\Delta}$-closure of the class which contains $\eta_\#((K_Q)_+ \to X_+)$ for Q upper distinguished and $\eta_\#((X \times \mathbf{A}^1)_+ \to X_+)$ for X in QP/G. By Theorem 4 it suffice to prove that morphisms of these two types are $\mathbf{A}^1$-equivalences. For morphisms of the first type it follows from the first half of the proposition. For the morphism of the second type it follows from the fact that morphisms $\eta_\#((X \times \mathbf{A}^1)_+ \to X_+)$ and $\eta_\#(X_+ \overset{Id\times\{0\}}{\to} (X \times \mathbf{A}^1)_+)$ are mutually inverse $\mathbf{A}^1$-homotopy equivalences. □

Define $\mathbf{L}\eta_\# : Ho_\bullet \to Ho_\bullet$ (and similarly on $Ho_{\mathbf{A}^1,\bullet}$) setting

$$\mathbf{L}\eta_\#(F) := \eta_\#(R(F))$$

where $R(F)$ is defined in Construction 29. Proposition 45 shows that $\mathbf{L}\eta_\#$ is well defined and that for a termwise ind-solid F one has $\mathbf{L}\eta_\#(F) \cong \eta_\#(F)$.

5.2 The Functor $X \mapsto X^W$

As in Sect. 3.1, we fix G and S. We also fix W in QP/G which is finite and flat over S.

For F a presheaf on QP/G, we define F^W to be the internal hom object $\underline{Hom}(h_W, F)$. Its value on U is $F(U \times_S W)$. If F is a sheaf, so is F^W.

Example 6. Take G and W deduced from the finite group S_n acting on $\{1, \ldots, n\}$ by permutations. In that case, if F is represented by X, with a trivial action of S_n, then F^W is represented by X^n, on which S_n acts by permutation of the factors.

Remark 1. If F is representable (resp. and represented by X smooth over S), so is F^W. More precisely, if F is represented by X in QP/G, consider the contravariant functor on Sch/S of morphisms of schemes from W to X, that is the functor

$$U \mapsto Hom_U(W \times_S U, X \times_S U)$$

This functor is representable, represented by some Y quasi-projective over S (resp. and smooth). This Y carries an obvious action ρ of G, and (Y, ρ) in QP/G represents F^W. Proof: by attaching to a morphism $W \to X$ its graph, one maps the functor considered into the functor of finite subschemes of $W \times_S X$, of the same rank as W, that is the functor

$$U \mapsto \left\{ \begin{array}{c} \text{subschemes of } (W \times_S X) \times_S U \text{ finite and flat over } U, \\ \text{with the same rank as } W \times_S U \text{ over } U. \end{array} \right\}$$

The later functor is represented by a quasi-projective scheme *Hilb*, by the theory of Hilbert schemes. The condition that $\Gamma \subset W \times_S X$ be the graph of a morphism from W to X is an open condition. This means: let $\Gamma \subset (W \times_S X) \times_S U$ be a subscheme finite and flat over U. There is U' open in U such that for any base change $V \to U$, the pull-back Γ_V of Γ is the graph of some V-morphism from $W \times_S V$ to $W \times_S X$ if and only if V maps to U'. This gives the existence of the required Y, and that it is open in *Hilb*. If X is smooth the smoothness of Y follows from the infinitesimal lifting criterion. The quasi-projectivity follows from that of *Hilb*. On the functors represented, the action $g(y) = gyg^{-1}$ of G is clear. For T in QP/G, one has

$$Hom_{QP/G}(T, Y) = Hom_G(T, \underline{Hom}(W, X)) = Hom_G(T \times_S W, X) =$$

$$= Hom_{QP/G}(T \times_S W, X) = F^W(T)$$

Let C be a class of open embeddings in $(QP/G)_{Nis}$. We will simply say "solid" for "C-solid".

Theorem 7. *If F is a solid sheaf on $(QP/G)_{Nis}$, so is F^W.*

If a morphism of sheaves $A \to F$ is open, i.e. relatively representable by open (equivariant) embeddings, there is a natural sequence of sheaves intermediate between A^W and F^W. In the case considered in Example 6, and for $h_U \to h_X$, they are represented by the open equivariant subschemes $(X, U)^n_k$ of X^n consisting of those n-uples $(x_1, \ldots, x_n)$ for which at least k of the x_i are in U. The formal definition is as follows.

A section of F^W over T is a section s of F over $W \times_S T$. Let $U(s)$ be the equivariant open subscheme of $W \times_S T$ on which s is in A. The sheaf $(F, A)^W_k$ is the subsheaf of F^W consisting of those s such that all fibers $U(s)_t$ of $U(s)$ over T are of degree at least k. The condition that the fiber at k be of degree $\geq k$ is open in t, and it follows that the inclusion of $(F, A)^W_k$ in F^W is open. For $k = 0$, $(F, A)^W_k$ is simply F^W. For k large, it is A^W.

Lemma 18. *Suppose that $A \to F$ is deduced by push-out from an open map $B \to G$, so that we have a cocartesian square*

$$\begin{array}{ccc} B & \longrightarrow & G \\ \downarrow & & \downarrow \\ A & \longrightarrow & F \end{array} \tag{45.2}$$

Then, for each k, the cartesian square

$$\begin{array}{ccc} \text{(fiber product)} & \longrightarrow & (A \coprod G, A)_k^W \\ \downarrow & & \downarrow \\ (F, A)_{k+1}^W & \longrightarrow & (F, A)_k^W \end{array} \tag{45.3}$$

is cocartesian as well.

Proof. The site $(QP/G)_{Nis}$ has enough points: as a consequence of Lemma 2, for each X in QP/G and $x \in X/G$, the functor

$$F \mapsto colim \quad F(X \times_{X/G} V),$$

the limit being taken over the Nisnevich neighborhoods of x in X/G, is a point (= a fiber functor). The class of all such functors is clearly conservative. Such a functor depends only on $Y := X \times_{X/G} (X/G)_x^h$, where $(X/G)_x^h$ is the henselization of X/G at x, and Y can be any equivariant S-scheme which is a finite disjoint union of local henselian schemes essentially of finite type over S, and for which Y/G is local. We call such a scheme G-local henselian, and write $F \mapsto F(Y)$ for the corresponding fiber functor.

We will show that (45.3) becomes cocartesian after application of any of the fiber functors $F \mapsto F(Y)$ defined above. It suffices to show that for any s in $(F, A)_k^W(Y)$, the fiber of (45.3)(Y) above s is cocartesian in Set. This fiber is of the form

$$\begin{array}{ccc} K \times L & \longrightarrow & K \\ \downarrow & & \downarrow \\ L & \longrightarrow & \{s\} \end{array}$$

and such a square is cocartesian if and only if whenever K or L is empty, the other is reduced to one element. Here, we also know that $L \to \{s\}$ is a injective. It hence suffice to check that if L is empty, then K is reduced to one element. Fix s in $(F, A)_k^W(Y)$, and view it as a section of F over $W \times_S Y$. Let $U \subset W \times_S Y$ be the open equivariant subset where it is in A. The assumption that s be in $(F, A)_k^W$ means that the degree of the fiber U_g of $U \to Y$ at a closed point y of Y is at least k. By G-equivariance of U, this degree is independent of the chosen y. We have to show that if s is not in $(F, A)_{k+1}^W(Y)$, that is if this degree is exactly k, then s is the image of a unique element of $(A \coprod G, A)_k^W$.

The scheme $W \times_S Y$ is a disjoint union of G-local henselian schemes $(W \times_S Y)_i$. By assumption, (45.2)($(W \times_S Y)_i$) is cocartesian, hence if s is not in A on $(W \times_S Y)_i$, then on $(W \times_S Y)_i$ it comes from a unique $\tilde{s}_i$ in G. Let $(W \times_S Y)'$ be the union of those $(W \times_S Y)_i$ on which s is in A, and $(W \times_S Y)''$ be the union of $(W \times_S Y)_i$ on which it is not. That s is in $(F, A)^W_k$ but not in $(F, A)^W_{k+1}$, means that $(W \times_S Y)'$ is of degree $d = k$ over Y. On $(W \times_S Y)'$, s is in A. On $(W \times_S Y)''$, it comes from a unique $\tilde{s}$ in G. The section

$$s_1 := (s \text{ in } A \text{ on } (W \times_S Y)', \tilde{s} \text{ on } (W \times_S Y)'')$$

of $A \coprod G$ over $W \times_S Y$ is a section of $(A \coprod G, A)^W_k$ on Y lifting s. It is the unique such lifting: any other lifting s_2, viewed as a section of $A \coprod G$ on $W \times_S Y$, can be in A at most on $(W \times_S Y)'$, hence must be in A on the whole of $(W \times_S Y)'$ which has just the required degree over Y. This determines s_2 uniquely on $(W \times_S Y)'$, where it is in A, as well as on $(W \times_S Y)''$, where it is the unique lifting of s to G. □

Proof of Theorem 7: The induction which works to prove Theorem 7 is the following. As F is solid, it sits at the end of a sequence

$$\emptyset \to F_1 \to \ldots \to F_n = F$$

where each $F_i \to F_{i+1}$ is a push-out of some open embedding in QP/G. We prove by induction on i that for any Y, $(F_i \coprod h_Y)^W$ is solid. For $i = 1$, F_1 is representable, hence so is $(F_1 \coprod h_Y)^W$ (1). A fortiori, $(F_1 \coprod h_Y)^W$ is solid. For the induction step one applies the following lemma to $F_i \coprod h_Y \to F_{i+1} \coprod h_Y$.

Lemma 19. *Let*

$$\begin{array}{ccc} h_U & \longrightarrow & h_X \\ \downarrow & & \downarrow \\ F & \longrightarrow & G \end{array}$$

be a cocartesian square with U open in X. Suppose that for any Z, $(F \coprod h_Z)^W$ is solid. Then $F^W \to G^W$ is solid.

Proof. As $F \to G$ is open by Proposition 33, the $(G, F)^W_j$ are defined. It suffices to prove that for each j, the open morphism $(G, F)^W_{j+1} \to (G, F)^W_j$ is solid.

By Lemma 18, this morphism sits in a cartesian and cocartesian square

$$\begin{array}{ccc} (\text{fiber product}) & \xrightarrow{[2]} & (F \coprod h_X, F)^W_k \\ \downarrow & & \downarrow \\ (G, F)^W_{j+1} & \xrightarrow{[1]} & (G, F)^W_j \end{array} \tag{45.4}$$

By assumption, $(F \coprod h_X)^W$ is solid. It follows that $(F \coprod h_X)^W_k$ is solid too (apply Proposition 37 to the open morphism $(F \coprod h_X)^W_k \to (F \coprod h_X)^W$). As [1] is open, so is [2], and by Proposition 36, [2] is solid. The map [1] is then solid as a push-out of a solid map. □

Example 7. It is not always true that if $f : A \to B$ is a solid morphism, so is f^W. Take G the trivial group and W two points (i.e. $S \coprod S$). Then $F^W = F^2$. For any sheaf F, the inclusion f of F in $F \coprod pt$ is solid (deduced by push-out from $\emptyset \to pt$), and applying $(-)^W$, we obtain $F^2 \to F^2 \coprod F \coprod F \coprod pt$. Pulling back by the natural map from F to one of the summands F (an open map), we see that if f^W is solid, so is F.

Corollary 46. *If $f : F \to G$ is open and G solid, then $f^W : F^W \to G^W$ is solid. In particular, if G is pointed solid, so is G^W.*

Proof. f^W is open and one applies Theorem 7 and Proposition 36. □

We now define for pointed sheaves on $(QP/G)_{Nis}$ a "smash" variant of the construction $F \mapsto F^W$. If we assume that the marked point $pt \to F$ is open, it is

$$F^{\wedge W} := F^W/(F, pt)_1^W$$

that is F^W with $(F, pt)_1^W$ contracted to the new base point. This definition can be repeated for any pointed sheaf if, for any monomorphism $A \to F$, we define$(F, A)_1^W$ to be the following subsheaf of F^W: a section s of $F^W(X) = F(X \times W)$ is in $(F, A)_1^W$ if for any non empty $X' \to X$, there exists a commutative diagram

$$\begin{array}{ccc} X'' & \longrightarrow & X \times W \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

with X'' non empty and s in A on X''.

Example 8. Under the assumptions of Example 6, if U is open in X with complement Z and if $F = h_X/h_U$, then $F^{\wedge W}$ is $h_{X^n}/h_{X^n - Z^n}$, where X^n has the natural action of S_n. In particular, if F is the Thom space of a vector bundle E over Y (that is, h_E with $h_{E - s_0(Y)}$ contracted to the base point), then $F^{\wedge W}$ is the Thom space of the S_n-equivariant vector bundle $\oplus pr_i^* E$ on Y^n.

Proposition 47. *If a pointed sheaf F is pointed solid, that is if $pt \to F$ is solid, then so is $F^{\wedge W}$.*

Proof. As F^W is solid, the open morphism $(F, pt)_1^W \to F^W$ is solid too (Proposition 36), and $pt \to F^{\wedge W}$ is deduced from it by push-out. □

The definition of $F^{\wedge W}$ immediately implies the following:

Lemma 20. *Let Y be a G-local henselian scheme. Then*

$$F^{\wedge W}(Y) = \bigwedge_i F((W \times Y)_i)$$

where $(W \times Y)_i$ are G-local henselian schemes such that

$$W \times Y = \coprod_i (W \times Y)_i$$

Proposition 48. *The functor $F^{\wedge W}$ respects local and $\mathbf{A}^1$-equivalences.*

Proof. Let $f : F \to H$ be a local equivalence. To check that $F^W \to H^W$ is a local equivalence it is enough to show that for any G-local henselian Y, the map

$$F^W(Y) = F(Y \times W) \to H(Y \times W) = H^W(Y)$$

is a weak equivalence of simplicial sets. This follows from Lemma 20.

Let f be an $\mathbf{A}^1$-equivalence. By the first part it is sufficient to show that $R(f)^W : R(F)^W \to R(H)^W$ is an $\mathbf{A}^1$-equivalence. We use the characterization of $\mathbf{A}^1$-equivalences given in Theorem 5. Since $F \mapsto F^{\wedge W}$ commutes with filtering colimits and preserves local equivalences it suffices to check that it takes $\mathbf{A}^1$-homotopy equivalences to $\mathbf{A}^1$-homotopy equivalences. This is seen using the natural map

$$F^{\wedge W} \wedge (h_{\mathbf{A}^1})_+ \to (F \wedge (h_{\mathbf{A}^1})_+)^{\wedge W}.$$

□

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